Bernstein Algebras With Zero Derivation Algebra*

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ABSTRACT

In a paper by A. Wörz the dimension of the derivation algebra of a finite dimensional Bernstein algebra is given and Bernstein algebras having a derivation algebra of maximal dimension are characterized as those with product zero in the kernel (trivial Bernstein algebras). This paper considers the opposite case, that is, Bernstein algebras having a zero derivation algebra. Some conditions needed for a Bernstein algebra to have zero derivation algebra are obtained, and some examples are given.

INTRODUCTION

A Bernstein algebra A is a commutative algebra over a field K, char $K \neq 2$, with a weight homomorphism, that is, a nonzero homomorphism of algebras $\omega : A \to K$ satisfying $(x^2)^2 = \omega(x)^2 x^2$ for every x in A. It is known that a Bernstein algebra has only one weight homomorphism and always has idempotent elements. If e is one of them, then A has a Peirce decomposition $A = Ke \dotplus U_e \dotplus Z_e$, where $U_e = \{x \in \operatorname{Ker} \omega : ex = \frac{1}{2}x\}$ and $Z_e = \{x \in \operatorname{Ker} \omega : ex = 0\}$. The relations

$$U_e U_e \subset Z_e, \qquad U_e Z_e \subset U_e, \quad \text{and} \quad Z_e Z_e \subset U_e \tag{II}$$

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are satisfied. Further information about Bernstein algebras can be found in [1], [6], [8], and [13].

DEFINITION. A derivation D of a Bernstein algebra is a linear application $D: A \to A$ satisfying $D(xy) = D(x)y + xD(y)\ \forall x, y \in A$.

Then $D(e) \in U$, and each derivation D of A defines and is defined by a triplet $(\bar{u}, f, g)$ where $\bar{u} \in U$, $f: U \to U$, and $g: Z \to Z$ are linear applications satisfying the relations:

$$g(uu') = f(u)u' + uf(u') \qquad \forall u, u' \in U, \qquad z, z' \in Z, \tag{1}$$

$$f(uz) = f(u)z + ug(z) + 2[D(e)u]z, \qquad \forall u, u' \in U, \qquad z, z' \in Z, \tag{2}$$

$$f(zz') = zg(z') + g(z)z' - 2\{[D(e)z]z' + [D(e)z']z\}. \tag{3}$$

So D is defined by $D(e) = \bar{u}$, $D(u) = f(u) + 2D(e)u\ \ \forall u \in U$, and $D(z) = -2D(e)z + g(z)\ \ \forall z \in Z$. (See [2], [5]. [7], and [9] about derivations in Bernstein algebras.)

It is known that $\dim U = r$, $\dim Z = s$ are invariants of the algebra (that is to say, they don't depend on the idempotent element e) and $\dim A = 1 + \dim \operatorname{Ker} \omega = 1 + (r + s)$.

The above observation lets us deduce that $\dim \operatorname{Der} A \leqslant r + r^2 + s^2$. A. Wörz proves that the Bernstein algebras having a derivation algebra of maximal dimension are the zero Bernstein algebras $[(\operatorname{Ker} \omega)^2 = 0]$, that is, $\dim \operatorname{Der} A = r + r^2 + s^2$ if and only if A is a zero Bernstein algebra.

Algebras having "many derivations" have been studied in [10] and [12]. In [12] it is obtained as a by-product that the derivation algebra of a finite dimensional nilpotent algebra is always nontrivial.

The aim of the present paper is to study the opposite extreme case, that is, the case in which $\operatorname{Der} A = 0$. Notice that a Bernstein algebra cannot be a nil algebra. In fact, it has a nil ideal of codimension one. So, considering the above-mentioned result of Röhrl and Walcher [12], the question is: Are there Bernstein algebras with $\operatorname{Der} A = 0$? In the affirmative case, what can we say about them?

RESULTS

Definition. $U^o = \{u \in U_e : uU_e = 0\}$.

It was seen in [6] that for any two idempotent elements e, f of A one has $\{u \in U_e : uU_e = 0\} = \{u \in U_f : uU_f = 0\}$; therefore, the definition doesn't depend on the idempotent element. U^o is an ideal of the algebra A with zero product, and it is the greatest ideal of square zero contained in every U_e, $e \in E(A)$.

