Arch. Math., Vol. 52, 539–543 (1989)

0003-889X/89/5206-0539 $ 2.50/0


# Semi-prime Bernstein algebras

By

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**1. Introduction.** Bernstein algebras were introduced by Holgate [2] in connection with a problem in genetics. Recently, Wörz-Busekros [3, 4] proved that trivial Bernstein algebras are Jordan and gave a complete classification in the case where the dimension is finite. A Bernstein algebra $A$ over a field $K$ is called trivial if $A = Ke + N$ (additive direct sum) where $e$ is an idempotent and $N$ is a nilpotent ideal of index two.

In this paper we study the semiprime case. That is, we consider Bernstein algebras that do not have nonzero nilpotent ideals of index two. We prove that any such algebra is Jordan. Furthermore, under the condition that the algebra is finitely generated, we show that it must be a field. The proofs require characteristic different from two. Our work implies that nearly all (finitely generated) Bernstein algebras possess nonzero ideals which are nilpotent of index two. The only ones which do not are the fields.

**2. Bernstein algebras.** Throughout this paper, $K$ will be a field of characteristic $\neq 2$, and $A$ will be a nonassociative commutative algebra over $K$. $A$ is called a Bernstein algebra if $A$ has a nontrivial algebra homomorphism $w: A \to K$ which satisfies the following identity.

(1) $$x^2 x^2 = w(x)^2 x^2 \quad \text{for all} \quad x \in A.$$

The homomorphism $w$ is uniquely determined [1, Lemma 1]. Let $N = \operatorname{Ker} w$. If $a \in A$ and $w(a) \neq 0$, then $e = a^2/w(a)^2$ is an idempotent of $A$ and we have the Peirce decomposition:

$$A = Ke + U + Z \quad \text{(additive direct sum)}$$

where $U = \{ex: x \in N\}$ and $Z = \{z \in A: ez = 0\}$. Note that $N = U + Z$. The subspaces $U$ and $Z$ satisfy the following identities (see [3]).

(2) $$U^2 \subseteqq Z, \quad UZ \subseteqq U; \quad Z^2 \subseteqq U, \quad UZ^2 = 0.$$

*) This paper was written while this author was visiting Iowa State University on a grant from the CNPq of Brazil.

Also, for any $u, u_1, u_2, u_3 \in U$ and $z, z_1, z_2 \in Z$, we have:

(3) $eu = \frac{1}{2}u$

(4) $u^3 = 0$ (4') $(u_1 u_2)u_3 + (u_2 u_3)u_1 + (u_3 u_1)u_2 = 0$

(5) $u(uz) = 0$ (5') $u_1(u_2 z) + u_2(u_1 z) = 0$

(6) $(uz)^2 = 0$ (6') $(u_1 z)(u_2 z) = 0$ (6'') $(uz_1)(uz_2) = 0$.

An algebra $A$ is called semiprime if it satisfies the following condition: If $I$ is an ideal of $A$ and $I^2 = 0$, then $I = 0$.

**Lemma.** *Let $A = Ke + U + Z$ be a semiprime Bernstein algebra. Then the following identities hold.*

(7) $Z^2 = 0$;

(8) $(uz)z = 0$ *for all* $u \in U,\ z \in Z$;

(8') $(uz_1)z_2 + (uz_2)z_1 = 0$ *for all* $u \in U, z_1, z_2 \in Z$;

(9) $n^3 = 0$ *for all* $n \in N$;

(9') $(n_1 n_2)n_3 + (n_2 n_3)n_1 + (n_3 n_1)n_2 = 0$ *for all* $n_1, n_2, n_3 \in N$.

Proof. Let $L = \{x \in U: xU = 0\}$. We claim that $L$ is an ideal of $A$ and $L^2 = 0$. If $x \in L$, then from Eq. (3), and the definition of $L$ we obtain $xe = \frac{1}{2}x \in L$, $xU = 0$. We will now prove that $xZ \subseteq L$. From Eq. (5') we have for any $u_1, u_2 \in U$, $z \in Z$, the condition $u_1(u_2 z) + u_2(u_1 z) = 0$. It follows then that $(u_1 Z)U \subseteq u_1(UZ)$. In our present circumstance, let $u_1 = x$. Then $(xZ)U \subseteq x(UZ) \subseteq xU = 0$ by Eq. (2) and definition of $L$. Since $xZ \subseteq U$ by Eq. (2) and we have just shown $(xZ)U = 0$, we have shown $xZ \subseteq L$. This shows that $LA \subseteq L$ and so $L$ is an ideal. Since $L^2 \subseteq LU = 0$, we have $L^2 = 0$. Since $A$ is semiprime, $L = 0$.

We now prove Eq. (7). By Eq. (2) $Z^2 \subseteq U$ and $Z^2 U = 0$. Thus $Z^2 \subseteq L$ and so $Z^2 = 0$.

