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ALGEBRAS, GROUPS AND GEOMETRIES

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DERIVATIVES IN Nth - ORDER BERNSTEIN ALGEBRAS, II[1]

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Abstract

Derivation in some special cases of nth-order Bernstein algebras are studied. Inner derivations for arbitrary nth-order Bernstein algebras are also treated.

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1 Introduction

In this paper all algebras considered will be commutative, non necessarily associative and finite dimensional over an infinite field K of char$(K) \neq 2$.

Let A be an algebra over a field K. As usual in non-associative algebras we will consider principal powers and plenary powers of an element $x \in A$, defined by $x^1 = x$ and $x^{n+1} = x^n x$ for $n \geq 1$ and $x^{[1]} = x$, $x^{[n+1]} = x^{[n]}x^{[n]}$, $n \geq 1$. If $X \subseteq A$, then $K(X)$ is the vector subspace of A generated by X.

If $\omega: A \longrightarrow K$ is a nonzero homomorphism, then the ordered pair (A, ω) will be called a baric algebra over K and ω its weight function. For $x \in A$, $\omega(x)$ is called the weight of x. The set $N = \{x \in A \mid w(x) = 0\}$ is an ideal of A of codimension 1. If $e \in A$ has weight 1 and $e^2 = e$, then e will be called idempotent element of A.

A *n^{th}-order Bernstein algebra* is a commutative baric algebra (A, ω) satisfying: $x^{[n+2]} = (\omega(x))^{2^n} x^{[n+1]}$, for all $x \in A$ and n is the smallest one with such property. If $n = 2$, that is, $x^{[4]} = \omega(x)^4 x^{[3]}$, then A is a *2^{nd}-order Bernstein algebra.*

Some general results about n^{th}-order Bernstein are the following (see [3], [7], [9] and [12]).

Proposition 1.1 *Let (A, ω) be a n^{th}-order Bernstein algebra, then:*

1. *The weight homomorphism is unique.*

2. *The set of idempotent elements of A is*
$$\Im(A) = \{x^{[n+1]} \mid x \in A \text{ and } \omega(x) = 1\}.$$

3. *If $e \in A$ is an idempotent element, then A has a Peirce decomposition, $A = Ke \oplus U_e \oplus Z_e$, where $U_e = \{x \in A \mid ex = \frac{1}{2}x\}$, $Z_e = \{x \in A \mid L_e^n(x) = 0\}$ and $L_e(x) = ex$. Moreover $L_e^n(x) \in U_e$ for all $x \in N$, $L_e(z) \in Z_e$ for all $z \in Z_e$.*

4. *$U_e^2 \subseteq Z_e$, $\dim U_e$ (and so $\dim Z_e$) does not depend on the particular idempotent element, and we can define type$(A) = (1 + r, d)$, where $r = \dim U_e$ and $d = \dim Z_e$.*

2 Preliminary results

Some general results about Derivations in n^{th}-order Bernstein are the following (see [1] and [5]).

Definition 2.1 *A derivation D in an algebra A is a linear application $D: A \longrightarrow A$ that verifies $D(xy) = D(x)y + xD(y)$ for every x, y in A.*

Definition 2.2 *Let A be an algebra and $e \in A$ be an idempotent element. Then $C_{je} = \ker L_e^j$, $j \geq 0$, where L_e^0 is the identity function on A.*

Proposition 2.1 *Let (A, ω) be a n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element. Then:*

1. *If $D: A \longrightarrow A$ is a derivation then*
   (a) *$D(e) \in U_e$*
   (b) *$D(A) \subseteq N$.*
2. *If $x \in A$ and L_x is a derivation then $x \in U_e \oplus C_{1e}$.*

The following definition appear in [3].

Definition 2.3 *Let A be a commutative algebra and $e \in A$ be an idempotent element. Let $G_{k,l,e}: A \longrightarrow A$ be given by: $G_{k,1,e}(a) = L_e^k(a)$ for all $k \geq 0$ and $a \in A$,*

$$G_{k,l,e}\left(a^{(r)}, b^{(s)}\right) = \sum_{i=0}^{k-1} 2^{i-1} L_e^{k-1-i}\Big(\sum_{j=1}^{l-1} \sum_{\substack{r'+r''=r \\ 0 \leq r' \leq j \\ 0 \leq r'' \leq l-j}} G_{i,j,e}\left(a^{(r')}, b^{(j-r')}\right) G_{i,l-j,e}\left(a^{(r'')}, b^{(l-j-r'')}\right)\Big),$$

if $2^k \geq l$ and $G_{k,l,e}\left(a^{(r)}, b^{(s)}\right) = 0$ if $2^k < l$, where $l \geq 2$; $r + s = l$; $k, r, s \geq 0$ and $a, b \in A$.

We will denote $G_e(a, b) = G_{n,2,e}(a, b) = \sum_{i=0}^{n-1} 2^{i-1} L_e^{n-1-i}\left(L_e^i(a) L_e^i(b)\right)$.

Theorem 2.1 *Let (A,ω) be a n^{th}-order Bernstein algebra, $e \in A$ be an idempotent element and $D: A \longrightarrow A$ be a derivation. Then D determine and is uniquely determined by a triple $(\tilde{u}, f, g)$ with $\tilde{u} = D(e)$, $f: U_e \longrightarrow U_e$, $g: Z_e \longrightarrow Z_e$ linear applications verifying for every $u, u' \in U_e$, and $z, z' \in Z_e$:*

$$D(u) = f(u) + 2^{n+1}G_e(D(e), u); \tag{1}$$

$$D(z) = g(z) - 2^{n+1}G_e(D(e), z); \tag{2}$$

1. $g(uu') = f(u)u' + uf(u') +$
$2^{n+1}\left[G_e(D(e), u)u' + uG_e(D(e), u') + G_e(D(e), uu')\right]$;
2. $2^n f(L_e^n(uz)) + g(uz - 2^n L_e^n(uz)) = f(u)z + ug(z) +$
$2^{n+1}[G_e(D(e), u)z - uG_e(D(e), z) + G_e(D(e), uz) - 2^{n+1}G_e(D(e), L_e^n(uz))]$;
3. $2^n f(L_e^n(zz')) + g(zz' - 2^n L_e^n(zz')) = g(z)z' + zg(z') -$
$2^{n+1}[G_e(D(e), z)z' + zG_e(D(e), z') - G_e(D(e), zz') + 2^{n+1}G_e(D(e), L_e^n(zz'))]$;
4. $g(ez) = eg(z) + D(e)z - 2^n L_e^n(D(e)z)$.

Moreover, the application $\varphi: \mathrm{Der}(A) \longrightarrow U_e \times End_K(U_e) \times End_K(Z_e)$ such that $\varphi(D) = (\tilde{u}, f, g)$ is a monomorphism of vector spaces, where $\mathrm{Der}(A)$ is the set of all derivations of A.

