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# Bernstein Superalgebras and Supermodules

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Bernstein superalgebras are introduced and irreducible Bernstein supermodules are classified. 

## 1. INTRODUCTION

A *Bernstein algebra* is a commutative algebra over a field $F$ with a nonzero algebra homomorphism $\sigma: B \to F$ satisfying the identity

$$(x^2)^2 = \sigma(x)^2 x^2.$$

These algebras were first introduced by Lyubich [6] and Holgate [4], in order to give an algebraic formulation of the Bernstein problem from mathematical genetics on the description of all the stationary heredity operators. Since then Bernstein algebras were studied by many authors (see [2, 5, 7, 8]).

The purpose of this work is to introduce Bernstein superalgebras and to describe irreducible Bernstein supermodules. Since Bernstein algebras do not form a variety of algebras, the definition of a Bernstein superalgebra is not so evident; it is not clear how "to superize" the defining identity which involves the homomorphism $\sigma$. To overcome this difficulty, we consider instead of Bernstein algebras their nucleus which does form a variety of



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graded algebras with respect to the grading induced by Peirce decomposition. We call these graded algebras $(U,W)$-graded Bernstein algebras. The notions of a $(U,W)$-graded Bernstein superalgebra and of supermodule over such a superalgebra are then defined in a natural way. We show that while there are no nontrivial simple or prime Bernstein superalgebras, nontrivial irreducible Bernstein supermodules do exist, and we describe all such supermodules.

Throughout the work, $F$ denotes a field of characteristic $\neq 2$, and all the algebras are considered over $F$.

## 2. $(U,W)$-GRADED BERNSTEIN ALGEBRAS

We recall some known facts about Bernstein algebras and their nucleus (see, for example, [2, 7, 8]).

Any Bernstein algebra $B$ has an idempotent $e$ $(\neq 0)$, and with respect to $e$ the algebra $B$ has a Peirce decomposition,

$$B = Fe \oplus U_e \oplus W_e, \tag{1}$$

where $U_e = \{x \in B \mid ex = \frac{1}{2}x\}$, $W_e = \{x \in B \mid ex = 0\}$, and $N = U_e \oplus W_e = \ker \sigma$. The ideal $N = U_e \oplus W_e$ is called a *nucleus* of a Bernstein algebra $B$. Furthermore, the Peirce components satisfy the relations

$$U_e^2 \subseteq W_e, \qquad W_e^2 \subseteq U_e, \qquad U_eW_e \subseteq U_e, \tag{2}$$

$$u^3 = 0, \qquad u(uw) = 0, \qquad uw^2 = 0, \qquad (u^2)^2 = 0, \qquad (uw)^2 = 0, \tag{3}$$

for any $u \in U_e$, $w \in W_e$.

If $e$ is a fixed idempotent of $B$ then the set $Id(B)$ of all the idempotents of $B$ is given by

$$Id(B) = \{e + u + u^2 \mid u \in U_e\}, \tag{4}$$

and for any idempotent $e' = e + u + u^2$ holds

$$U_{e'} = \{x + 2ux \mid x \in U_e\}, \tag{5}$$

$$W_{e'} = \{y - 2(u + u^2)y \mid y \in W_e\}. \tag{6}$$

Moreover, the dimensions of $U_e$ and $W_e$ do not depend on the idempotent $e$.

One of difficulties in the study of Bernstein algebras is caused by the fact that they do not form a variety of algebras. Now we give a definition of a $(U,W)$*-graded Bernstein algebra* that allows us to study instead of Bernstein algebras their graded nucleus, which does form a (graded) variety.

DEFINITION 2.1. Let $A = U + W$ be a commutative algebra over a field $F$ which is a direct sum of vector subspaces $U$ and $W$ such that

$$U^2 \subseteq W, \qquad W^2 \subseteq U, \qquad UW \subseteq U.$$

Then, $A$ is called a $(U, W)$-graded Bernstein algebra or simply a $(U, W)$-Bernstein algebra if $A$ satisfies the identities

$$u^3 = 0, \qquad u(uw) = 0, \qquad uw^2 = 0, \qquad (u^2)^2 = 0, \qquad (uw)^2 = 0,$$

for any $u \in U$, $w \in W$.

Observe that a $(U, W)$-Bernstein algebra $A = U + W$ is graded by the commutative groupoid $S = \{s, t \mid s^2 = t,\ t^2 = s,\ st = ts = s\}$, if we set $A_s = U$, $A_t = W$. This justifies the term *graded Bernstein algebra*.