We now prove Eq. (8). If $u \in U$ and $z \in Z$, we use Eq. (2), Eq. (5'), and Eq. (6') to get $(uz)z \in U$ and $[(uz)z]U \subseteq (Uz)(uz) = 0$. This means that $(uz)z \in L$ and so $(uz)z = 0$. The identity Eq. (8') is a linearization of Eq. (8).

We now prove Eq. (9). Let $n \in N$. Then $n = u + z$ where $u \in U$ and $z \in Z$. Then

$$n^3 = (u^2 + 2uz)(u + z) = u^3 + 2u(uz) + u^2 z + 2(uz)z = 0$$

since $z^2 = 0$ by Eq. (7), $u^3 = 0$ by Eq. (4), $u(uz) = 0$ by Eq. (5), and $(uz)z = 0$ by Eq. (8). The identity Eq. (9') is a linearized form of Eq. (9).

A Jordan algebra is a commutative nonassociative algebra which satisfies the additional identity $(x^2 y)x - x^2(yx) = 0$.

**Theorem 1.** *Let $A$ be a semiprime Bernstein algebra of characteristic $\neq 2$. Then $A$ is a Jordan algebra.*

Proof. Let $A = Ke + U + Z$. From [3, Theorem 3] we know that $A$ is a Jordan algebra if and only if Eq. (7), Eq. (8′) and the following identities hold for all $u, u_1, u_2 \in U$ and $z, z_1, z_2 \in Z$.

$$(u_1^2 u_2) z + 2((u_1 z) u_2) u_1 = 0 \tag{10}$$

$$((u z_1) z_2) z_1 = 0 \tag{11}$$

$$(u_1^2 u_2) u_1 = 0 \tag{12}$$

$$((u z_1) z_2) u = 0. \tag{13}$$

We will show that $A$ is Jordan by showing that Eq. (10) through Eq. (13) hold. By Eq. (2), Eq. (4′), Eq. (5′) and Eq. (8′) we obtain

$$(u_1^2 u_2) z = -(u_2 z)(u_1 u_1) = 2((u_2 z) u_1) u_1 = -2((u_1 z) u_2) u_1 .$$

This proves Eq. (10). By Eq. (2), Eq. (8), and Eq. (8′) we get $((u z_1) z_2) z_1 = -((u z_1) z_1) z_2 = 0$. This proves Eq. (11). By Eq. (2), Eq. (4), and Eq. (5′) we have $(u_1^2 u_2) u_1 = -(u_1^2 u_1) u_2 = 0$. This is Eq. (12). Finally, Eq. (2), Eq. (5′) and Eq. (6″) imply that $((u z_1) z_2) u = -(u z_1)(u z_2) = 0$. This proves Eq. (13).

**Theorem 2.** *Let $A$ be a semiprime Bernstein algebra of characteristic $\neq 2$. If $A$ is finitely generated, then $A$ is a field.*

Proof. Let $A = Ke + U + Z$. Remember that $N = U + Z$. We will first prove that $N$ is a special Jordan algebra. The endomorphisms of $(N, +)$ form an associative algebra which we denote by $E$. We construct the Jordan algebra $E^{(+)}$ by defining in $E$ the new operation "$\circ$" by $f \circ g = fg + gf$, where $fg$ and $gf$ mean the usual composition of functions. If $a \in N$, denote by $R_a$ the right multiplication by $a$. That is, $x R_a = x a$. By Eq. (9′), $-x(ab) = (xa) b + (xb) a$ for any $x, a, b$ in $N$. This means that

$$-R_{ab} = (-R_a)(-R_b) + (-R_b)(-R_a) = (-R_a) \circ (-R_b).$$

Therefore the map from $N$ to $E^{(+)}$ which sends the element $a$ to $-R_a$ is a homomorphism of algebras. The kernel of this map is an ideal of $N$ which squares to zero. If we can show that it is also an ideal of $A$, then it must be zero because $A$ is semiprime. Showing that it is an ideal of $A$ requires that we show that it absorbs multiplication by the idempotent $e$. This follows, since whenever an element $n$ is in the kernel, then the summands $n = u + z$ with $u \in U$ and $z \in Z$ must also be in the kernel as well from Eq. (7) and Eq. (2) and the fact that the sum $U + Z$ is an additive direct sum. Since the kernel of this map is zero, $N$ is isomorphic to a subalgebra of $E^{(+)}$, and this means that $N$ is a special Jordan algebra.

Now, we assert that $N$ is nilpotent. We already know that $N$ is a special Jordan algebra. Eq. (9) says that $N$ is a nil algebra of bounded index. It is known that a special Jordan nil algebra of bounded index is locally nilpotent [5, p. 114]. Thus, $N$ is locally nilpotent. This means that every finitely generated subalgebra of $N$ is nilpotent. Since $A$ is finitely generated, let $\{a_i, i = 1, \ldots, n\}$ be a finite set of generators of $A$. Decompose each $a_i$ into $a_i = k_i e + u_i + z_i$ where $k_i \in K$, $u_i \in U$, and $z_i \in Z$. Then the $u_i$'s and the $z_i$'s generate $N$. Thus $N$ itself is finitely generated and is nilpotent.