3 Derivations

3.1 The case $(\mathbf{U_e} \oplus \mathbf{Z_e})^2 \subseteq \mathbf{Z_e}$

In this section we will suppose that $\mathrm{char}(K) = 0$.

It's known that the condition $(U_e \oplus Z_e)^2 \subseteq Z_e$ is an invariant of the algebra A and $Z_e = Z_{e'}$ for all $e, e' \in \Im(A)$ (see [3]).

Lemma 3.1 *Let (A,ω) be a n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$. Then $G_e(u,z) = 0$, $L_e^n(uz) = 0$ and $L_e^n(zz') = 0$ for all $u \in U_e$ and $z, z' \in Z_e$*

Proof: For all $u \in U_e$ and $z \in Z_e$ we have, $L_e^n(uz), L_e^n(zz') \in U_e \cap Z_e$, so $L_e^n(uz) = L_e^n(zz') = 0$. By [3], $G_e(u,z) = G_{n,2,e}(u,z) = 0$. ∎

By Theorem 2.1 and Lemma 3.1, we have

Proposition 3.1 *Let (A,ω) be a n^{th}-order Bernstein algebra, $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$ and $D: A \longrightarrow A$ be a derivation. Then D determine and is uniquely determined by a triple $(\tilde{u}, f, g)$ with $\tilde{u} = D(e) \in U_e$, $f: U_e \longrightarrow U_e$, $g: Z_e \longrightarrow Z_e$ linear applications verifying for every $u, u' \in U_e$, and $z, z' \in Z_e$:*

$$D(u) = f(u) + 2^{n+1}G_e(D(e), u); \tag{3}$$

$$D(z) = g(z); \tag{4}$$

1. $g(uu') = f(u)u' + uf(u') + 2^{n+1}\left[G_e(D(e), u)u' + uG_e(D(e), u')\right]$;
2. $g(uz) = f(u)z + ug(z) + 2^{n+1}G_e(D(e), u)z$;
3. $g(zz') = g(z)z' + zg(z')$;
4. $g(ez) = eg(z) + D(e)z$.

Corollary 3.1 *Let (A,ω) be a n^{th}-order Bernstein algebra, $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$ and $D: A \longrightarrow A$ be a derivation. Then $D(Z_e) \subseteq Z_e$.*

Theorem 3.1 *Let Z be a finite dimensional algebra such that $Z^2 \neq 0$. If there exists $n \geq 3$ such that $z^{[n]} = 0$ for all $z \in Z$, then*

$$\dim_K \mathrm{Der}(Z) \leq d^2 - 2d + 2,$$

where $d = \dim_K Z$.

Proof: Let $z \in Z$ such that $z^2 \neq 0$. We will prove that $\{z, z^2\}$ is linearly independent. Given $\alpha, \beta \in K$ such that $\alpha z + \beta z^2 = 0$. Since $z^{[n]} = 0$ and $z^{[2]} = z^2 \neq 0$, there exists $k \geq 2$ such that $z^{[k]} \neq 0$ and $z^{[k+1]} = 0$. So $\alpha^{2^{k-1}} z^{[k]} = \beta^{2^{k-1}} z^{[k+1]} = 0$. Hence $\alpha = \beta = 0$.

Let $\{z_1 = z, z_2 = z^2, z_3, \ldots, z_d\}$ be a basis of Z. We define the linear functions $g_{i2}: Z \longrightarrow Z$ $(i = 1, \ldots, d)$ and $g_{1j}: Z \longrightarrow Z$ $(j = 3, \ldots, d)$ given by

$$g_{i2}(z_k) = \begin{cases} z_i & \text{if } k = i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g_{1j}(z_k) = \begin{cases} z_1 & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

We have $\{g_{12}, \ldots, g_{d2}, g_{13}, \ldots, g_{1d}\}$ is a linearly independent set. Moreover if $g \in K(g_{12}, \ldots, g_{d2}, g_{13}, \ldots, g_{1d})$ and g is a derivation, then $g = 0$. Hence $\dim_k \mathrm{Der}(Z) \leq d^2 - (2d-2)$. ■

Corollary 3.2 *Let (A, ω) be a n^{th}-order Bernstein algebra of type $(1+r, d)$ and $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$, $eZ_e = 0$ and $Z_2^2 \neq 0$. Then $\mathrm{Der}(A) \leq r + r^2 + d^2 - 2d + 2$.*

Remark 1 *There is an example of a 2^{nd}-order Bernstein algebra of type $(1+r, d)$, where $(U_e \oplus Z_e)^2 \subseteq Z_e$, $eZ_e = 0$, $Z_2^2 \neq 0$ and $\dim_k \mathrm{Der}(A) = r + r^2 + d^2 - 2d + 2$ (see [1]).*

3.2 The power-associative case

In this section we study derivation in power-associative n^{th}-order Bernstein algebras. In this section we will suppose that $n \geq 2$.

An algebra A is power-associative if $x^i x^j = x^{i+j}$ for every $i, j \geq 1$ and $x \in A$. It is known that, A is power-associative if and only if $x^2 x^2 = x^4$ for all $x \in A$.

Let (A, ω) be a power-associative n^{th}-order Bernstein algebra. By [5] and [12] we have the following results:

For all $e \in \Im(A)$, $u \in U_e$, $z \in Z_e$ we have:

$$U_e Z_e \subseteq U_e, \; Z_e^2 \subseteq Z_e, \; eZ_e = 0, \tag{5}$$

$$\Im(A) = \{e + u + u^2 \mid u \in U_e\} \tag{6}$$

$$U_{e'} = \{u' + 2uu' \mid u' \in U_e\}, \; Z_{e'} = \{z - 2uz \mid z \in Z_e\}, \text{ for all } e' = e + u + u^2 \in \Im(A), \text{ where } u \in U_e, \tag{7}$$

$$u^3 = 0, \tag{8}$$

$$2u(uz) = u^2 z, \tag{9}$$

$$2(uz)z = uz^2, \tag{10}$$

$$u^2(uz) = u(u^2 z) = 0, \tag{11}$$

$$2(uz)z^2 = 2(uz^2)z = uz^3, \tag{12}$$

$$2(uz)^2 + u(uz^2) = (u^2 z)z. \tag{13}$$

By Theorem 2.1, and (5), (8), (9) and (10), we have

Proposition 3.2 *Let (A, ω) be a power-associative n^{th}-order Bernstein algebra, $e \in A$ be an idempotent element and $D: A \longrightarrow A$ a derivation. Then D determine and is uniquely determined by a triple $(\widetilde{u}, f, g)$ with $\widetilde{u} = D(e)$, $f: U_e \longrightarrow U_e$, $g: Z_e \longrightarrow Z_e$ are linear applications verifying for every $u, u' \in U_e$, and $z, z' \in Z_e$:*

$$D(u) = f(u) + 2D(e)u; \tag{14}$$

$$D(z) = g(z) - 2D(e)z; \tag{15}$$

1. $g(uu') = f(u)u' + uf(u')$;

2. $f(uz) = f(u)z + ug(z)$;

3. $g(zz') = g(z)z' + zg(z')$.