PROPOSITION 2.1. (i) *For any Bernstein algebra B and any idempotent e the nucleus $N(B)$ is a $(U_e, W_e)$-Bernstein algebra.*

(ii) *Let $A = U + W$ be a commutative algebra which is a direct sum of subspaces U and W. Consider the vector space direct sum $A(e) = Fe + A$ and extend a multiplication from A to $A(e)$ by setting*

$$e^2 = e, \quad eu = \frac{1}{2}u, \quad ew = 0 \qquad \text{for any } u \in U, w \in W.$$

*Furthermore, define the mapping $\sigma\colon A(e) \to F$ by $\sigma(\alpha e + u + w) = \alpha$. Then, A is a $(U, W)$-Bernstein algebra if and only if $B = A(e)$ is a Bernstein algebra. In such case $N(B) = A$ and $U_e(B) = U$, $W_e(B) = W$.*

*Proof.* It suffices to apply properties (2) and (3) of the Peirce decomposition of $B$ with respect to $e$ and to check in a straightforward way that the algebra $A(e)$ satisfies the identity $(x^2)^2 = \sigma(x)^2 x^2$. ∎

A commutative algebra $A$ may carry nonisomorphic $(U, W)$-Bernstein structures, as the following examples show.

EXAMPLE 1. Let $A$ be a commutative algebra over $F$ with the basis $\{a, b, c, ac, bc\}$ and the only nonzero products $a \cdot c = ac$ and $b \cdot c = bc$. Then $A^3 = 0$, so identities (3) hold in $A$ for every grading. Consider the gradings:

(1) $U = \{ac, bc\}$ and $W = \{a, b, c\}$. We have $U^2 = 0$, $W^2 \subseteq U$, and $UW = 0$, hence $A$ is a $(U, W)$-Bernstein algebra.

(2) $U' = \{c, ac, bc\}$ and $W' = \{a, b\}$. In this case $U'^2 = 0$, $W'^2 = 0$, and $U'W' \subseteq U'$. Thus, $A$ is also a $(U', W')$-Bernstein algebra.

Since $\dim U = 2$ and $\dim U' = 3$, the graded algebras $A = U + W$ and $A = U' + W'$ are not isomorphic. Moreover, the corresponding Bernstein algebras $A(e)$ are not isomorphic.

The next example is more delicate.

EXAMPLE 2. Let $A$ be a commutative $F$-algebra with the basis $\{u_1, u_2, w_1, w_2\}$ and the only nonzero products $u_1 w_1 = u_2$, $u_2 w_2 = u_1$. Consider the gradings:

(1) $U = \{u_1, u_2\}$, $W = \{w_1, w_2\}$. Then $U^2 = W^2 = 0$, $UW \subseteq U$, and identities (3) are satisfied, so $A$ is a $(U, W)$-Bernstein algebra.

(2) $U' = U$, $W' = Fw'_1 + Fw'_2$, where $w'_1 = w_1 - 2u_2$, $w'_2 = w_2 - 2u_1$. Then $U'^2 = 0$, $W'^2 \subseteq U'$, $U'W' \subseteq U'$, and again identities (3) are satisfied, so $A$ is a $(U', W')$-Bernstein algebra. But in this case $W'^2 \neq 0$, so $A = U + W$ and $A = U' + W'$ are not isomorphic as graded algebras.

Nevertheless, in this example the corresponding Bernstein algebras $B = Fe + U + W$ and $B' = Fe' + U' + W'$ are isomorphic: the map $\phi$: $e \mapsto e' - (u_1 + u_2)$, $u \mapsto u$, $w_1 \mapsto w'_1 + 2u_2$, $w_2 \mapsto w'_2 + 2u_1$ is an isomorphism of algebras.

The last example suggests the following definition, that will provide an equivalence relation between $(U, W)$-graded Bernstein algebras, so that two such algebras are equivalent if and only if they come from the same Bernstein algebra.

DEFINITION 2.2. Let $A$ be a $(U, W)$-Bernstein algebra. For each $u \in U$ we define a $u$-isotope $A^{(u)} = U^{(u)} + W^{(u)}$ of $A$ as the same algebra $A$ with the new grading:

$$U^{(u)} = \{x + 2ux \mid x \in U\},$$
$$W^{(u)} = \{y - 2(u + u^2)y \mid y \in W\}.$$

PROPOSITION 2.2. *Let $A$ be a $(U, W)$-Bernstein algebra and $u \in U$. Then $A^{(u)}$ is a $(U^{(u)}, W^{(u)})$-Bernstein algebra.*

*Proof.* Consider the Bernstein algebra $A(e)$. By (4), we have $e' = e + u + u^2 \in Id(A(e))$, and so by (1), $A(e)$ has the following Peirce decomposition: $A(e) = Fe' + U_{e'} + W_{e'}$. By (5) and (6), $U_{e'} = U^{(u)}$ and $W_{e'} = W^{(u)}$. Now, by Proposition 2.1(i), $A^{(u)} = U^{(u)} + W^{(u)}$ is a $(U^{(u)}, W^{(u)})$-Bernstein algebra. ∎

DEFINITION 2.3. Let $A = U + W$ and $A' = U' + W'$ be two $(U, W)$-Bernstein algebras. We say that $A$ is isotopic to $A'$ if $A$ is isomorphic as a graded Bernstein algebra to $A'^{(u')}$ for some $u' \in U'$.