Lemma 1. *U^o is invariant for any derivation D of* A.

Proof. We know that $D(e) \in U_e$, so $uD(e) = 0$ if $u \in U^o$. So $eD(u) = D(eu) - uD(e) = D(eu) = D(\frac{1}{2}u) = \frac{1}{2}D(u)$. Therefore $D(u) \in U_e$.

For every u' in U_e we have $uu' = 0$, so $D(uu') = 0 = D(u)u' + uD(u') = D(u)u' + u[f(u') + 2D(e)u'] = D(u)u'$, because $uf(u') \in U^oU = 0$ and $u[D(e)u'] = -[uD(e)]u' - (uu')D(e) = 0$, using the linearization of $u^3 = 0$.

Consequently $D(u) \in U_e$ satisfies $D(u)U_e = 0$, that is, $D(u) \in U^o$. ■

Lemma 2. *If* A is *a Bernstein algebra that* is *not* Jordan, *then* $U^o \neq 0$.

Proof. We know that there is an idempotent e for which $Z_e^2 \neq 0$, because A is not a Jordan algebra. If z is an element of Z_e with $z^2 \neq 0$, it is clear that $z \in U^o$ and so $U^o \neq 0$.

We notice that for every u in U_e and z in Z_e the element $(uz)z \in U^o$, because $[(uz)z]u' = -(uz)(u'z) = 0$. Also, zz' for every z, z' in Z,. Consequently A/U^o is a Jordan-Bernstein algebra. ■

Lemma 3. *If A is a Bernstein algebra having zero derivation algebra, then there is no element* u *in* U *with* $u(U + Z) = 0$.

Proof. In [5] it was proved that the right multiplication $R_u, u \in U_e$, is a derivation if and only if $uZ = 0 = (uU)Z$, and that $R_z, z \in Z$, is a derivation if and only if $zZ = 0 = zZ^2$ and $(z, U, Z) = 0$. Consequently, if A has only zero derivations, it cannot have elements u in U_e or z in Z_e satisfying the above conditions. If $u(U + Z) = 0$, then the right multiplication R_u would be a derivation.

Consequently, for each u in U^o there must exist a z in Z_e with $uz \neq 0$. Even more, there is a z in Z_e with $(uz)z \neq 0$, because otherwise the triplet $(u, 0_U, 0_Z)$ would define a nonzero derivation of A. ■

PROPOSITION 1. *If A is a Bernstein algebra having zero derivation algebra, then A is not Jordan and so $U^o \neq 0$.*

Proof. In [11] inner derivations of a Jordan-Bernstein algebras were studied, and it was seen that the dimension of the algebra of inner derivations is always greater than or equal to r [each element $\bar{u} \in U_e$ defines an inner derivation by $D(e) = \frac{1}{2}\bar{u}$, $D(u) = \bar{u}u$, and $D(z) = -\bar{u}z$]. So if a Bernstein algebra has derivation algebra zero, then it cannot be a Jordan algebra. It was proved in [6] that a Bernstein algebra that is not Jordan has always an idempotent e with $Z_e^2 \neq 0$. ∎

THEOREM 1. *If A is a Bernstein algebra having zero derivation algebra, then* Ker ω *is not nilpotent, that is, A is not a genetic Bernstein algebra.*

Proof. Defining $(\text{Ker }\omega)^2 = (\text{Ker }\omega)(\text{Ker }\omega)$ and recursively $(\text{Ker }\omega)^{r+1} = (\text{Ker }\omega)^r(\text{Ker }\omega)$, we can see that $(\text{Ker }\omega)^r \neq 0$ $\forall r$, since, if $u_1 \in U^o$ we know that there exists z_1 in Z_e with $u_1 z_1 \neq 0$ and $u_1 z_1 \in U^o \cap (\text{Ker }\omega)$, so there exists z_2 in Z with $(u_1 z_1)z_2 \neq 0$. Following the above process, we can find for each $r \geqslant 1$ an element $0 \neq u_r \in U^o \cap (\text{Ker }\omega)^r$. ∎

COROLLARY 1. *If A is a Bernstein algebra having zero derivation algebra, and $U_e^2 = 0$ ($U^o = U$), then for every u in U_e there is a z in Z_e with $(uz)z \neq 0$.*