Lemma 3.2 *Let (A, ω) be a power-associative n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element. Then $Z_e^2 \neq 0$.*

Proof: Suppose that $Z_e^2 = 0$. By (9), (11) and (13) we have

$$u(uz) = 0, \; u^2(uz) = 0 \text{ and } (uz)^2 = 0.$$

Hence (A, ω) is a Bernstein algebra $(n = 1)$, it is a contradiction. ■

Corollary 3.3 *Let (A, ω) be a power-associative n^{th}-order Bernstein algebra of type $(1+r, d)$ and $e \in A$ be an idempotent element.*

1. If $U_e \neq 0$ then $\mathrm{Der}(A) \neq 0$,

2. If $U_e = 0$ and $Z_e^k = 0$ for some k, then $\mathrm{Der}(A) \neq 0$.

Proof: 1. By Proposition 3.2 we have $(\bar{u}, O_{U_e}, 0_{Z_e})$ define a derivation in A for all $\bar{u} \in U_e$. ■

2. By [6, Prop.2.5], we have $\mathrm{Der}(A) \neq 0$.

Corollary 3.4 *Let (A,ω) be a power-associative n^{th}-order Bernstein algebra. If $\mathrm{Der}(A) = 0$, then (A,ω) is not genetic algebra.*

Proof: By Corollary 3.3 we have $U_e = 0$ and $\ker\omega = Z_e$ is not nilpotent. Hence A is not genetic algebra. ■

By Theorem 3.1, Lemma 3.2 and Proposition 3.2 we have

Proposition 3.3 *Let (A,ω) be a power-associative n^{th}-order Bernstein algebra of type $(1+r,d)$. Then $\mathrm{Der}(A) \leq r + r^2 + d^2 - 2d + 2$.*

Theorem 3.2 *Let (A,ω) be a power-associative n^{th}-order Bernstein algebra. Suppose that there exists $e \in A$ an idempotent element such that $U_e = 0$, then A is determined, up to isomorphisms, by a commutative power-associative nil algebra Z of index 2^n. Moreover $\mathrm{Der}(A) \cong \mathrm{Der}(Z)$.*

Proof: Let $e' \in A$ other idempotent element, then $e' = (e+z)^{[n+1]}$ for some $z \in Z_e$, but $e' = (e+z)^{[n+1]} = e$, so (A,ω) has an unique idempotent element and the Peirce decomposition is unique too.

We will denote $Z_e = Z$. Since (A,ω) is power-associative n^{th}-order Bernstein algebra, then $0 = z^{[n+1]} = z^{2^n}$ for all $z \in Z$ and 2^n is the nil index of Z.

Let Z be a commutative power-associative algebra of nil index 2^n. For $A(Z) = Ke \oplus Z$ with multiplication:

$$(\alpha e + z)(\alpha' e + z') = \alpha\alpha' e + zz' \text{ for all } \alpha, \alpha' \in K \text{ and } z, z' \in Z,$$

and the weight homomorphism $\omega: A(Z) \longrightarrow K$ given by $\omega(\alpha e + z) = \alpha$, we have $(A(Z),\omega)$ is a power-associative n^{th}-order Bernstein algebra.

It is easy to see that, if Z and Z' are commutative power-associative algebras of nil index 2^n and $Z \cong Z'$, then $A(Z) \cong A(Z')$.

Given $D \in \mathrm{Der}(A)$. By Proposition 2.1 we have $D(e) \in U_e$ and $D(A) \subseteq \ker\omega$ so $D(e) = 0$ and $D(A) \subseteq Z$. Hence $D|_Z \in \mathrm{Der}(Z)$. Given $D \in \mathrm{Der}(Z)$. Let $D': A \longrightarrow A$ be given by $D'(\alpha e + z) = D(z)$, then $D' \in \mathrm{Der}(A)$. ■

Theorem 3.3 *Let Z be a finite dimensional power-associative nil algebra of index $k+1$. Then,*

$$\dim_K \mathrm{Der}(Z) \leq d^2 - 2(k-1)d + k(k-1),$$

where $d = \dim_K Z$.

Proof: Given $z \in Z$ such that $z^k \neq 0$. Since $z^{k+1} = 0$, then $\{z, \ldots, z^k\}$ is linearly independent set. Let $\{z_1 = z, \ldots, z_k = z^k, z_{k+1}, \ldots, z_d\}$ be a basis of Z.

We define the linear functions $g_{ij}: Z \longrightarrow Z$ $(i = 1, \ldots, d;\ j = 2, \ldots, k)$ and $g_{lj}: Z \longrightarrow Z$ $(l = 1, \ldots, k-1;\ j = k+1, \ldots, d)$ given by

$$g_{ij}(z_t) = \begin{cases} z_i & \text{if } t = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad g_{lj}(z_t) = \begin{cases} z_l & \text{if } t = j, \\ 0 & \text{otherwise.} \end{cases}$$

We have that $\{g_{ij}\}_{\substack{1 \leq i \leq d \\ 2 \leq j \leq k}} \cup \{g_{lj}\}_{\substack{1 \leq l \leq k-1 \\ k+1 \leq j \leq d}}$ is a linearly independent set. Moreover if $g \in K\left(\{g_{ij}\}_{\substack{1 \leq i \leq d \\ 2 \leq j \leq k}} \cup \{g_{lj}\}_{\substack{1 \leq l \leq k-1 \\ k+1 \leq j \leq d}}\right)$ and g is a derivation then $g = 0$.

Hence $\dim_K \mathrm{Der}(Z) \leq d^2 - [2(k-1)d - k(k-1)]$. ■

Corollary 3.5 *Let (A,ω) be a power-associative n^{th}-order Bernstein algebra of type $(1+r,d)$, $e \in A$ be an idempotent element and $k+1$ be the nil index of Z_e. Then*

$$\dim_K \mathrm{Der}(Z) \leq r + r^2 + d^2 - 2(k-1)d + k(k-1).$$

Example 3.1 *Let V be a vector space and $T: V \longrightarrow V$ be a linear function such that $T^{2^n} = 0$ and $T^{2^n-1} \neq 0$. Let U be a vector space of $\dim_K U = r$. In the vector space $A = Ke \oplus U \oplus K(T, \ldots, T^{2^n-1})$ we define a following multiplication:*

$$(\alpha e + u + z)(\alpha' e + u' + z') = \alpha\alpha' e + \tfrac{1}{2}(\alpha' u + \alpha u') + zz',$$

for all $\alpha, \alpha' \in K$, $u, u' \in U$ and $z, z' \in K(T, \ldots, T^{2^n-1})$ and the weight homomorphism $\omega: A \longrightarrow K$ given by $\omega(\alpha e + u + z) = \alpha$. Then (A,ω)

is a power-associative n^{th}*-order Bernstein algebra of type* $(1+r,d)$, *where* $d = 2^n - 1$. *The nil index of* $Z_e = K(T, \ldots, T^{2^n-1})$ *is* $k+1 = 2^n$. *In this case,* $\dim_K \operatorname{Der}(A) = r + r^2 + d^2 - 2(k-1)d + k(k-1) = 2^n - 1$.