PROPOSITION 2.3. *Let $A$ and $A'$ be $(U, W)$-Bernstein algebras. Then $A$ is isotopic to $A'$ if and only if $A(e)$ is isomorphic to $A'(e')$.*

*Proof.* Let $A = U + W$, $A' = U' + W'$. Assume first that $A$ is isotopic to $A'$. Then there exist $u' \in U'$ and an isomorphism of algebras $f\colon A \to A'$ such that $f(U) = U'^{(u')}$, $f(W) = W'^{(u')}$. Consider $A(e)$ and $A'(e')$. Then $\bar{e} = e' + u' + u'^2 \in Id(A'(e'))$ by (4), and $U_{\bar{e}} = U'^{(u')}$, $W_{\bar{e}} = W'^{(u')}$ by (5) and (6). Therefore, $A'(e') = F\bar{e} + U_{\bar{e}} + W_{\bar{e}}$. Now, consider $\bar{f}\colon A(e) \to A'(e')$ which extends $f$ and sends $e$ to $\bar{e}$. Evidently, $\bar{f}$ is an isomorphism of algebras, so $A(e)$ is isomorphic to $A'(e')$.

Conversely, let $f\colon A(e) \to A'(e')$ be an isomorphism of algebras. Then $f(e) \in Id(A'(e'))$ and so by (4) there exists $u' \in U'$ such that $f(e) = e' + u' + u'^2$. Therefore,

$$f(U) = f(U_e) = U_{f(e)} = U_{e'}^{(u')} = U'^{(u')},$$

$$f(W) = f(W_e) = W_{f(e)} = W_{e'}^{(u')} = W'^{(u')}.$$

Consider $A'^{(u')}$. The restriction $f|_A\colon A \to A'^{(u')}$ is an isomorphism of graded Bernstein algebras, hence $A$ is isotopic to $A'$. ∎

COROLLARY 2.1. *Isotopy of $(U, W)$-Bernstein algebras is an equivalence relation.*

Thus the classification of Bernstein algebras up to isomorphism is equivalent to the classification of $(U, W)$-Bernstein algebras up to isotopy.

## 3. BERNSTEIN SUPERALGEBRAS

Recall that a superalgebra, in general, is a $Z_2$-graded algebra, that is, an algebra $A = A_0 + A_1$ which is a direct sum of subspaces $A_0$ (the even part) and $A_1$ (the odd part), with a product satisfying $A_i A_j \subseteq A_{i+j \pmod 2}$. We will define Bernstein superalgebras according to the general definition of a superalgebra in an arbitrary homogeneous variety of algebras (see [10]).

Let $\Gamma = \mathrm{alg}\langle 1, g_i \mid i = 1, 2, \dots \rangle$ be the *Grassmann* (or *exterior*) *F-algebra* over a countable number of generators $g_i$, with $g_i^2 = 0$, $g_i g_j = -g_j g_i$ for $i \neq j$. The elements $1, g_{i_1} g_{i_2} \cdots g_{i_r}$, $i_1 < i_2 < \cdots < i_r$ form an $F$-basis of $\Gamma$. If we denote by $\Gamma_0$ (respectively by $\Gamma_1$) the span of the products of even length (respectively of odd length), then $\Gamma = \Gamma_0 + \Gamma_1$ is a superalgebra. One can easily check that this superalgebra is *supercommutative*, that is, satisfies the identity

$$ab = (-1)^{\delta a\, \delta b} ba,$$

where $a, b \in \Gamma_0 \cup \Gamma_1$ and for $a \in A_i$, $\delta a = i$.

For a superalgebra $A = A_0 + A_1$, the subalgebra $\Gamma(A) = \Gamma_0 \otimes A_0 + \Gamma_1 \otimes A_1$ of the tensor product $\Gamma \otimes A$ is called the *Grassmann envelope* of $A$. One can also consider the Grassmann envelope $\Gamma(B)$ of any homogeneous subspace $B$ of $A$, which in general is not a subalgebra but a subspace of $\Gamma(A)$.

DEFINITION 3.1. A superalgebra $A = A_0 + A_1$, with a homogeneous $(U, W)$-grading $A_i = U_i + W_i$, $i = 0, 1$, where $U = U_0 + U_1$ and $W = W_0 + W_1$, is called a $(U, W)$-Bernstein superalgebra, or simply a Bernstein superalgebra, if its Grassmann envelope $\Gamma(A)$ is a $(\Gamma(U), \Gamma(W))$-Bernstein algebra.

