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2.1

L a Lie algebra

ideal $I \subset L$

$$[I, L] \subset I$$

center of L $Z(L) = \{x \in L : [x, L] = 0\}$ is an ideal

$$L \text{ is abelian} \Leftrightarrow Z(L) = L$$

derived algebra $[L, L] = \text{Span} \{[x, y] : x, y \in L\} \subseteq L$

(it is an ideal)

$$L \text{ is abelian} \Leftrightarrow [L, L] = 0$$

The classical Lie algs A_n, B_n, C_n, D_n satisfy $[L, L] = L$.
(exercise 9 in Assignment 1)
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L is simple if $0, L$ are the only ideals, and $[L, L] \neq 0$

$$L \text{ is simple} \Rightarrow Z(L) = 0 \text{ and } L = [L, L]$$

Example $L = \mathfrak{sl}(2, F)$ is simple

fill in the proof (see p. ① of § 2 revisited)

P.7 If I is an ideal in L , L/I is a Lie algebra

$$[x+I, y+I] = [xy] + I$$

$$x+I + y+I = x+y + I$$

Some definitions for later use

If $K \subset L$ is a subspace, the normalizer of K in L is

$$N_L(K) = \{x \in L : [x, K] \subset K\}$$

$N_L(K)$ is a subalgebra of L , the largest subalgebra of L which contains K as an ideal (if K is a ^{sub-}algebra to begin with)

K is self-normalizing if $K = N_L(K)$

If $X \subset L$ is a subset, the centralizer of X in L

is $C_L(X) = \{x \in L : [x, X] = 0\}$. It is also

a subalgebra, and for example $C_L(L) = Z(L)$.

2.2

homomorphism

$$\phi : L \rightarrow L'$$

monomorphism

$$\text{Ker } \phi = 0$$

epimorphism

$$\text{Im } \phi = L'$$

isomorphism

Prop (a) $L/\text{Ker } \phi \cong \text{Im } \phi$

(b) $I \subset J \subset L \Rightarrow L/I/J/I \cong L/J$

(c) $(I+J)/J \cong I/I \cap J$

"homomorphism theorems"

Exercise 9 asks to prove these.

representation of L is a homomorphism $\phi: L \rightarrow \mathfrak{gl}(V)$
(V a vector space)

example: $\text{ad}: L \rightarrow \mathfrak{gl}(L)$. $\text{Ker ad} = Z(L)$

If L is simple, it is isomorphic to a linear Lie algebra.
since $Z(L) = 0$

2.3

$\text{Aut}(L)$ is the group of all automorphisms of L

example 1 If L is a linear Lie algebra $L \subset \mathfrak{gl}(V)$

and $g \in \mathfrak{gl}(V)$ is invertible

and if $gLg^{-1} = L$

($GL(V)$ denotes the group of invertible elements in $\mathfrak{gl}(V)$)

then $x \rightarrow gxg^{-1}$ is an automorphism of L .

(every $g \in GL(V)$ defines an automorphism of $\mathfrak{gl}(V)$ and of $\mathfrak{sl}(V)$.) (see Exercise 12)

see p. ② of § 2 revisited for more detail

example 2

Suppose $x \in L$ (a Lie alg) and suppose $\text{ad } x$ is nilpotent, i.e. $(\text{ad } x)^k = 0$ for some $k > 0$

Then $\exp(\text{ad } x) = I + \text{ad } x + \frac{(\text{ad } x)^2}{2!} + \dots + \frac{(\text{ad } x)^{k-1}}{(k-1)!}$

Proposition If δ is a nilpotent derivation, then $\exp \delta$ is an automorphism

(proof is on pp 8-9)
see pp ③-④ of § 2 revisited

fill in the proof of the Proposition (see pp ②-④ of §2 Revised)

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inner automorphism is $\exp(\text{ad } x)$ where $\text{ad } x$ is nilpotent

$\text{Int } L =$ the set of inner automorphisms is a normal subgroup of $\text{Aut } L$.

$$\phi(\text{ad } x) \phi^{-1} = \text{ad } \phi(x) \quad \text{if } \phi \in \text{Aut } L$$

(see p. ⑤ of §2 Revised)

example $L = \mathfrak{sl}(2, F)$ standard basis x, y, h

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{see Exercise 3 in Assign 1})$$

$\text{ad } x, \text{ad } y$ are nilpotent

see p. ⑤ of §2 Revised define $\sigma = \exp(\text{ad } x) \cdot \exp(\text{ad } -y) \cdot \exp(\text{ad } x)$

$\sigma(x) = -y, \sigma(y) = -x, \sigma(h) = -h$ (Exercise 10 on p. 10)

$$\sigma \circ \sigma = \text{Id}_L$$

see p. ⑥ of §2 Revised

$\exp x, \exp(-y) \in \text{SL}(2, F)$ ($:=$ the group of 2×2 matrices of determinant 1)

$S := \exp x \exp(-y) \exp x$ satisfies $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and ① $z \rightarrow S z S^{-1}$ is an automorphism of L

② $S z S^{-1} = \sigma(z)$

Proposition If L is a linear Lie algebra, $L \subset \mathfrak{gl}(V)$,

and $x \in L$ is nilpotent, then

see pp ①-⑤ of §2 Revised

$$(\exp x) y (\exp x)^{-1} = \exp(\text{ad } x)(y) \quad \forall y \in L$$

fill in the proof on page 9