L a vector space / F $\qquad [x,x]=0$

p. 1 $\quad [\cdot,\cdot] : L\times L \to L$ bilinear $\quad [x,[y,z]] = [[x,y],z] + [y,[x,z]]$

$\boxed{1.1}$ definition Lie algebra (restrict to finite dimensional) over a field F

Jacobi identity

isomorphism

subalgebra

p. 2

$\boxed{1.2}$ End (V) $\quad V$ finite dim'l vector space, dimension n

dimension n^2

End (V) is an associative algebra

Notation: $\quad gl(V) = \text{End}(V)^-$ $\quad [x,y] = xy - yx$ general linear algebra

<u>def.</u> <u>linear Lie algebra</u> = a subalg of $gl(V)$

- $gl(V)$ is isomorphic to $M_n(F)^-$ (which is denoted $gl(n,F)$)

"Classical Lie algebras" $\quad A_\ell, B_\ell, C_\ell, D_\ell \quad \ell \geq 1$

$\boxed{A_\ell}$: $\dim V = \ell+1$ $\quad \boxed{sl(V)} = sl(\ell+1, F) = \{x \in \text{End}(V) : tr(x) = 0\}$

a subalgebra of $gl(V)$ of dimension $(\ell+1)^2 - 1$

TRACE: www.math.uci.edu/~brusso/meyberg40-46NOTESbis.pdf

(Math 199B W17)

added 4/15/17

Since $sl(V) \neq gl(V)$, $\dim sl(V) \leq (\ell+1)^2 - 1$

$h_1 = e_{11} - e_{22}$, $h_2 = e_{22} - e_{23}$, ..., $h_\ell = e_{\ell\ell} - e_{\ell+1,\ell+1}$ $\in sl(V)$

$i \neq j$, $e_{ij} \in sl(V)$

$$\begin{matrix}1&0&0&0\\0&-1&0&0\\0&0&0&0\end{matrix}, \quad \begin{matrix}0&0&0&0\\0&1&0&0\\0&0&-1&0\end{matrix} \quad \begin{matrix}0&0&0&0\\0&0&0&0\\0&0&1&0\end{matrix}$$

lin. indep $(\ell+1)^2 - (\ell+1) + \ell$

(even)

$\boxed{C_\ell}$: $\dim V = 2\ell$ bases $(v_1, \ldots, v_{2\ell})$

let $v = \sum_{i=1}^{2\ell} \alpha_i v_i$ $\quad w = \sum_{i=1}^{2\ell} \beta_i v_i \in V$

$(v, w) = \sum_{i=1}^{2\ell} \alpha_i \beta_i$, etc.

$v = v' + v''$ $\quad w = w' + w''$

$= \sum_{i=1}^{\ell} \alpha_i v_i + \sum_{i=\ell+1}^{2\ell} \alpha_i v_i$

let f be the non-degenerate skew symmetric form defined by the matrix $S = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$

$$f(v,w) = \left(\begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix} \begin{pmatrix} v' \\ v'' \end{pmatrix}, \begin{pmatrix} w' \\ w'' \end{pmatrix} \right) = (S(v), w)$$

$$= \left(\begin{pmatrix} v'' \\ -v' \end{pmatrix}, \begin{pmatrix} w' \\ w'' \end{pmatrix} \right) = (v'', w') - (v', w'')$$

$sp(V) \subset sl(V \oplus V)$

so $f(v,w) = -f(w,v)$

$\boxed{sp(V)} := \{ x \in \mathrm{End}(V) : \; f(x(v), w) = -f(v, x(w)) \}$

$sp(V)$ is a Lie subalgebra of $gl(V)$ $\quad (:= sp(2\ell, F))$

If $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$, $m, n, p, q \in gl(\ell, F)$ $\qquad S = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$

then $x \in sp(V) \iff Sx = -x^t S$

The dimension of $sp(2\ell, F)$ is $2\ell^2 + \ell$

$\boxed{B_\ell}$: $\dim V = 2\ell+1$ (odd). Let f be the non-degenerate symmetric bilinear form on V defined by the matrix $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}$

$\boxed{\mathfrak{o}(V)} = \mathfrak{o}(2\ell+1, F) := \{ x \in \mathrm{End}(V) : f(x(v), w) = -f(v, x(w)) \}$
(same as C_ℓ)