PROPOSITION 3.1. *A superalgebra $A = A_0 + A_1$, with $A_i = U_i + W_i$ for $i = 0, 1$, is a $(U, W)$-Bernstein superalgebra if and only if it is supercommutative and satisfies the relations*:

(1) $U_iU_j \subseteq W_{i+j}$, $W_iW_j \subseteq U_{i+j}$, $U_iW_j \subseteq U_{i+j}$, *for any* $i, j \in \{0, 1\}$,

(2) $a^3 = (a^2)^2 = 0$, *for any* $a \in U_0$,

(3) $u_1u_2 \cdot u_3 + (-1)^{\delta u_2\, \delta u_3} u_1u_3 \cdot u_2 + u_1 \cdot u_2u_3 = 0$,

(4) $u_1 \cdot u_2w + (-1)^{\delta u_1\, \delta u_2} u_2 \cdot u_1w = 0$,

(5) $u \cdot w_1w_2 = 0$,

(6) $u_1u_2 \cdot u_3u_4 + (-1)^{\delta u_2\, \delta u_3} u_1u_3 \cdot u_2u_4 + (-1)^{\delta u_4(\delta u_2 + \delta u_3)} u_1u_4 \cdot u_2u_3 = 0$,

(7) $u_1w_1 \cdot w_2u_2 + (-1)^{\delta w_1\, \delta w_2} u_1w_2 \cdot w_1u_2 = 0$,

*for any* $u, u_i \in U_0 \cup U_1$, $w, w_i \in W_0 \cup W_1$.

The proof is straightforward.

Note that identities (2) are superfluous if char $F \neq 3$.

The next proposition is a superanalogue of the corresponding result from [1, 3] for Bernstein algebras.

PROPOSITION 3.2. *Let $A = A_0 + A_1 = U + W$ be a Bernstein superalgebra. Consider $L = L(A) = \{u \in U \mid uU = 0\}$. Then*

(i) *$L$ is a graded ideal of $A$ and $L^2 = 0$.*

(ii) *The quotient superalgebra $A/L$ satisfies the graded identities*

$$a^3 = 0, \qquad \text{for any even } a, \tag{7}$$

$$x_1x_2 \cdot x_3 + (-1)^{\delta x_2\, \delta x_3} x_1x_3 \cdot x_2 + x_1 \cdot x_2x_3 = 0, \tag{8}$$

*for any homogeneous elements $x_i$. In particular, the superalgebra $A/L$ is Jordan.*

*Proof.* It is clear that $L$ is homogeneous and that $0 = LU \subset L$. Furthermore, it follows from Proposition 3.1(4) that $U \cdot LW \subseteq L \cdot UW \subseteq LU = 0$, hence $LW \subseteq L$ and $L$ is an ideal of $A$. Besides, $L^2 \subseteq LU = 0$, so (i) is proved.

To prove (ii), observe first that $A/L$ satisfies the graded identities (7) and (8) if and only if its Grassmann envelope $\Gamma(A/L)$ satisfies the identity $x^3 = 0$. It is easily seen that $\Gamma(A/L)$ is isomorphic to the quotient algebra $\Gamma(A)/\Gamma(L)$. We claim that $\Gamma(L) = L(\Gamma(A)) = \{x \in \Gamma(U) \mid x\Gamma(U) = 0\}$. In fact, clearly $\Gamma(L) \subseteq L(\Gamma(A))$. Conversely, let $x = \sum_i g_i \otimes u_i \in L(\Gamma(A))$, with the elements $g_i \in \Gamma$ homogeneous and linearly independent. For any $u \in U_0 \cup U_1$ choose $g \in \Gamma_0 \cup \Gamma_1$ such that $\delta u = \delta g$ and the elements $gg_i$ are still linearly independent. Then $g \otimes u \in \Gamma(U)$, so $x(g \otimes u) = \sum_i gg_i \otimes uu_i = 0$, which yields $uu_i = 0$ for every $i$. Therefore $u_i \in L(A)$ for all $i$, and $x \in \Gamma(L)$ as required.

It remains to note that the quotient algebra $\Gamma(A)/L(\Gamma(A))$ satisfies the identity $x^3 = 0$ by [3]. It is well known that a commutative algebra satisfying this identity is Jordan, hence the algebra $\Gamma(A/L)$ is Jordan and so is the superalgebra $A/L$. ∎

Note that (7) follows from (8) if char $F \neq 3$.

THEOREM 3.1. *There are no semiprime Bernstein superalgebras over a field $F$ of characteristic* $\neq 2, 3$.

*Proof.* Assume that $A$ is a semiprime Bernstein superalgebra. Since $L = L(A)$ is a graded ideal of $A$ with $L^2 = 0$, we have $L = 0$. Therefore $A$ satisfies (8) and is Jordan superalgebra. Consequently $\Gamma(A)$ is a Jordan algebra satisfying $x^3 = 0$. By [9], $\Gamma(A)^2$ is nilpotent, which clearly forces that $A^2$ is nilpotent. It is easily seen that a semiprime superalgebra satisfying (8) does not contain nilpotent ideals, so $A^2 = 0$ and $A = 0$, a contradiction. ∎

COROLLARY 3.1. *There are no simple Bernstein superalgebras over a field $F$ of characteristic* $\neq 2, 3$.

