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On continuous time models in genetic and Bernstein algebras

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Abstract. We discuss the long-time behavior of Andreoli's differential equation for genetic algebras and for Bernstein algebras and show convergence to an equilibrium in both cases. For a class of Bernstein algebras this equilibrium is determined explicitly.

Key words: Continuously overlapping generations – Genetic algebra – Bernstein algebra

0 Continuous time models

Andreoli [1] introduced the differential equation

$$\dot{x} = x^2 - x,$$
$$x(0) = y$$

in the standard simplex of a genetic algebra with genetic realization to model the time dependence of the genotype frequencies of a population in the limiting case of continuously overlapping generations. While Andreoli restricted his attention to three-dimensional systems, Heuch [5] later considered genetic algebras of arbitrary (finite) dimension and showed that this equation can be solved by elementary functions; see also Wörz-Busekros [9].

The long-time behavior of the solutions seems to have been unknown so far in the general case, although it was known for special classes of algebras [9]. We show in this note that for any genetic algebra with genetic realization every solution in the standard simplex tends to a stationary point for $t \to \infty$.

Using different methods, we show that the same holds for Bernstein algebras (which are not necessarily genetic).

In the remainder of this section we derive some preliminary results about the differential equation $\dot{x} = x^2 - x$ in a real or complex commutative algebra A. The first one is proved in [7]:

Lemma 0.1 *Let $G(y, t)$ resp. $S(y, t)$ be the solution of $\dot{x} = x^2$ resp. $\dot{x} = x^2 - x$ with $x(0) = y$. Then $S(y, t) = e^{-t} \cdot G(y, 1 - e^{-t})$, wherever both sides are defined.*

Now let $G(y, t) = \sum_{k \geqslant 0} t^k g_k(y)$ be the Taylor series expansion about $t = 0$. From [7] it is known that the g_k satisfy the recursion

$$(k+1)g_{k+1}(y) = Dg_k(y)y^2$$

for all $k \geqslant 0$. (In particular $g_0(y) = y$, $g_1(y) = y^2$, $g_2(y) = y^3$ and $g_3(y) = \frac{1}{3}(2y^4 + y^2y^2)$.) The following result is not new, we include it for sake of completeness:

Theorem 0.2 *With the notations above the following is true:*

$$\lim_{t \to \infty} S(y, t) = \lim_{k \to \infty} g_k(y),$$

whenever the right hand side exists.

Proof. Let $g(y) := \lim_{k \to \infty} g_k(y)$. By virtue of Lemma 0.1 and the preceding remarks we have

$$S(y, t) = e^{-t} \cdot \sum_{k \geqslant 0} g_k(y)(1 - e^{-t})^k.$$

Having in mind the identity $\sum_{k \geqslant 0} (1 - e^{-t})^k = e^t$ for $t \geqslant 0$ we get

$$S(y, t) - g(y) = e^{-t} \cdot \sum_{k \geqslant 0} (g_k(y) - g(y))(1 - e^{-t})^k$$

and

$$\|S(y, t) - g(y)\| \leqslant e^{-t} \cdot \sum_{k \geqslant 0} \|g_k(y) - g(y)\|(1 - e^{-t})^k$$

for any norm $\|\cdot\|$ on A. Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\|g_k(y) - g(y)\| < \varepsilon$ for all $k > N$. Then

$$\|S(y, t) - g(y)\| \leqslant e^{-t} \cdot \sum_{k=0}^{N} \|g_k(y) - g(y)\|(1 - e^{-t})^k + e^{-t} \cdot \varepsilon \cdot \sum_{k=N+1}^{\infty} (1 - e^{-t})^k.$$

Now the second term equals $e^{-t} \cdot [(1 - e^{-t})^{N+1}/e^{-t}] \cdot \varepsilon < \varepsilon$. For t sufficiently large, the first term also will be less than ε, hence $\|S(y, t) - g(y)\| < 2\varepsilon$ for all sufficiently large t.