Proposition 3.4 *Let* (A,ω) *be a power-associative* n^{th}*-order Bernstein algebra. If* J *is the Etherington's ideal of* A, *that is, the ideal of* A *generated by* $\{x^2 - \omega(x)x \mid x \in A\}$, *then* $J = U_eZ_e \oplus Z_e$ *for all idempotent element* $e \in A$.

Proof: Let $e \in A$ be an idempotent element. It is easy to see that $U_eZ_e \oplus Z_e$ is an ideal of A. If $x = \alpha e + u + z \in A$ with $\alpha \in K$, $u \in U_e$ and $z \in Z_e$, then

$$x^2 - \omega(x)x = 2uz + u^2 + z^2 - \alpha z \in U_eZ_e \oplus Z_e.$$

If $z \in Z_e$, for $x_1 = z$ and $x_2 = e - z$ we have $x_1^2 - \omega(x_1)x_1 = z^2 \in J$ and $x_2^2 - \omega(x_2)x_2 = z + z^2 \in J$ so $z \in J$. For all $u \in U_e$ and $z \in Z_e$ we have $uz \in J$ because J is an ideal. ■

Corollary 3.6 *Let* (A,ω) *be a power-associative* n^{th}*-order Bernstein algebra. Then the dimension of* U_eZ_e *and* $U_e^2 + Z_e^2$ *are invariants.*

Proof: Since $\dim_K Z_e$ is invariant, then by Proposition 3.4, $\dim_K(U_eZ_e)$ is invariant so $\dim(U_e^2 + Z_e^2)$ is invariant too, because $(\ker\omega)^2 = U_eZ_e \oplus (U_e^2 + Z_e^2)$. ■

Proposition 3.5 *Let* (A,ω) *be a power-associative* n^{th}*-order Bernstein algebra of type* $(1+r,d)$. *Then*

$$\dim_K \operatorname{Der}(A) \leq r + s^2 + r(r-s) + t^2 + d(d-t),$$

where $s = \dim_K(U_eZ_e)$ *and* $t = \dim_K(U_e^2 + Z_e^2)$.

Proof: By Proposition 3.2 the triple $(D(e), f, g)$ determine a derivation if and only if

$$g(uu') = f(u)u' + uf(u');\ f(uz) = f(u)z + ug(z);\ g(zz') = g(z)z' + zg(z'),$$

for all $u, u' \in U_e$ and $z, z' \in Z_e$. It follows that $f(U_eZ_e) \subseteq U_eZ_e$ and $g(U_e^2 + Z_e^2) \subseteq U_e^2 + Z_e^2$.

Hence, $\dim_K \operatorname{Der}(A) \leq r + s^2 + r(r-s) + t^2 + d(d-t)$. ■

4 Inner derivations

4.1 General case

Definition 4.1 *A derivation* $D\colon A \longrightarrow A$ *is an inner derivation if* $D \in \mathcal{L}(A)$, *where* $\mathcal{L}(A)$ *is the Lie algebra generated by* L_x *and* R_x *with* $x \in A$

We will see what conditions determine that L_u and L_z are derivations in a n^{th}-order Bernstein algebra.

Lemma 4.1 *Let* (A,ω) *be a* n^{th}*-order Bernstein algebra and* $e \in A$ *be an idempotent element. If* $x \in N$ *verifies that* $x + 2ex = 0$, *then* $x = 0$.

Proof: Given $u \in U_e$ and $z \in Z_e$ such that $x = u+z$ then $u+z+u+2ez = 0$, so $u = 0$ and $x \in Z_e$. Therefore $x = -2L_e(x) = (-2)^2 L_e^2(x) = \cdots = (-2)^n L_e^n(x) = 0$. ■

Proposition 4.1 *Let* (A,ω) *be a* n^{th}*-order Bernstein algebra and* $e \in A$ *be an idempotent element. If* $\overline{u} \in U_e$, *then* $L_{\overline{u}}$ *is a derivation if and only if:*

1. $\overline{u}U_e \subseteq C_{1e}$;
2. $\overline{u}Z_e = 0$;
3. $(\overline{u}u)u' + (\overline{u}u')u = 0$ *for all* $u, u' \in U_e$. *In particular* $(\overline{u}u)u = 0$ *for all* $u \in U_e$;
4. $(\overline{u}u)z = \overline{u}(uz)$ *for all* $u \in U_e$ *and* $z \in Z_e$;
5. $\overline{u}Z_e^2 = 0$.

In this case the triple associated to $L_{\overline{u}}$ *is* $\left(\frac{1}{2}\overline{u}, 0_{U_e}, 0_{Z_e}\right)$.

Proof: Suppose that $L_{\overline{u}}$ is a derivation. For all $x, x' \in A$ we have $L_{\overline{u}}(xx') = L_{\overline{u}}(x)x' + xL_{\overline{u}}(x')$ then $\overline{u}(xx') = (\overline{u}x)x' + x(\overline{u}x')$.

If $x = e$ and $x' = u$, we have $e(\overline{u}u) = 0$ and so $\overline{u}u \in C_{1e}$.

If $x = e$ and $x' = L_e^{n-1}(z)$, we have $\overline{u}L_e^{n-1}(z) + 2e\left(\overline{u}L_e^{n-1}(z)\right) = 0$ and by Lemma 4.1 $\overline{u}L_e^{n-1}(z) = 0$. We will by induction that $\overline{u}L_e^{n-k}(z) = 0$. If $k = 1$ the result holds. So assume that it holds for $k-1$. If $x = e$ and $x' = L_e^{n-k}(z)$ we have $\overline{u}L_e^{n-k}(z) + 2e\left(\overline{u}L_e^{n-k}(z)\right) = 0$ and so $\overline{u}L_e^{n-k}(z) = 0$. If $k = n$ then $\overline{u}z = 0$.

If we take $x = u$ and $x' = u'$ or $x = u$ and $x' = z$ or $x = z$ and $x' = z'$ respectively and we consider $\overline{u}Z_e = 0$ we have 3, 4 and 5.

Conversely a calculation shows that $L_{\overline{u}}$ is a derivation.

Let's see which is the triple $(\tilde{u}, f, g)$ associated to $L_{\overline{u}}$, $\tilde{u} = L_{\overline{u}}(e) = \frac{1}{2}\overline{u}$.