In the next section we will introduce Bernstein supermodules and will show that irreducible Bernstein supermodules do exist.

## 4. BERNSTEIN SUPERMODULES

Let $A = A_0 + A_1$ be a superalgebra over $F$ and $M$ be an $A$-bimodule. Then $M$ is a *superbimodule* over $A$ if it has a $Z_2$-grading $M = M_0 + M_1$ such that the corresponding split extension $E = A + M$ is a superalgebra with the grading $E_i = A_i + M_i$, $i = 0, 1$. In other words, $M = M_0 + M_1$ is a superbimodule over $A = A_0 + A_1$ if

$$A_i M_j + M_i A_j \subseteq M_{i+j\,(\mathrm{mod}\,2)}.$$

A superbimodule $M$ is *supercommutative* if $ma = (-1)^{\delta a\,\delta m} am$ for any homogeneous $m \in M$ and $a \in A$.

DEFINITION 4.1. Let $A = U + W$ be a $(U, W)$-Bernstein superalgebra and $M$ a supercommutative $A$-supermodule. Assume that $M$ admits a direct vector space decomposition $M = X + Y$ which is compatible with the $Z_2$-grading on $M$, that is, $M_i = X_i + Y_i$ for $X_i = X \cap M_i$, $Y_i = Y \cap M_i$, $i = 0, 1$. Then, $M$ is called an $(X, Y)$-Bernstein supermodule over $A$ if the split extension $E = A + M$ is a $(U + X, W + Y)$-Bernstein superalgebra, with $E_i = A_i + M_i$, $i = 0, 1$.

Applying the definition of Bernstein superalgebra, we get:

PROPOSITION 4.1. *Let $A = U + W$ be a $(U, W)$-Bernstein superalgebra and $M$ a supercommutative $A$-supermodule with an $(X, Y)$-decomposition $M = X + Y$. Then $M$ is an $(X, Y)$-Bernstein supermodule over $A$ if and only if*

(1) $XU \subseteq Y,\ YW \subseteq X,\ YU \subseteq X,\ XW \subseteq X,$

(2) $xu_1 \cdot u_2 + (-1)^{\delta u_1\,\delta u_2} xu_2 \cdot u_1 + x \cdot u_1 u_2 = 0,$

(3) $x \cdot wu + xw \cdot u = 0,$

(4) $yu_1 \cdot u_2 + (-1)^{\delta u_1\,\delta u_2} yu_2 \cdot u_1 = 0,$

(5) $x \cdot w_1 w_2 = 0,$

(6) $yw \cdot u = 0,$

(7) $xu_1 \cdot u_2 u_3 + (-1)^{\delta u_1\,\delta u_2} xu_2 \cdot u_1 u_3 + (-1)^{\delta u_3(\delta u_1 + \delta u_2)} xu_3 \cdot u_1 u_2 = 0,$

(8) $xw_1 \cdot w_2 u + (-1)^{\delta w_1\,\delta w_2} xw_2 \cdot w_1 u = 0,$

(9) $yu_1 \cdot u_2 w + (-1)^{\delta u_1\,\delta u_2} yu_2 \cdot u_1 w = 0,$

*for any $u, u_1, u_2, u_3 \in U$, $w, w_1, w_2 \in W$, $x \in X$, $y \in Y$.*

COROLLARY 4.1. *Let $A$ be a $(U, W)$-Bernstein superalgebra with $U = 0$ (or, equivalently, $A = W$). Then any supercommutative $A$-supermodule $M$ with an $(X, Y)$-decomposition $M = X + Y$ such that $MA \subseteq X$ is an $(X, Y)$-Bernstein $A$-supermodule.*

*Proof.* Indeed, in this case conditions (1)–(9) are obviously satisfied. ∎

COROLLARY 4.2. *Let $M = X + Y$ be an $(X, Y)$-Bernstein supermodule over $A = U + W$. Denote by $M^s$ the same module $M$ but with the opposite grading, that is, $(M^s)_0 = M_1$, $(M^s)_1 = M_0$, and with the new action of $A$,*

$$a \cdot m = (-1)^{\delta a} am, \qquad m \cdot a = ma.$$

*Then $M^s$ is again an $(X, Y)$-Bernstein supermodule over $A$, which is irreducible if and only if so is $M$.*

The proof is straightforward. ∎

Notice that the supermodule $M^s$ in general may be not isomorphic to $M$.

Before giving examples of irreducible Bernstein supermodules, we will prove a general result on irreducible supercommutative supermodules.