1 Genetic algebras with genetic realization

Let $\mathbb{K}$ be the field of real or complex numbers and A a finite dimensional commutative $\mathbb{K}$-algebra A with (left) multiplication $L(x)$. Then A is called a *genetic algebra* if there is a nontrivial homomophism $\omega : A \to \mathbb{K}$ and the coefficients of the characteristic polynomial of any transformation $f(L(x_1), \ldots, L(x_s))$ only depend on $\omega(x_1), \ldots, \omega(x_s)$ for any polynomial f in s noncommuting indeterminates; see [9] for the details.

Let A be a real genetic algebra of dimension $n + 1$. A is said to have a *genetic realization* if there are coordinates $x_0, \ldots, x_n$ such that $\omega(x) = x_0 + \cdots + x_n$ and the standard simplex

$$S = \{x \in A : x_1 \geqslant 0, \omega(x) = 1\}$$

is closed under the multiplication in A. The motivation for this is as follows; cf. [9] for a detailed account: Consider a population with genotypes $a_0, \ldots, a_n$ and assume that random mating of genotypes a_i and a_j produces an offspring of genotype a_k with probability γ_{ijk}. Taking $(a_0, \ldots, a_n)$ as the basis of a vector space over $\mathbb{R}$, the γ_{ijk} serve as the structure coefficients of a commutative algebra. Genetic realization represents the facts that the genotype frequencies are non-negative and add up to 1 in any generation.

From the definition of a genetic algebra we have in particular that the characteristic roots of $L(x)$ only depend on $\omega(x)$ and thus are $\lambda_0\omega(x), \ldots, \lambda_n\omega(x)$, the numbers $\lambda_0, \ldots, \lambda_n$ being called the *train roots* of A. As 1 is always among them, we set $\lambda_0 = 1$.

It is known that genetic algebras with genetic realization satisfy $|\lambda_i| \leqslant \frac{1}{2}$ for all $i = 1, \ldots, n$ and that there is an idempotent in S; cf. [9, Theorems 4.11 and 4.14]. Using this and the fact that the complexification of a real genetic algebra A permits coordinates such that $L(x)$ is lower triangular for all $x \in A$ and strict lower triangular for all $x \in \operatorname{Ker} \omega$, we have:

There is an idempotent c_0 and (at least in the complexification) a basis $(c_0, \ldots, c_n)$ such that

$$
\begin{aligned}
c_0^2 &= c_0, \\
c_0 c_i &= \lambda_i c_i + \sum_{k=i+1}^{n} \lambda_{0ik} c_k \quad \text{for } 1 \leqslant i \leqslant n, \\
c_i c_j &= \sum_{l=\max\{i,j\}+1}^{n} \lambda_{ijl} c_l \quad \text{for } 1 < i, j \leqslant n
\end{aligned}
$$

with $\lambda_i, \lambda_{0ik}, \lambda_{ijl} \in \mathbb{C}$ (cf. [9, Theorem 3.13]). Thus the c_k-component of x^2 for $x = x_0c_0 + \cdots + x_nc_n$ is

$$
2\lambda_k x_0 x_k + 2 \cdot \sum_{i=1}^{k-1} \lambda_{0ik} x_0 x_i + \sum_{i,j=1}^{k-1} \lambda_{ijk} x_i x_j
$$

for $k = 1, \ldots, n$.

We now consider the initial value problem

$$
\dot{x} = x^2 - x, \qquad x(0) = y \quad \text{with } \omega(y) = 1 \tag{$*$}
$$

in a genetic algebra A with genetic realization (thus $|\lambda_i| \leqslant \frac{1}{2}$ for all $i = 1, \ldots, n$) and denote the solution by $S(y, t)$. Choosing coordinates as above, the system reads as follows:

$$
\begin{aligned}
\dot{x}_0 &= x_0^2 - x_0 \\
\dot{x}_k &= \mu_k x_0 x_k + 2 \sum_{i=1}^{k-1} \lambda_{0ik} x_0 x_i + \sum_{i,j=1}^{k-1} \lambda_{ijk} x_i x_j
\end{aligned}
$$

for $k = 1, \ldots, n$, where $\mu_k = 2\lambda_k - 1$. Note that $\operatorname{Re} \mu_k \leqslant 0$ for $k = 1, \ldots, n$. Using $\omega(y) = y_0 = 1$ we get $x_0(t) = 1$ for all t. The general solution can be written as follows:

Lemma 1.1 *Define recursively the sets* $A_1 := \{\mu_1\}$, $A_{k-1} := A_k \cup \{A_k + A_k\} \cup \{\mu_{k+1}\}$. *Then there are polynomials* $p_{k,\alpha}$, $k = 1, \ldots, n$, $\alpha \in A_k$, *such that*

$$
x_k(t) = \sum_{\alpha \in A_k} p_{k,\alpha}(t) \cdot e^{\alpha t}
$$

for $k = 1, \ldots, n$.

Proof. For $k = 1$ we have $\dot{x}_1 = \mu_1 x_1$ with solution $y_1 \cdot e^{\mu_1 t}$. Now assume $k > 1$. Using the induction hypothesis we have

$$\dot{x}_k = \mu_k x_k + 2 \sum_{i=1}^{k-1} \lambda_{0ik} \left(\sum_{\alpha \in A_i} p_{i,\alpha}(t) e^{\alpha t} \right)$$
$$+ \sum_{i,j=1}^{k-1} \lambda_{ijk} \left(\sum_{\alpha \in A_i} p_{i,\alpha}(t) e^{\alpha t} \right) \cdot \left(\sum_{\beta \in A_i} p_{j,\beta}(t) e^{\beta t} \right).$$

Expanding and collecting terms yields

$$\dot{x}_k = \mu_k x_k + \sum_{\alpha \in A_{k-1}} q_{k,\alpha}(t) e^{\alpha t} + \sum_{\alpha \in A_{k-1} + A_{k-1}} r_{k,\alpha}(t) e^{\alpha t}$$

with polynomials $q_{k,\alpha}$ and $r_{k,\alpha}$. Applying standard facts about linear differential equations in one variable gives the result. □

We now ask for the long time behavior of the system (*).

Lemma 1.2 *Let $y \in A$ such that $\omega(y) = 1$ and $S(y, t)$ is bounded for $t \to \infty$. Then $S(y, t)$ converges to an idempotent.*

Proof. We may assume that the ground field is $\mathbb{C}$. Otherwise consider the complexification of A. Note that the real part of μ_k is nonpositive: $\operatorname{Re} \mu_k \leqslant 0$, and $\operatorname{Re} \mu_k = 0$ if and only if $\mu_k = 0$ resp. $\lambda_k = \frac{1}{2}$. Similarly, $\operatorname{Re} \alpha \leqslant$ holds for all $\alpha \in A_k$, and $\operatorname{Re} \alpha = 0$ if and only if $\alpha = 0$. Using (1.1), we see

$$x_k(t) = p_{k,0}(t) + \sum_{\alpha \in A_k \setminus \{0\}} p_{k,\alpha}(t) e^{\alpha t}.$$

For $t \to \infty$, the second term tends to zero, whereas $p_{k,0}$ must be constant, as it is a polynomial bounded for $t \to \infty$. Thus $\lim_{t \to \infty} x_k(t) = p_{k,0}$ for all k. It is well-known that the limit c is a stationary point, thus $c^2 = c$ and $c \neq 0$ follows from $\omega(c) = 1$.

Using the fact that $S(y, t) \in S$ for all $t \geqslant 0$ whenever $y \in S$ we get

Theorem 1.3 *Let A be a genetic algebra with genetic realization. Then for every $y \in S$ the solution of $\dot{x} = x^2 - x$, $x(0) = y$, converges to an idempotent of A.*

It would be interesting to have an explicit expression for the limit in terms of the initial value. Yet we have not been able to find such an expression.

It is worth noting that in the limiting case of continuous time the behavior of the system is, in general, simpler than in the case of discrete generations. For instance, the sequence of plenary powers in a genetic algebra does not necessarily converge for every initial value in S; see Gonshor [2] for an investigation of the problem, and also Wörz-Busekros [9].