Since for all $u \in U_e$, $\overline{u}u \in C_{1e}$ we have $2^{n+1}G_e(\frac{1}{2}\overline{u}, u) = \overline{u}u$ and by $L_{\overline{u}}(u) = f(u) + 2^{n+1}G_e(\frac{1}{2}\overline{u}, u)$ we obtain $f(u) = 0$.

Let $z \in Z_e$, since $\overline{u}Z_e = 0$, then $G_e(\frac{1}{2}\overline{u}, z) = 0$ and by $L_{\overline{u}}(z) = g(z) - 2^{n+1}G_e(\frac{1}{2}\overline{u}, z)$, we obtain $g(z) = 0$. ■

Proposition 4.2 *Let (A, ω) be a n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element. If $\overline{z} \in Z_e$, then $L_{\overline{z}}$ is a derivation if and only if:*

1. $e\overline{z} = 0$;
2. $\overline{z}U_e \subseteq U_e$;
3. $(ez)\,\overline{z} = e\,(z\overline{z})$ *for all* $z \in Z_e$;
4. $u^2\overline{z} = 2u\,(u\overline{z})$ *for all* $u, \in U_e$;
5. $\overline{z}\,(uz) = (\overline{z}u)\,z + u\,(z\overline{z})$ *for all* $u \in U_e$ *and* $z \in Z_e$;
6. $z^2\overline{z} = 2z\,(z\overline{z})$ *for all* $z \in Z_e$.

In this case the triple associated to $L_{\overline{z}}$ is $\left(0, L_{\overline{z}|U_e}, L_{\overline{z}|Z_e}\right)$.

Proof: Suppose that $L_{\overline{z}}$ is a derivation. By Proposition 2.1, we have $\overline{z} \in C_{1e}$. For all $x, x' \in A$ we have $\overline{z}(xx') = (\overline{z}x)\,x' + x\,(\overline{z}x')$.

If we take $x = e$ and $x' = u$, or $x = e$ and $x' = z$ respectively and we consider that $e\overline{z} = 0$ we have 2 and 3.

Taking $x = x' = u$, or $x = u$ and $x' = z$, or $x = x' = z$, we obtain the others.

Conversely if we have the conditions, then a calculation shows that $L_{\overline{z}}$ is a derivation.

Let's see which is the triple $(\tilde{u}, f, g)$ associated to $L_{\overline{z}}$, $\tilde{u} = L_{\overline{z}}(e) = 0$ and $G_e(0, b) = 0$, so $L_{\overline{z}}(u) = f(u)$ and $L_{\overline{z}}(z) = g(z)$ for all $u \in U_e$, $z \in Z_e$. ■

4.2 The case $(\mathbf{U}_e \oplus \mathbf{Z}_e)^2 \subseteq \mathbf{Z}_e$

In this section we will suppose that $\text{char}(K) = 0$.

By Propositions 4.1 and 4.2, we have

Proposition 4.3 *Let (A, ω) be a n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$.*

1. *Let $\overline{u} \in U_e$. Then $L_{\overline{u}}$ is a derivation if and only if:*

   (a) $\overline{u}U_e \subseteq C_{1e}$;

   (b) $\overline{u}Z_e = 0$;

   (c) $(\overline{u}u)\,u' + (\overline{u}u')\,u = 0$ *for all* $u, u' \in U_e$. *In particular* $(\overline{u}u)\,u = 0$ *for all* $u \in U_e$;

   (d) $(\overline{u}U_e)\,Z_e = 0$.

   In this case the triple associated to $L_{\overline{u}}$ is $\left(\frac{1}{2}\,\overline{u}, 0_{U_e}, 0_{Z_e}\right)$.

2. *Let $\overline{z} \in Z_e$. Then $L_{\overline{z}}$ is a derivation if and only if:*

   (a) $e\overline{z} = u\overline{z} = u^2\overline{z} = 0$ *for all* $u, \in U_e$;

   (b) $(ez)\,\overline{z} = e\,(z\overline{z})$ *for all* $z \in Z_e$;

   (c) $\overline{z}\,(uz) = u\,(z\overline{z})$ *for all* $u \in U_e$ *and* $z \in Z_e$;

   (d) $z^2\overline{z} = 2z\,(z\overline{z})$ *for all* $z \in Z_e$. *In particular* $\overline{z}^3 = 0$.

   In this case the triple associated to $L_{\overline{z}}$ is $\left(0, 0_{U_e}, L_{\overline{z}|Z_e}\right)$.

Corollary 4.1 *Let (A, ω) be a n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$. If $L_{\overline{z}}$ is a derivation, where $\overline{z} \in Z_e$, then*

$$L_e^{n-1}(z\overline{z}) = L_e^{n-1}(z)\overline{z} = G_e(u, z)\overline{z} = 0 \text{ for all } u \in U_e \text{ and } z \in Z_e.$$

Proof: By [3] and 2 (a) of Proposition 4.3 we have,

$$0 = G_e(z,\bar{z}) = \sum_{i=0}^{n-1} 2^{i-1} L_e^{n-1-i}\left(L_e^i(z)L_e^i(\bar{z})\right) = 2^{-1}L_e^{n-1}(z\bar{z}).$$

By 2 (b) of the same Proposition and induction,

$$L_e^k(z\bar{z}) = L_e^k(z)\bar{z} \text{ for all } k \geq 1 \text{ and } z \in Z_e. \tag{16}$$

By 2 (c) and (16),

$$uL_e^k(z\bar{z}) = \bar{z}(uL_e^k(z)) \text{ for all } u \in U_e \text{ and } z \in Z_e. \tag{17}$$

By [3],

$$G_{n,3,e}(u,z,z) = \sum_{i=0}^{n-1} 2^i L_e^{n-1-i}\Big(L_e^i(u)G_{i,2,e}(z,z) + L_e^i(z)G_{i,2,e}(u,z)\Big) = 0, \tag{18}$$

where $G_{0,2,e}(a,b) = 0$. Linearizing (18) we have, $\sum_{i=1}^{n-1} L_e^{n-1-i}\Big(uL_e^{i-1}(z\bar{z})\Big) = 0$.