PROPOSITION 4.2. *Let $A = A_0 + A_1$ be a supercommutative superalgebra over $F$ and $M = M_0 + M_1$ an irreducible supercommutative $A$-supermodule. Then $M$ is an irreducible $A$-module.*

*Proof.* Let $N$ be a proper submodule of $M$. Consider the projections

$$\pi_i : M \to M_i, \qquad i \in \{0, 1\},$$

given by $\pi_0(m_0 + m_1) = m_0$, $\pi_1(m_0 + m_1) = m_1$. Since $M$ is irreducible as a graded module, we have $N \cap M_0 = N \cap M_1 = 0$. It follows easily that $\pi_i(N) \neq 0$ for both $i = 0, 1$, and the graded submodule $\pi_0(N) + \pi_1(N)$ is nonzero. Therefore $\pi_0(N) + \pi_1(N) = M$, and hence $\pi_0(N) = M_0$ and $\pi_1(N) = M_1$.

Let $n = m_0 + m_1$ be an arbitrary element from $N$. For any $x \in A_1$ we have

$$xn = xm_0 + xm_1 \in N,$$
$$nx = m_0 x + m_1 x = xm_0 - xm_1 \in N.$$

Subtracting the second equality from the first one, we get $2xm_1 \in N \cap M_0 = 0$, which yields $0 = A_1 \pi_1(N) = A_1 M_1$. Consequently, $xn = xm_0 + 0 \in N \cap M_1 = 0$, and as before, $A_1 M_0 = 0$. Resuming, we have $A_1 M = 0$. Thus $M$ is an irreducible $A_0$-module, which yields $M_1 = 0$ or $M_0 = 0$, a contradiction. ∎

Consider now two examples of irreducible Bernstein supermodules.

(1) *Supermodules of Type $M(K, V, \alpha)$.* Let $K$ be a field extension of $F$, $V$ an $F$-subspace of $K$, and $\alpha \in K$. Consider $A = A_1 = Vu$ with $A^2 = 0$ and $M = Kx + Ky$, where $M_0 = Kx$ and $M_1 = Ky$. Define an action of $A$ on $M$ by extending $K$-linearly the conditions $xu = ux = y$, $yu = -uy = \alpha x$. Then a trivial verification shows that $M = M(K, V, \alpha)$ is an $(X, Y)$-Bernstein supermodule over $A = U + 0$, with $X = X_0 = Kx$, $Y = Y_1 = Ky$.

Moreover, it can be easily seen that the corresponding split extension $E = A + M$ satisfies superidentities (7), (8), hence $E$ is a Jordan superalgebra and $M$ is a Jordan $A$-supermodule.

Observe that the opposite supermodule $M^s$ with $M_0^s = Ky$, $M_1^s = Kx$, and $yu = uy = \alpha x$, $xu = -ux = y$, is isomorphic to $M$ as a Jordan supermodule but not as Bernstein supermodule.

PROPOSITION 4.3. *The supermodule $M = M(K, V, \alpha)$ is irreducible if and only if $K = \mathrm{alg}_F\langle \alpha V^2 \rangle$.*

*Proof.* Assume that $K = \mathrm{alg}_F\langle \alpha V^2 \rangle$, and let $N \neq 0$ be a subsupermodule of $M$. Obviously, there exists $\beta x \in N$ for some $0 \neq \beta \in K$. We have $(xA)A = (\alpha V^2)x$, so $(\alpha V^2)\beta x = (\beta x A)A \subseteq (NA)A \subseteq N$. Hence $Kx = (K \cdot \beta)x = \mathrm{alg}_F\langle \alpha V^2 \rangle \beta x \subseteq N$ and $Ky = (Kx)A \subseteq N$. Therefore $M = N$.

Conversely, if $M$ is irreducible then the submodule $N$ generated by $x$ coincides with $M$, hence $N_0 = Kx$. On the other hand, it is easily seen that $N_0 \subseteq \mathrm{alg}_F\langle \alpha V^2 \rangle x$. Thus $K = \mathrm{alg}_F\langle \alpha V^2 \rangle$. ∎

*Remark* 4.1. If $F$ is algebraically closed then every supermodule $M = M(K, V, \alpha)$ of finite dimension over $F$ and with $\alpha \neq 0$ is isomorphic to $M(F, F, 1)$.

*Proof.* Indeed, in this case $\dim_F K < \infty$ and $K = V = F$. Moreover, considering $u' = (1/\sqrt{\alpha})u$ and $y' = (1/\sqrt{\alpha})y$, we can always suppose $\alpha = 1$. ∎

(2) *Supermodules of Type $M(W, X)$.* Let $M$ be a $Z_2$-graded vector space and $A$ a graded subspace of $\mathrm{End}(M)$. Consider $A$ as a superalgebra with $A^2 = 0$ and define an action of $A$ on $M$ by setting

$$am = (-1)^{\delta m\, \delta a} ma = a(m), \qquad \text{for any homogeneous } a \in A,\ m \in M.$$

Then $M$ is a supercommutative faithful $A$-supermodule. Set $A = W$, $M = X$. Then it follows from Corollary 4.1 that $M$ is an $(X, 0)$-Bernstein supermodule over the $(0, W)$-Bernstein superalgebra $A$. We will call this supermodule *a supermodule of type $M(W, X)$*.