COROLLARY 2. *If A is a Bernstein algebra having zero derivation algebra, then $U_e Z_e \neq 0$ for any idempotent e of A.*

COROLLARY 3. *If A is a Bernstein algebra having zero derivation algebra, then A is not nuclear, that is, $U_e^2 \neq Z_e$.*

Proof. If $u \in U^o$, then $uU_e = 0 = uU_e^2$. Therefore, if $Z_e = U_e^2$ we would have that $uZ_e = 0$ $\forall u \in U^o$, and R_u would be a nonzero derivation. ∎

We could have seen this by using the result of [3] that assures that a nuclear Bernstein algebra (that is, $U_e^2 = Z_e$) is genetic and applying Theorem 1 above.

LEMMA 4. *If A is a Bernstein algebra having zero derivation algebra, there cannot exist an idempotent element e with $U_e^2 = 0$ and $Z_e^2 = 0$.*

Proof. If we had such an idempotent element e, then considering the Peirce decomposition with respect to e, the triplet $(0, 1_U, 0_Z)$ would define a nonzero derivation of A. ∎

PROPOSITION 2. *If A is a Bernstein algebra having zero derivation algebra and* $U_e^2 = 0$, *then* $\dim Z_e > 1$ (*and consequently* $\dim A \geqslant 4$).

Proof. In fact, $\dim Z = 1$ implies that a derivation D is given by an element of U, say $\bar{u}$ (whose determination needs r parameters, the coordinates with respect to the fixed basis in U), a linear application $f: U \to U$ (for which we need r^2 parameters, the r coordinates of the images of each element in the basis of U), and a linear application $g: Z \to Z$ (given by one parameter). The above parameters must satisfy (1), (2), and (3). In the present case, (1) is an identity $0 = 0$; (2) is equivalent to the relations $f(u_i z) = f(u_i)z + u_i g(z)$, where $\{u_1, \dots, u_r\}$ is a basis of U and $\{z\}$ is a basis of Z. Each one of the above relations yields r equations (making the coefficients of u_i equal). Then the above parameters must satisfy r^2 homogeneous equations as a consequence of (2).

Finally, (3) produces in this case r equations $[f(z^2) = 2zg(z) - 2(\bar{u}z)z]$. In conclusion, to give a derivation D we need to find $r^2 + r + 1$ parameters satisfying $r^2 + r$ homogeneous equations. As we know, there is always a nontrivial solution, because $r^2 + r < r^2 + r + 1$. So we can always find a nonzero derivation. ■

LEMMA 5. *If A is Bernstein algebra having zero derivation algebra and* $U_e^2 = 0$, *then there is no element z in* Z_e *with* $z \operatorname{Ker} \omega = 0$.

Proof. If such a z existed, we could define a linear application $g: Z \to Z$, $g \neq 0$, with $\operatorname{Im} g = Kz$, and then the triplet $(0, 0_U, g)$ would define a nonzero derivation.

PROPOSITION 3. *If A is a Bernstein algebra having zero derivation algebra and* $U_e^2 = 0$, *then* $\dim U > 1$.

Proof. If $A = Ke + K(u) + K(z_1, \dots, z_s)$, we can use the same argument of Proposition 2 and we see that a derivation is given by $1 + 1 + s^2$ parameters satisfying s equations associated with (2) $[f(uz_i) = f(u)z_i + ug(z_i)]$ and $\frac{1}{2}s(s+1)$ equations associated with (3) $[f(z_i z_j) = z_i g(z_j) + z_j g(z_i) - 2\{(\bar{u}z_i)z_j) + (\bar{u}z_j)z_i\}, i \leqslant j]$. So it is always possible to determine $s^2 + 2$ parameters (not all of them equal to zero) satisfying $s + \frac{1}{2}s(s+1)$ equations, that is, there is always a nonzero derivation. ■

EXAMPLE 1. There are Bernstein algebras with $U^2 = 0$, $\operatorname{Der} A = 0$ and minimal dimension, that is, 5. Let $A = Ke + K(u_1, u_2) + K(z_1, z_2)$ with the product given by $U^2 = 0$, $z_1^2 = u_1$, $z_2^2 = u_2$, $z_1 z_2 = 0$, $z_1 u_1 = 0$, $z_1 u_2 = u_2$, $z_2 u_1 = u_1$, $z_2 u_2 = 0$.