Since $L_e^{n-1}(z\bar{z}) = 0$ and by (16) and (17) then $G_e(u,z)\bar{z} = 0$. ■

Corollary 4.2 *Let (A,ω) be a n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$ and $z^3 = 0$ for all $z \in Z_e$. Then $L_{\bar{z}}$ is a derivation if and only if*

1. $e\bar{z} = u\bar{z} = u^2\bar{z} = 0$ *for all* $u \in U_e$;
2. $(ez)\bar{z} = e\,(z\bar{z})$ *for all* $z \in Z_e$;
3. $\bar{z}\,(uz) = u\,(z\bar{z})$ *for all* $u \in U_e$ *and* $z \in Z_e$;
4. $z^2\bar{z} = 0$ *for all* $z \in Z_e$

Proof: By the Proposition 4.3 it's suffices to prove that $z^2\bar{z} = 0$. Since $z^3 = 0$, we obtain $z^2z' + 2z\,(zz') = 0$ for all $z, z' \in Z_e$ so $z^2\bar{z} + 2z\,(z\bar{z}) = 0$. Since $L_{\bar{z}}$ is a derivation we have $z^2\bar{z} = 2z\,(z\bar{z})$ so $z^2\bar{z} = 0$. Conversely by the Proposition 4.3 it's suffices to prove that $2z\,(z\bar{z}) = 0$, but this is immediate since $z^2\bar{z} + 2z\,(z\bar{z}) = 0$ and $z^2\bar{z} = 0$. ■

Corollary 4.3 *Let (A,ω) be a Power-Associative n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$ then*

1. $L_{\bar{u}}$ *is a derivation for all* $\bar{u} \in U_e$
2. *If* $\bar{z} \in Z_e$, *then* $L_{\bar{z}}$ *is a derivation if and only if* $z^2\bar{z} = 2z\,(z\bar{z})$ *for all* $z \in Z_e$.

Proof: In this case $C_{1e} = Z_e$, $U_eZ_e = 0$ and $(uu')\,z = u\,(u'z) + u'\,(uz) = 0$, so hold the conditions of Proposition 4.3. ■

There are n^{th}-order Bernstein algebra with $(U_e \oplus Z_e)^2 \subseteq Z_e$ in which there exist derivations $L_{\bar{u}}$ and $L_{\bar{z}}$ non null.

Example 4.1 *Let $A = Ke + K(u) + K(z_1, z_2, z_3, w_1, \ldots, w_n)$ the n^{th} -order Bernstein algebra with product given by:*

$$e^2 = e,\ eu = \tfrac{1}{2}u,\ ew_i = w_{i+1}\ i = 1, \ldots, n-1,\ u^2 = z_3,\ z_1z_2 = z_3.$$

L_u is a derivation and $L_u(u) = z_3 \neq 0$.

Example 4.2 *Let $A = Ke + K(z_1, z_2, z_3, z_4, z_5, w_1, \ldots, w_n)$ the n^{th}-order Bernstein algebra with product given by:*

$$e^2 = e,\ ew_i = w_{i+1}\ (i = 1, \ldots, n-1),\ z_1z_2 = z_3,\ z_1z_3 = z_4,\ z_1z_5 = -z_3,$$
$$z_2z_3 = z_5,\ z_2z_4 = z_3.$$

L_{z_3} is a derivation and $L_{z_3}(z_1) = z_4 \neq 0$.

If (A,ω) is a Bernstein algebra and L_x is a derivation with $x = \bar{u} + \bar{z}$, then $L_{\bar{u}}$ and $L_{\bar{z}}$ are derivations too. But this is not hold if (A,ω) be a n^{th}-order Bernstein algebra with $n \geq 2$ as show this example.

Example 4.3 *Let $A = Ke + K(u_1, u_2) + K(z_1, z_2, z_3, w_1, \ldots, w_n)$ the n^{th}-order Bernstein algebra such that $(U_e \oplus Z_e)^2 \subseteq Z_e$, with product given by:*

$$e^2 = e,\ eu_i = \tfrac{1}{2}u_i,\ ew_i = w_{i+1}\ (i = 1, \ldots, n-1),\ u_i^2 = z_i,\ u_1u_2 =$$
$$z_3,\ u_2z_3 = \tfrac{1}{2}z_1,\ z_2^2 = z_1.$$

L_{u_1} isn't a derivation because if $u = \alpha_1 u_1 + \alpha_2 u_2$ we obtain $(u_1 u)u = \frac{1}{2}\alpha_2^2 z_1 \neq 0$ if $\alpha_2 \neq 0$. and yet it's suffices to make the calculations to see that $L_{u_1+z_1}$ is a derivation.

Proposition 4.4 *Let (A,ω) be a n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$. If $x = \bar{u} + \bar{z} \in U_e \oplus Z_e$, then L_x is a derivation if and only if for all $u, u' \in U_e$ and $z, z' \in Z_e$ we have:*

1. $e\bar{z} = 0$;
2. $\bar{z}u = \sum_{i=1}^{n-1} 2^i L_e^i(u\bar{u})$;
3. $\bar{u}z = \sum_{i=1}^{2n-1} 2^i [L_{\bar{z}}, L_e]_i(z)$,

   where $[L_a, L_b] = L_a L_b - L_b L_a = [L_a, L_b]_1$ *and* $[L_a, L_b]_{k+1} = \big[[L_a, L_b]_k, L_b\big]$,
4. $\bar{u}(uu') + \bar{z}(uu') = (\bar{u}u)u' + (\bar{u}u')u + (\bar{z}u)u' + (\bar{z}u')u$;
5. $\bar{u}(uz) + \bar{z}(uz) = (\bar{u}u)z + (\bar{u}z)u + (\bar{z}u)z + (\bar{z}z)u$;
6. $\bar{u}(zz') + \bar{z}(zz') = (\bar{u}z)z' + (\bar{u}z')z + (\bar{z}z)z' + (\bar{z}z')z$.

Proof: Suppose that L_x is a derivation, then clearly we have 4, 5 and 6.

1. Since e is an idempotent element, then $L_x(e) = L_x(e^2) = 2eL_x(e)$ so $e\bar{z} = 0$.

2. Since $eu = \frac{1}{2}u$, then $\frac{1}{2}L_x(u) = L_x(eu) = eL_x(u) + uL_x(e)$ so

$$\bar{z}u = 2e(\bar{u}u) + 2e(\bar{z}u) = 2L_e(\bar{u}u) + 2L_e(\bar{z}u). \tag{19}$$

Hence $\bar{z}u = \sum_{i=1}^{n-1} 2^i L_e^i(u\bar{u}) + 2^{n-1}L_e^{n-1}(\bar{z}u)$. By (19), $L_e^{n-1}(\bar{z}u) = 0$.

3. Since $L_x(ez) = eL_x(z) + zL_x(e)$ we have $L_{\bar{u}}(z) = 2[L_{\bar{u}}, L_e](z) + 2[L_{\bar{z}}, L_e](z)$.

Considering $L_{\bar{u}}$ restrict to Z_e then

$$L_{\bar{u}} = 2[L_{\bar{u}}, L_e] + 2[L_{\bar{z}}, L_e]. \tag{20}$$

So $L_{\bar{u}} = \sum_{i=1}^{k} 2^i [L_{\bar{z}}, L_e]_i + 2^k [L_{\bar{u}}, L_e]_k$ for all $k \geq 1$. But

$$[L_a, L_e]_k(z) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} L_e^{k-i}(aL_e^i(z)),$$

for all $a \in N$, $z \in Z_e$ and $k \geq 1$. It is easy to see that $[L_a, L_e]_{2n}(z) = 0$ for all $z \in Z_e$ because, if $0 \leq i \leq n$ then $n \leq 2n - i \leq 2n$ so $L_e^{2n-i}(aL_e^i(z)) = 0$. If $n < i \leq 2n$ we have $L_e^i(z) = 0$. Hence $L_{\bar{u}} = \sum_{i=1}^{2n} 2^i [L_{\bar{z}}, L_e]_i = \sum_{i=1}^{2n-1} 2^i [L_{\bar{z}}, L_e]_i$.