Observe that any graded $A$-submodule of $M = M(W, X)$ is obviously also a graded Bernstein submodule, so $M$ is irreducible as a Bernstein supermodule if and only if it is an irreducible $A$-supermodule, which, in turn, by Proposition 4.2 is equivalent to the irreducibility of $M$ as $A$-module. The last condition is given by requiring the algebra $\mathrm{alg}_{\mathrm{End}(M)}\langle W \rangle$ to be dense in $\mathrm{End}(M)$. In particular, if $M$ is finite dimensional over $F$, this is equivalent to $\mathrm{alg}\langle W \rangle = \mathrm{End}(M)$.

Recall that an $A$-(super)module $M$ is called *almost faithful* if its annihilator $\mathrm{Ann}\, M = \{a \in A \mid Ma = 0\}$ does not contain nonzero ideals of $A$.

THEOREM 4.1. *Let $A$ be a $(U, W)$-Bernstein superalgebra over a field $F$ of characteristic $\neq 2, 3$ and $M$ be an almost faithful irreducible $(X, Y)$-Bernstein supermodule over $A$. Then $M$ is isomorphic to a supermodule of one of the types*: $M(K, V, \alpha)$, $M(K, V, \alpha)^s$, $M(W, X)$.

*Proof.* Consider the split extension $E = A + M$, and let $L(M) = L(E) \cap M = \{x \in X \mid x(U + X) = 0\}$. By Proposition 3.2, $L(E)$ is a graded ideal of $E$, hence $L(M)$ is a graded Bernstein submodule of $M$. Therefore, either $L(M) = 0$ or $L(M) = M$.

Suppose first that $L(M) = 0$, then $L(E)M \subseteq M \cap L(E) = L(M) = 0$, and since $M$ is an almost faithful supermodule, we have $L(E) = 0$. By Proposition 3.2(ii), $E$ is a Jordan superalgebra satisfying the identity (8), and the Grassmann envelope $\Gamma(E)$ satisfies the identity $x^3 = 0$. Since $\mathrm{char} F \neq 2, 3$, by [9] we have

$$\left(\left(\left(\Gamma(E)^2\right)^2 \Gamma(E)^2\right)\Gamma(E)^2\right)\Gamma(E)^2 = 0.$$

A standard passage from $E$ to $\Gamma(E)$ now implies (see, for example, [10])

$$\left(\left((E^2)^2 E^2\right)E^2\right)E^2 = 0,$$

and so $(((MA \cdot A^2)A^2)A^2)A^2 = 0$. Since $AM \neq 0$ is a Bernstein subsupermodule of $M$, we have $AM = M$ and so

$$(((M \cdot A^2)A^2)A^2)A^2 = 0. \tag{9}$$

By (8), for any $m \in M$, $a \in A^2$, $b \in A$ holds

$$ma \cdot b = -(-1)^{\delta a\, \delta b} mb \cdot a - m \cdot ab.$$

Therefore $MA^2 \cdot A \subseteq MA \cdot A^2 + M \cdot A^2 A$ and $MA^2$ is a submodule of $M$ which is evidently graded and Bernstein. Since $M$ is irreducible and almost faithful, either $A^2 = 0$ or $MA^2 = M$.

If $MA^2 = M$ then $M = 0$ by (9), a contradiction. Thus $A^2 = 0$, $\mathrm{Ann}\, M = 0$, and $A$ satisfies

$$ma \cdot b + (-1)^{\delta a\, \delta b} mb \cdot a = 0, \tag{10}$$

for any $m \in M$, $a, b \in A$. Taking here $m = x \in X$, $a = u \in U$, $b = w \in W$, we obtain $xu \cdot w = \pm xw \cdot u \in XW \cdot U \subseteq XU \subseteq Y$. On the other hand, $xu \cdot w \in XU \cdot W \subseteq YW \subseteq X$. Since $X \cap Y = 0$, we deduce that $XU \cdot W = XW \cdot U = 0$. Furthermore, by Proposition 4.1(6), we have $YW \cdot U = 0$, hence by (10) also $YU \cdot W = 0$. Thus

$$M = MA = MU + MW = MA \cdot U + MA \cdot W = MU \cdot U + MW \cdot W.$$

It can be easily seen that $M_U = MU \cdot U$ and $M_W = MW \cdot W$ are Bernstein subsupermodules of $M$ with $M_U \cap M_W = 0$, hence either $M = M_U$ or $M = M_W$.

Assume first that $M = M_W$, then $MU = 0$ and so $U = 0$. Moreover, $M_W = MW \cdot W = XW \cdot W + YW \cdot W \subseteq XW \subseteq X$, hence $M = X$ and we have $L(M) = M$, a contradiction.