$\mathfrak{o}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$ of dimension $2\ell^2+\ell$
(same as C_ℓ)

added 4/15/17 $\quad x = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$

$sx = -x^t s \iff a = 0,\ c_1 = -b_2^t \quad c_2 = -b_1^t \quad q = -m^t \quad n^t = -n$
$p^t = -p$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} \begin{array}{c} \begin{array}{ccc} 1 & \ell & \ell \end{array} \\ \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix} \end{array} \begin{array}{c} 1 \\ \ell \\ \ell \end{array} = \begin{matrix} a & b_1 & b_2 \\ c_2 & p & q \\ c_1 & m & m \end{matrix}$$

$$\begin{pmatrix} a & c_1^t & c_2^t \\ b_1^t & m^t & p^t \\ b_2^t & n^t & q^t \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix} = a$$

$\begin{pmatrix} & p & q\ m^t \\ & m & n \end{pmatrix}$

$a = 0$

contribution to dimension

a	0
b_1, c_2	ℓ
b_2, c_1	ℓ
n	$\frac{\ell^2-\ell}{2}$
p	$\frac{\ell^2-\ell}{2}$
m $q = m^t$	ℓ^2

$2\ell^2+\ell$

$\boxed{D_\ell}$: (same construction as B_ℓ) $\dim V = 2\ell$

f is the bilinear form defined by $s = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$

$$\mathcal{O}(2\ell, F) = \{ x \in \mathrm{End}(V) : f(x(v), w) = -f(v, x(w)) \}$$

also called $\mathcal{O}(V)$

is a Lie subalgebra of $gl(V)$ of dimension $2\ell^2 - \ell$
(exercise 8)

$\vec{t}(n,F)$ = upper triangular matrices $a_{ij} = 0 \quad i > j$

$\vec{n}(n,F)$ = strictly upper triangular matrices $a_{ij} = 0 \quad i \geq j$

$\vec{\delta}(n,F)$ = diagonal matrices $a_{ij} = 0 \quad i \neq j$

These are subalgebras of $gl(n,F)$ such that

$$\vec{t}(n,F) = \vec{\delta}(n,F) \oplus \vec{n}(n,F) \quad \text{(vector space direct sum)}$$

$$[\vec{\delta}(n,F), \vec{n}(n,F)] = \vec{n}(n,F)$$

$$[\vec{t}(n,F), \vec{t}(n,F)] = \vec{n}(n,F)$$

(exercise 5)

$\boxed{1.3}$ **F-algebra**: a vector space A over F with a bilinear operation $A \times A \to A$, usually denoted ab but for Lie algebras $[a,b]$ or $[ab]$

derivation: is a linear map $\delta : A \to A$ with $\delta(ab) = a\,\delta(b) + \delta(a)\,b$

(Exercise 11) Der A is a Lie subalgebra of $gl(A)$.

$\operatorname{ad} x$ is the endomorphism $y \to [xy]$ in a Lie algebra L

$\operatorname{ad} x$ is in fact a derivation of L, called **inner**

adjoint representation: $x \to \operatorname{ad} x \quad L \to \operatorname{Der}(L)$

is a homomorphism.

$\operatorname{ad}(x+y) = \operatorname{ad} x + \operatorname{ad} y$

$[\operatorname{ad} x, \operatorname{ad} y] = \operatorname{ad}[x,y]$

$\boxed{1.4}$ abelian Lie algebra $\quad [xy] = 0 \quad \forall x, y$

structure constants (depends on the bases)

$x_1, \dots, x_n$ bases $\quad [x_i x_j] = \sum_{k=1}^{n} a_{ij}^k x_k$

dimension 0,1 trivial

dimension 2 : basis x, y for L

$$[\alpha x+\beta y, \alpha' x+\beta' y] = \alpha\alpha'[xx] + \alpha\beta'[xy] + \beta\alpha'[yx] + \beta\beta'[yy]$$

(*) $$= (\alpha\beta' - \beta\alpha')[xy]$$

if $[xy]=0$ L is Abelian (all products are zero)

if $[xy]\neq 0$
Consider the basis $x'=[x,y]$, y'

$$[x'y'] = [[xy], y'] \underset{(\text{by } *)}{=} ax' \qquad a\neq 0$$
(otherwise Abelian)

Change basis to x', $a^{-1}y' = y''$

$$[x'y''] = a^{-1}[x'y'] = x'$$

Check : Example (1) Basis u, v with $[u,v]=u$ defines a Lie algebra of dimension 2 L_2

Theorem (2) Every Lie algebra of dimension 2 is either Abelian or isomorphic to L_2