Assume now $1, \ldots, 6$ true.

a) Clearly, $L_x(e) = \frac{1}{2}\bar{u} = 2eL_x(e)$.

b) We need to prove that $L_x(eu) = eL_x(u) + uL_x(e)$ but it is equivalent to $\bar{z}u = 2L_e(\bar{u}u) + 2L_e(\bar{z}u)$.

But $\bar{z}u = \sum_{i=1}^{n-1} 2^i L_e^i(u\bar{u})$ then $2L_e(\bar{z}u) = \sum_{i=1}^{n-1} 2^{i+1} L_e^{i+1}(u\bar{u}) = \sum_{i=2}^{n} 2^i L_e^i(u\bar{u}) = \sum_{i=2}^{n-1} 2^i L_e^i(u\bar{u})$. Hence $2L_e(\bar{u}u) + 2L_2(\bar{z}u) = \sum_{i=1}^{n-1} 2^i L_e^i(u\bar{u}) = \bar{z}u$.

c) Similarly to b) we prove that

$$2[L_{\bar{u}}, L_e](z) + 2[L_{\bar{z}}, L_e](z) = L_{\bar{u}}(z)$$

that is equivalent to $L_x(ez) = eL_x(z) + zL_x(e)$. ∎

Proposition 4.5 *Let (A,ω) be a n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$ and $eZ_e = 0$. If $x = \bar{u} + \bar{z} \in U_e \oplus Z_e$, then*

1. *L_x is a derivation if and only if for all $u \in U_e$ and $z \in Z_e$ we have:*

   (a) $\bar{z}U_e = 0$;

   (b) $\bar{u}Z_e = 0$;

   (c) $\bar{z}u^2 = 2u(u\bar{u})$;

(d) $\bar{z}(uz) = (\bar{u}u)z + (\bar{z}z)u$;

(e) $\bar{z}z^2 = 2z(z\bar{z})$;

2. If L_x is a derivation, then $\bar{u}^2, \bar{z}^2 \in \mathrm{Ann}(Z_e)$ *and* $\bar{u}^2 + \bar{z}^2 \in \mathrm{Ann}(A)$;

3. If $U_e^2 = 0$ *or* $U_eZ_e = 0$ *or* $Z_e^2 = 0$, *then L_x is a derivation if and only if $L_{\bar{u}}$ and $L_{\bar{z}}$ are derivations.*

Proof: 1. Suppose that L_x is a derivation.

By 2 and 3 of Proposition 4.4, $\bar{z}u = 0$ for all $u \in U_e$ and $\bar{u}z = 0$ for all $z \in Z_e$.

By (a), (b) and 4, 5, 6 of Proposition 4.4, we have

$$\bar{z}u^2 = 2u(\bar{u}u),\ \bar{z}(uz) = (\bar{u}u)z + (\bar{z}z)u \text{ and } \bar{z}z^2 = 2z(\bar{z}z).$$

The converse is clear.

2. For $u = \bar{u}$ in (d) we have $\bar{u}^2 z = 0$ for all $z \in Z_e$.

Linearizing (e) we have $\bar{z}(zz') = z(z'\bar{z}) + z'(z\bar{z})$. If $z' = \bar{z}$ then $\bar{z}(z\bar{z}) = z\bar{z}^2 + \bar{z}(z\bar{z})$, so $\bar{z}^2 z = 0$ for all $z \in Z_e$.

Linearizing (c) we have $\bar{z}(uu') = u(\bar{u}u') + u'(\bar{u}u)$. If $u' = \bar{u}$ then $\bar{z}(u\bar{u}) = u\bar{u}^2$. For $z = \bar{z}$ in (d), $0 = (\bar{u}u)\bar{z} + \bar{z}^2 u$ then $\bar{z}^2 u = -\bar{z}(u\bar{u})$. Hence $(\bar{u}^2 + \bar{z}^2)u = 0$ for all $u \in U_e$.

3. Suppose that L_x is a derivation.

Then $L_{\bar{u}}$ and $L_{\bar{z}}$ are derivation if and only if

$$L_{\bar{u}}(yy') = L_{\bar{u}}(y)y' + yL_{\bar{u}}(y') \text{ and } L_{\bar{z}}(yy') = L_{\bar{u}}(y)y' + yL_{\bar{u}}(y'),$$

for all $y, y' \in A$.

i) If $y = y' = e$ it is clear. If $y = e$ and $y' = u$ or z too.

ii) If $y = y' = u$ we have

$L_{\bar{u}}(u^2) = 2uL_{\bar{u}}(u)$ if and only if $\bar{u}u^2 = 2u(u\bar{u})$. By (b), $\bar{u}u^2 = 0$. If $U_e^2 = 0$ or $U_eZ_e = 0$ then $u(\bar{u}u) = 0$. If $Z_e^2 = 0$ then by (c), $2u(u\bar{u}) = \bar{z}u^2 = 0$.

$L_{\bar{z}}(u^2) = 2uL_{\bar{z}}(u)$ if and only if $\bar{z}u^2 = 2u(u\bar{z})$. By (a), $u\bar{z} = 0$. If $U_e^2 = 0$ or $Z_e^2 = 0$ then $zu^2 = 0$. If $U_eZ_e = 0$ then by (c), $zu^2 = 2u(\bar{u}u) = 0$.

iii) For $y = u$ and $y' = z$ we have

$L_{\bar{u}}(uz) = L_{\bar{u}}(u)z + uL_{\bar{u}}(z)$ if and only if $\bar{u}(uz) = u(\bar{u}z) + z(\bar{u}u)$. By (b), $\bar{u}z = 0 = \bar{u}(uz)$. If $U_e^2 = 0$ or $Z_e^2 = 0$ then $z(\bar{u}u) = 0$. If $U_eZ_e = 0$ then by (c), $\bar{z}(u\bar{u}) = u\bar{u}^2 + \bar{u}(u\bar{u}) = 0$.

$L_{\bar{z}}(uz) = L_{\bar{z}}(u)z + uL_{\bar{z}}(z)$ if and only if $\bar{z}(uz) = u(\bar{z}z) + z(\bar{z}u)$. By (a), $\bar{z}u = 0$. If $Z_e^2 = 0$ or $U_eZ_e = 0$ then $\bar{z}(uz) = u(\bar{z}z) = 0$. If $U_e^2 = 0$ and by (d), $\bar{z}(uz) = (\bar{z}z)u$.

iv) If $y = y' = z$ we have

$L_{\bar{u}}(z^2) = 2zL_{\bar{u}}(z)$ if and only if $\bar{u}z^2 = 2z(\bar{u}z)$. By (b), $\bar{u}z^2 = 0 = \bar{u}z$.