If D is a derivation given by the triplet $(\alpha u_1 + \beta u_2, f, g)$, where $f : U \to U$ is a linear application given by $f(u_1) = t_1 u_1 + t_2 u_2$, $f(u_2) = s_1 u_1 + s_2 u_2$, and $g : Z \to Z$ is a linear application given by $g(z_1) = \gamma_1 z_1 + \gamma_2 z_2$, $g(z_2) = \delta_1 z_1 + \delta_2 z_2$, they must satisfy:

(1) $f(z_1^2) = f(u_1) = 2z_1 g(z_1) - 4[D(e)z_1]z_1 = 2\gamma_1 u_1 - 4\beta u_2$, whence

$$t_1 = 2\gamma_1, \qquad t_2 = -4\beta;$$

$f(u_2) = f(z_2^2) = 2z_2 g(z_2) - 4[D(e)z_2]z_2 = 2\delta_2 u_2 - 4\alpha_2 u_1$, whence

$$s_1 = 4\alpha, \qquad s_2 = 2\delta_2;$$

$0 = f(z_1 z_2) = g(z_1)z_2 + z_1 g(z_2) - 2\{[D(e)z_1]z_2 + [D(e)z_2]z_1\} = \gamma_2 u_2 + \delta_1 u_1$, whence $\delta_1 = 0 = \gamma_2$;

(2) $0 = f(u_1 z_1) = f(u_1)z_1 + u_1 g(z_1) = t_2 u_2 + \gamma_2 u_1$, whence

$$t_2 = 0 = \gamma_2;$$

$0 = f(u_2 z_2) = f(u_2)z_2 + u_2 g(z_2) = s_1 u_1 + \delta_1 u_2$, whence

$$s_1 = 0 = \delta_1;$$

$f(u_1) = f(u_1 z_2) = f(u_1)z_2 + u_1 g(z_2) = t_1 u_1 + \delta_2 u_1$, whence

$$t_1 = t_1 + \delta_2 \quad (\delta_2 = 0) \quad \text{and} \quad t_2 = 0;$$

$f(u_2) = f(u_2 z_1) = f(u_2)z_1 + u_2 g(z_1) = s_2 u_2 + \gamma_1 u_2$, whence

$$s_1 = 0 \quad \text{and} \quad s_2 = s_2 + \gamma_1 \quad (\gamma_1 = 0).$$

So $D = 0$.

Similarly, the algebra $Ke + K(u_1, u_2) + K(z_1, z_2)$ with the product $U^2 = 0$, $z_1^2 = u_1 - u_2$, $z_1 z_2 = 2u_1$, $z_2^2 = 3u_2$, $u_1 z_1 = u_1$, $u_1 z_2 = u_1 + u_2$, $u_2 z_1 = 0$, $u_2 z_2 = -u_2$ has derivation algebra equal to zero.

EXAMPLE 2. It may be that for a Bernstein algebra A one has $\dim U \geqslant 2$, $\dim Z \geqslant 2$, $U_e^2 = 0$, $U_e Z_e \neq 0 \neq Z_e^2$ for every idempotent element e, but $\operatorname{Der} A \neq 0$. For instance, the algebra $A = Ke + K(u_1, u_2) + K(z_1, z_2)$ with the multiplication given by $U^2 = 0$, $u_1 z_1 = u_1$, $u_1 z_2 = u_2$, $u_2 z_1 = 0$, $u_2 z_2 = u_1$, $z_1^2 = u_1$, $z_1 z_2 = 0 = z_2^2$ satisfies all the above conditions. It is easy to see that $\dim \operatorname{Der} A = 1$, and every derivation D of A is of the form

$$D(e) = 0, \qquad D(u_1) = \beta u_1, \qquad D(u_2) = \beta u_2,$$

$$D(z_1) = \tfrac{1}{2}\beta z_1, \quad \text{and} \quad D(z_2) = 0.$$

EXAMPLE 3. Now the natural question is: Are there Bernstein algebras with derivation algebra equal to zero and $U_e^2 \neq 0$? The answer is yes, as the following example proves.