Finally, let $I$ and $J$ be ideals of $A$ with the property that $I$ and $J$ are both contained in $N$. Eq. (9′) yields

$$(IJ)N \subseteqq (JN)I + (NI)J \subseteqq IJ.$$

Thus $IJ$ absorbs multiplication from $N$. We will now show that $IJ$ absorbs multiplication from $e$ as well. Let $x = u + z \in I$ and $y = u' + z' \in J$ where $u, u' \in U$ and $z, z' \in Z$. We have that $u = 2ex \in I$ by Eq. (3), and so also $z \in I$. Analogously, $u'$ and $z'$ are elements of $J$. By Eq. (2), Eq. (3) and Eq. (7) we have

$$(xy)e = (uu' + uz' + u'z)e = 1/2(uz' + u'z) \in IJ.$$

It follows that $(IJ)A = (IJ)(Ke + N) \subseteqq IJ$. Thus, $IJ$ is an ideal of $A$. Therefore, $N^{(0)} = N$, $N^{(1)} = NN, \ldots, N^{(n+1)} = N^{(n)}N^{(n)}, \ldots$ are ideals of $A$. Since $N$ is nilpotent, $N^{(r)} = 0$ for some $r$. This means that $N^{(r-1)}$ is an ideal which squares to zero. Since $A$ is semiprime it forces $N^{(r-1)} = 0$. Continuing with this process we end up with the fact that $N = 0$. This implies that $A = Ke$ and the result of the theorem that $A$ is a field follows.

**Theorem 3.** *Let $A = Ke + N$ be a Bernstein algebra where $N$ is the kernel of $w$. If $A$ is finitely generated, then $N$ is solvable. That is $N^{(k)} = 0$ for some positive integer $k$.*

Proof. Any Bernstein algebra $A$ can be decomposed as $A = Ke + N$ (additive direct sum) where $N$ is an ideal of $A$. Let $L = \{x \in U : xU = 0\}$ and $\mathrm{Ann} = \{x \in N : xN \subseteqq L\}$. As in the proof of Theorem 2, $N/L$ is a nil Jordan algebra of index three, and $\mathrm{Ann}/L$ is the kernel of the homomorphism of $N/L$ into $\mathrm{End}\,(N/L)^{(+)}$. When $A$ is finitely generated, $N/\mathrm{Ann}$ is a finitely generated special Jordan nil algebra of bounded index; thus $N/\mathrm{Ann}$ is nilpotent. It follows that for some positive integer $r$,

$$N^{(r+2)} \subseteqq \mathrm{Ann}^{(2)} \subseteqq L^{(1)} = 0.$$

Therefore, $N$ is solvable.

With slightly more work one can show that $N/L$ is nilpotent. However, $N$ itself may not be nilpotent as the following example shows.

Example. Let $C$ be any commutative algebra over a field $K$. The Cartesian product $K \times C \times C$ can be made into a Bernstein algebra by defining addition coordinatewise and defining multiplication by

$$(k_1, a_1, b_1)(k_2, a_2, b_2) = (k_1 k_2, \tfrac{1}{2} k_1 a_2 + \tfrac{1}{2} k_2 a_1 + a_1 b_2 + a_2 b_1 + b_1 b_2, 0).$$

The function $w$ is defined by: $w(k, a, b) = k$.

The idempotents of this algebra are elements of the form $(1, a, 0)$. For the idempotent $e = (1, 0, 0)$, $U = (0, C, 0)$, and $Z = (0, 0, C)$. Since $(0, c_1, 0)(0, 0, c_2) = (0, c_1 c_2, 0)$, in this new algebra $N$ is nilpotent if and only if $C$ is nilpotent.

For a specific example let $K = C =$ the integers mod 5. Now $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ generate the algebra, so the algebra is finitely generated. (It is even finite dimensional.) Since $(0, 1, 0)(0, 0, 1) = (0, 1, 0)$, right multiplication by $(0, 0, 1)$ leaves $(0, 1, 0)$ fixed. Therefore right multiplication by $(0, 0, 1)$ is not nilpotent, so $N$ is not nilpotent.

This example is not semiprime. $L = (0, C, 0)$ is a trivial ideal. This example is not Jordan since for $x = (0, 0, 1)$ and $e = (1, 0, 0)$, $(x^2 e)x - x^2(ex) = (0, \frac{1}{2}, 0)$. This example shows that if we drop completely, the semiprime hypothesis from Theorem 1, the conclusion of Theorem 1 does not follow. Our example is a Bernstein algebra which is not Jordan. Since our example was not Jordan, it certainly was not a field. It is finitely generated though, so the semiprime hypothesis cannot be removed from Theorem 2 either.

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Eingegangen am 9. 12. 1987

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