Thus $M = M_U$ and $W = 0$ as above. It follows from (10) that for any homogeneous $u \in U$ the subset $Mu$ is a graded Bernstein submodule of $M$, hence either $Mu = M$ or $Mu = 0$. If $u$ is even then by (10), $Mu \cdot u = 0$, hence $Mu = 0$. By faithfulness of $M$, $U_0 = 0$ and so $A = U_1$. Besides, since $X_i U_1 \subseteq Y_{1-i}$, $Y_i U_1 \subseteq X_{1-i}$, $i = 0, 1$, it follows that $X_0 + Y_1$ and $X_1 + Y_0$ are Bernstein subsupermodules of $M$, and therefore either $M = X_0 + Y_1$ or $M = X_1 + Y_0$. It can be now easily seen that any graded submodule of $M$ would automatically be a graded Bernstein submodule of $M$, and so $M$ is irreducible as an ordinary (not Bernstein) $A$-supermodule. Moreover, by Proposition 4.2, $M$ is just an irreducible $A$-module. In particular, the centroid $C(M)$ of the $A$-module $M$ is a division algebra.

For any $u \in A = U_1$ consider the mapping $R_u$: $M \to M$ defined by $R_u(m) = mu$. It follows from (10) that $[R_u, R_{u'}] = 0$ for any $u' \in A$ and therefore $R_u$ belongs to the centroid $C(M)$. Observe that $R_u \neq 0$ for any $0 \neq u \in A$. Denote by $K$ the $F$-subalgebra of $C(M)$ generated by the set $\{R_u R_{u'} \mid u, u' \in A\}$. We claim that $K$ is a field. Indeed, $K$ is clearly a commutative domain. Let $0 \neq \alpha \in K$, $0 \neq m \in M_0$, then it is easy to check that the submodule $N$ generated by the element $m\alpha$ is given by $N = (m\alpha)K + (m\alpha)K \cdot A$. Since $N = M$, it follows $m \in N$, and therefore, since $m \in M_0$, there exists $\beta \in K$ such that $m\alpha\beta = m$. Then $\alpha\beta = 1$ and $K$ is a field.

Fix $0 \neq u \in A = U_1$. Then $0 \neq R_u^2 = \alpha \in K$ and for any $u' \in A$ holds $R_{u'}\alpha = R_{u'}R_u^2 = R_u R_{u'} R_u \in R_u K$. Thus $R_A \subseteq R_u K$, and it follows from the above that $M = m \cdot K + mu \cdot K$ for any fixed $0 \neq m \in M_0$. Consider $V = \{\gamma \in K \mid \gamma R_u \in R_A\}$ which is clearly an $F$-subspace of $K$. Since $VR_u = R_A$, we have $R_A R_A = V^2 R_u^2 = V^2\alpha$, so $K = \text{alg}\langle R_A R_A \rangle = \text{alg}\langle V^2\alpha \rangle$. Finally, we can identify $A$ with $R_A$ and get $A = VR_u$.

Now, if $M = X_0 + Y_1$ then it is evidently isomorphic to $M(K, V, \alpha)$, and if $M = X_1 + Y_0$ then it is isomorphic to the opposite supermodule $M(K, V, \alpha)^s$. This finishes the case $L(M) = 0$.

Consider now the case $L(M) = M$. By definition of $L(M)$ we have $M = X$ and $MU = 0$. Let $I = U + U^2$. Then $IU \subseteq U^2 + WU \subseteq I$ and $IW \subseteq UW + W^2 \subseteq U$, hence $I$ is an ideal of $A$. Furthermore,

$$MI = MU^2 = L(M)U^2 \subseteq$$

(by Proposition 4.1(2))

$$\subseteq L(M)U \cdot U = 0,$$

and since $M$ is almost faithful, $U = 0$. Thus $A$ is a $(0, W)$-Bernstein algebra. Since $A^2 = 0$, Ann $M$ is an ideal of $A$ and is therefore zero. Thus $M$ is faithful and we can suppose without loss of generality that $A$ is a graded subspace of End $M$. Therefore, $M$ is a module of the type $M(W, X)$, and the theorem is proved. ∎

The following corollary gives a description of irreducible Bernstein modules.

COROLLARY 4.3. *Let $A$ be an $(U, W)$-Bernstein algebra and $M$ be an almost faithful irreducible $(X, Y)$-Bernstein $A$-module. Then $U = 0$, $Y = 0$, and $M$ is isomorphic to a module of type $M(W, X)$ where the subalgebra* $\mathrm{alg}\langle W\rangle$ *is dense in* $\mathrm{End}(M)$.

*Proof.* The algebra $A$ and the module $M$ can be considered as superalgebra and supermodule, with zero odd parts. Now it follows from Theorem 4.1 that the only possibility for $M$ is to be isomorphic to a module of type $M(W, X)$. ∎

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