Let $A = Ke + K(u_1, u_2, u_3) + K(z_1, z_2, z_3)$ with the product $u_1^2 = z_1$, $u_2 U = u_3 U = 0 = z_1 U$, $u_1 z_2 = u_2$, $u_2 z_2 = u_3$, $u_3 z_2 = 0$, $u_1 z_3 = u_2 z_3 = u_3 z_3 = u_2$, $z_1^2 = 0$, $z_1 z_2 = u_2 + u_3$, $z_2^2 = u_3$, $z_3^2 = u_2$, $z_1 z_3 = -u_2$, $z_2 z_3 = u_2 - u_3$. This algebra satisfies $UZ = U^o = Z^2$, and A is orthogonal, because $U^2 U = 0$ (it is even totally orthogonal, because $U^2 U^2 = 0$; see [4]).

It is in fact a Bernstein algebra, because if $x = \alpha e + u + z$, then $x^2 = \alpha^2 e + \alpha u + 2uz + z^2 + u^2$ with $u^2 \in Kz_1$, $2uz + z^2 \in K(u_2, u_3) = U^o$. So $(x^2)^2 = \alpha^4 e + \alpha^2(\alpha u + 2uz + z^2) + \alpha^2 u^2$, since $u^2 u = 0 = u(2uz + z)$, $(2uz + z)^2 = 0 = (u^2)^2 = u^2(2uz + z)$, because $z_1 U = 0$.

It is easy to see that if $(\bar{u}, f, g)$ defines a derivation, the conditions on the product assure that $\bar{u} = 0$, $f = 0$, and $g = 0$.

THEOREM 2. *Suppose that* $\operatorname{Der} A = 0$ *and there is an idempotent* e *such that* $U_e^2 \neq 0$, $Z_e^2 = 0$. *Then* A *is not orthogonal and* $\dim A \geqslant 8$.

Proof. If $U_e^3 = 0$, then we can consider $1_U : U \to U$. Since $U^2 \subset Z$ but $U^2 \neq Z$, we can consider any linear application $g : Z \to Z$ satisfying $g(z) = 2z$ if $z \in U^2$ and $g(Z) \subset U^2$. By the hypothesis, $(0, 1_U, g)$ defines a derivation.

So $U_e^3 \neq 0$, and using [4], A is not orthogonal, that is, there is no idempotent element f for which $U_f^3 = 0$. In this case (3) is always satisfied, and (1) and (2) are now

(1) $g(uu') = 2(uu') = f(u)u' + uf(u')$,

(2) $f(uz) = uz = f(u)z + ug(z) = uz + 0$, since $ug(z) \in uU^2 \subset U^3 = 0$.

The above is always true if $\dim U = 2$. In fact, since $U^2 \neq 0$, we know that $1 \leqslant \dim U^o < \dim U = 2$; then we must have $\dim U^o = 1$. Let $\{u_1\}$ be a basis of U^o that we extend to a basis $\{u_1, u_2\}$ of U. Then it is immediate to prove that $U^3 = 0$.

If $\dim U = 3$, then $\dim U^o = 2$ or $\dim U^o = 1$.

If $\dim U^o = 2$, we can see again that $U^3 = 0$. (This is always true when $\dim U^o = \dim U - 1$.)

If $\dim U^o = 1$, $\{u_1\}$ a basis of U^o, there cannot exist $u_2 \notin K(u_1)$ with $u_2^2 = 0$. In fact, if we had such an element, we would be able to choose a basis $\{u_1, u_2, u_3\}$ of U with $u_1^2 = 0 = u_1u_2 = u_1u_3 = u_2^2$, $u_2u_3 = z_1$, $u_3^2 = z_2$. Then $\{z_1, z_2\}$ would be linearly independent, because otherwise we would have $U^3 = 0$ and consequently $\operatorname{Der} A \neq 0$, as we have seen before. Also $\dim U^2 < \dim Z$; then $\dim Z \geqslant 3$. For every z in Z we have $u_1 z \in U^o$, so $u_1 z = \alpha_z u_1$. Also $(u_2 z)^2 = (u_3 z)^2 = 0$; therefore $u_2 z = \beta_z u_1 + \gamma_z u_2$, $u_3 z = \delta_z u_1 + \mu_z u_2$. But $u_3(u_3 z) = 0$ implies $\mu_z = 0$. Similarly $u_2(u_2 z) = 0$ implies $\gamma_z = 0$. So $UZ = Ku_1 = U^o$. Using the same argument as before, we can see that the determination of a derivation $(0, f, g)$ needs $1 + 3 + 3 = 7$ parameters for f and s parameters for g satisfying the $3s$ relations $f(u_i z_j) = f(u_i)z_j + u_i g(z_j)$, and consequently $3s$ equations (let us remember that $UZ = Ku_1$) together with the 6 relations and 6 equations $g(u_i u_j) = u_i g(u_j) + u_i g(u_j)$ [because $U^2 = K(z_1, z_2)$]. Consequently, we need to find $s^2 + 7$ parameters satisfying $3s + 6$ homogeneous equations. Since $s \geqslant 3$, it is always true that $s^2 + 7 > 3s + 6$, and consequently there are always nontrivial solutions, that is, there are always nonzero derivations.