$L_{\bar{z}}(z^2) = 2zL_{\bar{z}}(z)$ if and only if $\bar{z}z^2 = 2z(\bar{z}z)$. But it is the condition (e).

Suppose that $L_{\bar{u}}$ and $L_{\bar{z}}$ are derivations.

For $x = \bar{u} + 0$ in 1 of Proposition 4.5 we have

$$\bar{u}Z_e = 0,\ u(\bar{u}u) = 0 \text{ and } (\bar{u}u)z = 0.$$

For $x = 0 + \bar{z}$ we have

$$\bar{z}U_e = 0,\ \bar{z}u^2 = 0,\ \bar{z}(uz) = (\bar{z}z)u \text{ and } \bar{z}z^2 = 2z(\bar{z}z).$$

Therefore $L_{\bar{u}+\bar{z}}$ is a derivation. ∎

Corollary 4.4 *Let (A, ω) be a n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element such that $(U_e \oplus Z_e)^2 \subseteq Z_e$, $eZ_e = 0$ and $\dim_K U_e = 1$. Then for $x = \bar{u} + \bar{z} \in U_e \oplus Z_e$ we have L_x is a derivation if and only if $L_{\bar{u}}$ and $L_{\bar{z}}$ are derivations.*

Proof: We can suppose that $\bar{u} \neq 0$. Since $L_{\bar{u}}$ is a derivation then $\bar{u}Z_e = 0$ so $U_eZ_e = 0$. Therefore by Proposition 4.5 we have the end of this proof. ∎

4.3 The power-associative case

By Propositions 4.1 and 4.2, we have

Proposition 4.6 *Let (A, ω) be a power-associative n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element. If $\bar{u} \in U_e$ and $z \in Z_e$, then:*

1. *$L_{\bar{u}}$ is a derivation if and only if $\bar{u}Z_e = 0$,*

   In this case the triple associated is $\left(\frac{1}{2}\bar{u}, 0_{U_e}, 0_{Z_e}\right)$.

2. *$L_{\bar{z}}$ is a derivation if and only if $(\bar{z}U_e)Z_e = 0$ and $L_{\bar{z}}|_{Z_e}$ is a derivation.*

   In this case the triple associated is $(0, L_{\bar{z}}|_{U_e}, L_{\bar{z}}|_{Z_e})$.

The proof of the followings results is the same as [1],

Proposition 4.7 *Let (A,ω) be a power-associative n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element. If there exists $\bar{u} \in U_e$ such that $\bar{u}Z_e = 0$, then $L_{\bar{u}u'}$ is a derivation for all $u' \in U_e$.*

Theorem 4.1 *Let (A,ω) be a power-associative n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element. If $x = \bar{u} + \bar{z}$, where $\bar{u} \in U_e$ and $\bar{z} \in Z_e$, then L_x is a derivation if and only if $L_{\bar{u}}$ and $L_{\bar{z}}$ are derivations.*

5 The graded n^{th}-order Bernstein algebras

The proof of followings results in this section is the same as [1].

The algebra A is $\mathbb{Z}_2$-graded if $A = A_0 + A_1$, where A_0, A_1 are subspaces of A and $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}_2$. It is known that, if A is $\mathbb{Z}_2$-graded then $\operatorname{Der}(A)$ is $\mathbb{Z}_2$-graded too, where

$$D_0(A) = \{D \in \operatorname{Der}(A) \mid D(A_0) \subseteq A_0,\ D(A_1) \subseteq A_1\} \text{ and}$$
$$D_1(A) = \{D \in \operatorname{Der}(A) \mid D(A_0) \subseteq A_1,\ D(A_1) \subseteq A_0\}.$$

Proposition 5.1 *Let (A,ω) be a power-associative n^{th}-order Bernstein algebra. Then A is $\mathbb{Z}_2$-graded. Moreover if $e \in A$ is an idempotent element, then $A_0 = Ke \oplus Z_e$, $A_1 = U_e$, $D_0(A) = \{D \in \operatorname{Der}(A) \mid D(e) = 0\}$ and $D_1(A) = \{(\bar{u}, 0_{U_e}, 0_{Z_e}) \mid \bar{u} \in U_e\}$.*

Corollary 5.1 *Let (A,ω) be a power-associative n^{th}-order Bernstein algebra of type $(1+r,d)$. Then $\dim_K D_1(A) = r$.*

We will denote by $\operatorname{Inn}(A)$ the set of all inner derivations of A.

Corollary 5.2 *Let (A,ω) be a power-associative n^{th}-order Bernstein algebra. Then $D_1(A) \subseteq \operatorname{Inn}(A)$ and $\operatorname{Inn}(A) = D_1(A) \oplus (D_0(A) \cap \operatorname{Inn}(A))$.*

A commutative algebra A is a Jordan algebra if $(x^2y)x = x^2(yx)$ for all $x, y \in A$.

Corollary 5.3 *Let (A,ω) be a Jordan n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element. Then:*

1. $\operatorname{Der}(A) \neq 0$,

2. If $U_e \neq 0$ there exist $\bar{u} \in U_e$ such that $L_{\bar{u}} \neq 0$.

It is known that, if A is a Jordan algebra then

$$\mathcal{L}(A) = \left\{ L_x + \sum_{i=1}^{k} [L_{x_i}, L_{y_i}] \mid k \geq 1, x, y_i, z_i \in A,\ L_x \text{ is a derivation} \right\}.$$

Proposition 5.2 *Let (A,ω) be a Jordan n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element. Then*

$$D_0(A) \cap \operatorname{Inn}(A) = \Big\{ L_{\bar{z}} + \sum_{i=1}^{k} [L_{u_i}, L_{u'_i}] + \sum_{j=1}^{t} [L_{z_j}, L_{z'_j}] \mid k \geq 1, t \geq 1,$$
$$u_i, u'_i \in U_e, \bar{z}, z_j, z'_j \in Z_e, (\bar{z}U_e)Z_e = 0, L_{\bar{z}}|_{Z_e} \textit{ is a derivation} \Big\}.$$

Corollary 5.4 *Let (A,ω) be a Jordan n^{th}-order Bernstein algebra and $e \in A$ be an idempotent element. If $U_eZ_e = 0$, then*

$$\operatorname{Inn}(A) = L_{U_e} \oplus \Big\{ L_{\bar{z}} + \sum_{j=1}^{t} [L_{z_j}, L_{z'_j}] \mid t \geq 1, \bar{z}, z_j, z'_j \in$$
$$Z_e, L_{\bar{z}}|_{Z_e} \textit{ is a derivation} \Big\},$$

where $L_{U_e} = \{L_u \mid u \in U_e\}$.

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