2 Bernstein algebras

A finite dimensional commutative $\mathbb{K}$-algebra A is called a *Bernstein algebra* if there is a nontrivial homomorphism $\omega : A \to \mathbb{K}$ and the identity

$$(x^2)^2 = \omega(x)^2 x^2$$

holds in A. In terms of genetics this models a population whose distribution of genotypes is stationary after two generations, provided that there is no interaction between different generations.

These algebras were introduced by Holgate [6]. A summary of known results is given in [9]. Bernstein algebras are not necessarily genetic algebras, cf. [9] for examples.

In [7] the general solution of the differential equation in question and its long time behavior were determined for Bernstein algebras satisfying a train equation. Here we drop this constraint.

Let $c \in A$ be an idempotent (which always exists) and

$$A = \mathbb{K}c \oplus N_{1/2} \oplus N_0$$

the corresponding Peirce decomposition with $N_\alpha := \{x \in \operatorname{Ker} \omega : cx = \alpha x\}$ with Peirce relations $N_{1/2}^2 \subset N_0$, $N_0 \cdot N_{1/2} \subset N_{1/2}$ and $N_0^2 \subset N_{1/2}$; cf. Wörz-Busekros [10]. The subspace

$$J := \{x \in N_{1/2} : x \cdot N_{1/2} = 0\}$$

is of particular interest in Bernstein algebras:

Theorem 2.1 (Hentzel and Peresi [4]) *Let A be a Bernstein algebra.*

(a) *J is an ideal of A.*

(b) *The algebra $\bar{A} := A/J$ is a Bernstein algebra (with nontrivial homomorphism $\bar{\omega} : \bar{A} \to \mathbb{K}$) and a Jordan algebra, i.e. the identity*

$$\bar{x}^3 - \bar{\omega}(\bar{x})\bar{x}^2 = \bar{0}$$

holds in $\bar{A}$.

It should be noted that (b) is somewhat sharper than stated by Hentzel and Peresi: They prove that A/J is a Bernstein and Jordan algebra, using results of [10] on the Peirce decomposition. The remaining assertion follows from Walcher [8].

Corollary 2.2 *In any Bernstein algebra*

$$x^3 - \omega(x)x^2 \in J$$

for all $x \in A$.

We will call J the *Hentzel–Peresi-Ideal* of a Bernstein algebra A.

In [7] the following polynomials were shown to be of some importance. Letting

$$f_l(x) := (L(x) - \tfrac{1}{2}\omega(x)\mathrm{Id})^{l-3}(x^3 - \omega(x)x^2)$$

for $l \geqslant 3$, the identity

$$g_k(x) - \omega(x)g_{k-1}(x) = \frac{2^{k-1}}{k!} f_{k+1}(x)$$

is satisfied for all $k \geqslant 2$.

We now ask for the solution $G(y, t)$ of $\dot{x} = x^2$, $x(0) = y$, where y lies in the hyperplane $H := \{y \in A : \omega(y) = 1\}$ in an arbitrary Bernstein algebra A. The following result improves Lemma 3.1 of [7]:

Proposition 2.3 *Let $y = c + u + z \in H$ with $u \in N_{1/2}$ and $z \in N_0$. Then*

$$G(y, t) = y + \frac{t}{1-t} y^2 + \frac{t^2}{1-t}\left(\sum_{j=0}^{\infty} \frac{2^{j+1}}{(j+2)!} t^j L(z)^j\right) f_3(y).$$

Proof. From the definition of the Hentzel–Peresi-Ideal it follows that

$$(L(y) - \tfrac{1}{2}\omega(y)\mathrm{Id})w = L(z)w$$

for $w \in J$ and $y = c + u + z \in H$ like above. By virtue of Corollary 2.2 we therefore get $f_l(y) = L(z)^{l-3} f_3(y)$ for all $l \geqslant 3$ and $y \in H$. Thus, by induction we have

$$g_l(y) = y^2 + \left(\sum_{k=2}^{l} \frac{2^{k-1}}{k!} L(z)^{k-2}\right) f_3(y).$$

Using this and the Taylor series expansion of $G(y, t)$ show the results. □

Of course, we also want to investigate the long time behavior here:

Lemma 2.4 *Let $y \in H$. Then $\lim_{k \to \infty} g_k(y)$ exists.*

Proof. Let $y = c + u + z$ be the decomposition of $y \in H$ with $u \in N_{112}$ and $z \in N_0$, furthermore $\| \cdot \|$ be any norm on A. We denote the corresponding operator norm on End A by the same symbol. Using standard arguments and results cited above, we get for $k \geqslant 2$

$$\begin{aligned}\|g_{k+l}(y) - g_k(y)\| &\leqslant \sum_{i=0}^{l-1} \|g_{k+l-i}(y) - g_{k+l-i-1}(y)\| \\ &= \sum_{i=0}^{l-1} \frac{2^{k+l-i-1}}{(k+l-i)!} \|f_{k+l-i+1}(y)\| \\ &= \sum_{i=0}^{l-1} \frac{2^{k+l-i-1}}{(k+l-i)!} \|L(z)^{k+l-i-2} f_3(y)\| \\ &\leqslant \sum_{i=0}^{l-1} \frac{2^{k+l-i-1}}{(k+l-i)!} \|L(z)\|^{k+l-i-2} \|f_3(y)\|.\end{aligned}$$

Now let c, $M > 0$ be constants such that $\|f_3(y)\| \leqslant c$ and $\|L(z)\| \leqslant M$, whence

$$\begin{aligned}\|g_{k+l}(y) - g_k(y)\| &\leqslant \left(\sum_{i=0}^{l-1} \frac{2^{k+l-i-1}}{(k+l-i)!} M^{k+l-i-2}\right) \cdot c \\ &= \frac{c}{2M} \sum_{i=0}^{l-1} \frac{2^{k+l-i}}{(k+l-i)!} M^{k+l-i}.\end{aligned}$$

Applying Cauchy's criterion shows convergence. □

Without any further work we get from Theorem 0.2

Theorem 2.5 *Let A be a Bernstein algebra and $S(y, t)$ the solution of $\dot{x} = x^2 - x$ for $x(0) = y$, $\omega(y) = 1$. Then $\lim_{t \to \infty} S(y, t)$ exists and is idempotent.*

We remark that by virtue of Lemma 0.1 and Proposition 2.3 it follows that

$$\lim_{k\to\infty} g_k(y) - y^2 \in J$$

for all $y \in H$. In the general case, it again seems difficult to express the limit in closed form. This, however, does not apply to the special case of Bernstein algebras satisfying $A^2 = A$. Due to a theorem of Grishkov [3], this condition implies that $N := \operatorname{Ker} \omega$ is nilpotent (and A is genetic).

Proposition 2.6 *Let A be a Bernstein algebra satisfying $A^2 = A$. Then*

$$\lim_{t\to\infty} S(y,t) = y^3$$

for $y \in H$.

Proof. The Peirce relations cited above imply

$$A^2 = \mathbb{K}c \oplus N_{1/2}^2 \oplus N_{1/2}.$$

Note that $N_{1/2}^2 \subset N_0$. Thus $A^2 = A$ if and only if $N_0 = N_{1/2}^2$. By definition of the Hentzel–Peresi-Ideal $J \cdot N_{1/2} = \{0\}$, but $J \cdot N_{1/2}^2 = \{0\}$ also holds, as follows from linearizing the relation $u^3 = 0$ for all $u \in N_{1/2}$; cf. [9, Theorem 9.6]. Putting pieces together, $A^2 = A$ implies $J \cdot N = \{0\}$. Applying this to the formula for $g_l(y)$ in the proof of (2.3) we see

$$g_l(y) = y^2 + f_3(y) = y^3$$

for $l \geqslant 3$ and $y \in H$. Lemma 0.1 not completes the proof. □

(Incidentally, the proof of Proposition 2.6 shows that the identity $f_4(y) = 0$ holds in every Bernstein algebra satisfying $A^2 = A$.)

Note that if we consider a sequence of discrete generations, each of them generated by its predecessor, the corresponding limit equals (by the defining property of Bernstein algebras) y^2 instead of y^3 for all $y \in H$.

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