Finally, let us suppose that an element u of U has square equal to zero if and only if it belongs to Ku_1, which implies that $UZ = Ku_1$. If $\dim U^2 = 1$, then we would have again $U^3 = 0$ and, as we know, A would have nonzero derivations.

If $\dim U^2 = 2$, then $\dim Z = s \geqslant 3$, and in order to have a derivation $(0, f, g)$ we would need $s^2 + 7$ parameters satisfying the $3s$ relations $f(u_i z_j) = f(u_i)z_j + u_i g(z_j)$ and consequently $3s$ equations (because $\dim UZ = 1$) as well as the 3 relations $g(u_i u_j) = u_i g(u_j) + u_i g(u_j)$ yielding 3×2 equations ($\dim U^2 = 2$). But $3s + 6 < s + 7$, so there is always a nonzero derivation.

If $\dim U^2 = 3$, then $\dim Z = s \geqslant 4$, and, as before, to give a derivation $(0, f, g)$ we need $s^2 + 7$ parameters satisfying $3s + 9$ relations. As before, there is always a nonzero derivation. ■

The above argument assures that if there is a Bernstein algebra A with $\operatorname{Der} A = 0$ and $U_e^2 \neq 0$ and $Z_e^2 = 0$ for some idempotent element, then $\dim U > 3$, $\dim Z \geqslant 3$, that is, $\dim A \geqslant 8$. It can even be proved that $\dim Z > 4$ when $\dim U = 4$. But is there any such algebra?

We can suggest the following conjecture: Let A be a Bernstein algebra with Der $A = 0$. Is it true that $Z_e^2 \neq 0$ for every idempotent element? We know that this is the case when $U_e^2 = 0$, but we don't know in the case $U_e^2 \neq 0$.

EXAMPLE A. For each $r \geqslant 2$, there is a Bernstein algebra A (not Jordan) with $U^2 = 0$, $\dim U = \dim Z = r$, and Der $A = 0$.

In fact, it suffices to consider the algebra $A = Ke + K(u_1, \dots, u_r) + K(z_1, \dots, z_r)$ with the product given by $U^2 = 0$, $u_i z_i = 0$, $u_i z_j = u_i$, $z_i^2 = u_i$, $z_i z_j = 0$ if $i \neq j \in \{1, 2, \dots, r\}$.

EXAMPLE B. The Bernstein algebra $A = Ke + K(u_1, u_2, u_3) + K(z_1, z_2)$ with $U^2 = 0$, $z_1^2 = u_2$, $z_2^2 = u_3$, $z_1 z_2 = 0$, $u_1 z_1 = u_3$, $u_1 z_2 = u_2$, $u_2 z_1 = 0$, $u_2 z_2 = u_2$, $u_3 z_1 = u_3 = u_3 z_2$ satisfies Der $A = 0$. In this example $\dim U > \dim Z$ and $\dim(UZ + Z^2) < \dim U$.

EXAMPLE C. $A = Ke + K(u_1, u_2) + K(z_1, z_2, z_3)$ with the product given by $U^2 = 0$, $z_i^2 = u_1$, $z_i z_j = 0\ (i \neq j)$, $u_1 z_1 = u_2$, $u_1 z_2 = 0$, $u_1 z_3 = u_1$, $u_2 z_1 = u_1$, $u_2 z_2 = u_2$, $u_2 z_3 = 0$ is a Bernstein algebra with Der $A = 0$. In this example, $\dim Z^2 < \dim U < \dim Z$.

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