I

p. 1

$\boxed{1.1}$ definition Lie algebra (restrict to finite dimensional) over a field F

Jacobi identity

isomorphism

subalgebra

p. 2

$\boxed{1.2}$ End (V) V finite dim'l vector space, dimension n

dimension n^2

End (V) is an associative algebra

Notation: $gl(V) = \text{End}(V)^-$ general linear algebra

def. linear Lie algebra = a subalg of $gl(V)$

- $gl(V)$ is isomorphic to $M_n(F)^-$ (which is denoted $gl(n, F)$)

"Classical Lie algebras" $A_\ell, B_\ell, C_\ell, D_\ell$ $\ell \geq 1$

$\boxed{A_\ell}$: $\dim V = \ell+1$ $sl(V) = sl(\ell+1, F) = \{x \in \text{End}(V) : tr(x) = 0\}$

a subalgebra of $gl(V)$ of dimension $(\ell+1)^2 - 1$

TRACE: www.math.uci.edu/~brusso/meyberg40-46NOTESbis.pdf

(Math 199B W17)

(even)

$\boxed{C_\ell}$: $\dim V = 2\ell$ basis $(v_1, .., v_{2\ell})$

let $v = \sum_{i=1}^{2\ell} \alpha_i v_i$ $w = \sum_{i=1}^{2\ell} \beta_i v_i$ $\in V$

$(v, w) = \sum_{i=1}^{2\ell} \alpha_i \beta_i$, etc.

$v = v' + v''$ $w = w' + w''$

$= \sum_{i=1}^{\ell} \alpha_i v_i + \sum_{i=\ell+1}^{2\ell} \alpha_i v_i$

let f be the non-degenerate skew symmetric form defined by the matrix $S = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$

$$f(v,w) = \left(\begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix} \begin{pmatrix} v' \\ v'' \end{pmatrix}, \begin{pmatrix} w' \\ w'' \end{pmatrix} \right)$$

$$= \left(\begin{pmatrix} v'' \\ -v' \end{pmatrix}, \begin{pmatrix} w' \\ w'' \end{pmatrix} \right) = (v'', w') - (v', w'')$$

so $f(v,w) = -f(w,v)$

$sp(V) := \{ x \in \mathrm{End}(V) : \ f(x(v), w) = -f(v, x(w)) \}$

$sp(V)$ is a Lie subalgebra of $gl(V)$ $\left(:= sp(2\ell, F) \right)$

If $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$, $m, n, p, q \in gl(\ell, F)$

then $x \in sp(V) \iff Sx = -x^t S$

$S = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$

The dimension of $sp(2\ell, F)$ is $2\ell^2 + \ell$

$\boxed{B_\ell}$: $\dim V = 2\ell+1$ (odd) . Let f be the non-degenerate symmetric bilinear form on V defined by the matrix $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}$

$$o(V) = o(2\ell+1, F) := \{ x \in \mathrm{End}(V) : f(x(v), w) = -f(v, x(w)) \}$$

(same as C_ℓ)

$o(V)$ is a Lie subalgebra of $gl(V)$ of dimension $2\ell^2+\ell$

(same as C_ℓ)

$\boxed{D_\ell}$: (same construction as B_ℓ) $\dim V = 2\ell$

f is the bilinear form defined by $s = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$

$o(2\ell, F) = \{x \in \mathrm{End}(V) : f(x(v), w) = -f(v, x(w))\}$

is a Lie subalgebra of $gl(V)$ of dimension $2\ell^2 - \ell$
(exercise 8)

$\vec{t}(n,F) =$ upper triangular matrices $a_{ij} = 0 \quad i > j$

$\vec{n}(n,F) =$ strictly upper triangular matrices $a_{ij} = 0 \quad i \geq j$

$\vec{\delta}(n,F) =$ diagonal matrices $a_{ij} = 0 \quad i \neq j$

These are subalgebras of $gl(n,F)$ such that

$\vec{t}(n,F) = \vec{\delta}(n,F) \oplus \vec{n}(n,F)$ (vector space direct sum)

$[\vec{\delta}(n,F), \vec{n}(n,F)] = \vec{n}(n,F)$

$[\vec{t}(n,F), \vec{t}(n,F)] = \vec{n}(n,F)$

(exercise 5)

p. 4

$\boxed{1.3}$ **F-algebra** : a vector space A over F with a bilinear operation $A \times A \to A$, usually denoted ab but for Lie algebras $[a,b]$ or $[ab]$

derivation : is a linear map $\delta : A \to A$ with $\delta(ab) = a\,\delta(b) + \delta(a)\,b$

(Exercise 11) Der A is a Lie subalgebra of $gl(A)$.

ad x is the endomorphism $y \to [xy]$ in a Lie algebra L

ad x is in fact a derivation of L, called **inner**

adjoint representation : $x \to \operatorname{ad} x$ $\quad L \to \operatorname{Der}(L)$

is a homomorphism.

$$\operatorname{ad}(x+y) = \operatorname{ad} x + \operatorname{ad} y$$

$$[\operatorname{ad} x, \operatorname{ad} y] = \operatorname{ad}[x,y]$$

$\boxed{1.4}$ abelian Lie algebra $\quad [xy] = 0 \quad \forall x, y$

structure constants (depends on the bases)

$x_1, \dots, x_n$ basis $\quad [x_i x_j] = \sum_{k=1}^{n} a_{ij}^{k} x_k$

dimension 0,1 trivial

dimension 2 : basis x, y for L

$$[\alpha x + \beta y, \alpha' x + \beta' y] = \alpha\alpha' [xx] + \alpha\beta' [xy] + \beta\alpha' [yx] + \beta\beta' [yy]$$

$$(*) \qquad = (\alpha\beta' - \beta\alpha')[xy]$$

if $[xy] = 0$ L is Abelian (all products are zero)

if $[xy] \neq 0$
Consider the basis $x' = [x, y]$, y'

$$[x'y'] = [[xy], y'] = ax' \quad (\text{by } (*)) \qquad a \neq 0 \ (\text{otherwise Abelian})$$

Change basis to x', $a^{-1}y' = y''$

$$[x'y''] = a^{-1}[x'y'] = x'$$

Check : (1) Example Basis u, v with $[u, v] = u$ defines a Lie algebra of dimension 2 L_2

(2) Theorem Every Lie algebra of dimension 2 is either Abelian or isomorphic to L_2