§ 3 Revisited   humphreys 10-14 NOTES bis. pdf (1)

5/5/17

Notes on § 3 (p. 10)

$L^{(i)}$ is an ideal. $\quad i=1 \quad [L,[L,L]] \subset [L,L]$

let $x \in L$, $[y,z] \in [L,L]$

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \in [L,L]$$

assume induction hypothesis for $L^{(i)}$

$$\text{Then } [L, L^{(i+1)}] = [L,[L^{(i)},L^{(i)}]]$$

$$\subseteq [[L,L^{(i)}],L^{(i)}] + [L^{(i)},[L,L^{(i)}]]$$

$$\subseteq [L^{(i)},L^{(i)}] = L^{(i+1)}.$$

if L is simple $\quad L \supset L^{(1)} \supset L^{(2)} \cdots \supseteq L^{(k)}$

$[L,L] \neq 0$ so $[L,L]=L$ i.e. $L^{(1)}=L$

$$L^2 = [L^{(1)},L^{(1)}] = [L,L] = L = L^{(1)}$$

so $L^k = L$ for all k.

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basis for $L = \vec{t}(n,F)$. e_{ij} $\quad i \leq j$

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj} \quad \text{for all matrix units}$$

if $i<l$ $\quad [e_{ii}, e_{il}] = \delta_{ii} e_{il}$ $\quad$ so $\vec{n}(n,F) \subset [L,L]$

(step 1)

we know (obvious) $\vec{t}(n,F) = \vec{n}(n,F) + \underbrace{\vec{\delta}(n,F)}$

diagonal

$\therefore$ abelian

infact by Exercise 1.5

$L^{(1)} = [L,L] = \vec{n}(n,F)$ (step 2)

$\vec{n}$ has basis e_{ij} $i<j$

$$\begin{array}{c|c} & \begin{array}{ccc} \quad & j & \quad l \end{array} \\ \begin{array}{c} i \\ k \end{array} & \begin{bmatrix} 0 & \cdots & e_{ij} & & \\ & & & & e_{kl} \\ \cdots & & & \ddots & \\ & & & & 0 \end{bmatrix} \end{array}$$

let $i<j$, $k<l$

$$[e_{ij}, e_{kl}] = \begin{cases} e_{il} & \text{if } j=k \quad i\neq l \\ 0 & \quad\ \ j\neq k \quad i\neq l \\ 0 & \text{if } j=k \quad i=l \\ 0 & \text{if } j\neq k \quad i\neq l \end{cases}$$

This is 0 unless $j=k, i\neq l$

so $[L,L]$ is spanned by e_{ij}

$0 \; e_{12} \; e_{13} \; e_{14}$ level 3 $\quad 3=4-1$

$0 \; e_{23} \; e_{24}$ level 2

$0 \; e_{34}$

$0 \quad 0$

level 1

let $e_{\alpha\beta}$ have "level" $\beta-\alpha$

$$2 \le \beta \le n$$
$$n-1 \ge \alpha \ge 1$$
$$1-n \le -\alpha \le -1$$
$$2-n \le \beta-\alpha \le n-1$$

$\cancel{j-i+l-k}$

For $i<l$,

$e_{il} = [e_{ij}, e_{jl}]$ $\qquad [e_{12}, e_{13}]$

level $\qquad$ levels

$l-i$ $\qquad \underbrace{j-i,\ l-j}_{\text{add to } l-i}$

Thus $L^{(2)}$ is spanned by $\{ [e_{ij}, e_{kl}] : i<j,\ k<l \}$

~~elements of level ≥ 2~~

~~for suppose~~ i.e. by e_{il} where

level of e_{il} = level of $[e_{ij}, e_{kl}]$

suppose $l-i=1$

$$1 = j-i + l-j$$

$L^{(2)}$ is spanned by $[e_{ij}, e_{jl}]$

which has level $\underbrace{(j-i)+(l-j)}_{= \, l-i} \geq 2$

$$L^{(2)} \subset \begin{bmatrix} 0 & 0 & & & * \\ & 0 & 0 & & \\ & & \ddots & & \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix}$$

~~e_{ij}, e_{jk}~~ $L^{(3)} = [L^{(2)}, L^{(2)}]$

$$= sp\{ [e_{ij}, e_{kl}] : \begin{array}{l} j-i \geq 2 \\ l-k \geq 2 \end{array}$$

$$= sp\{ [e_{ij}, e_{jl}] \quad \begin{array}{l} j-i \geq 2 \\ l-j \geq 2 \end{array}$$

$$\overset{\shortparallel}{e_{il}} \qquad \underset{l-i}{\cancel{\ }} \geq 2+2 = 4 = 2^2$$

$$L^{(4)} = [L^{(3)}, L^{(3)}] = sp\,[e_{ij}, e_{jl}] \qquad j-i \geq 4, \quad l-j \geq 4$$

$$\qquad\qquad\qquad\qquad\qquad\quad \overset{\shortparallel}{e_{il}} \qquad l-i \geq 8 = 2^3$$

$$L^{(i+1)} = [L^{(i)}, L^{(i)}] = sp\,[e_{ij}, e_{jl}] \qquad j-i \geq 2^{i-1}, \quad l-j \geq 2^{i-1}$$

$$l-i \geq 2 \cdot 2^{i-1} = 2^i$$

but $l - i \leq n-1$

so $n-1 \geq l-i \geq 2^i$

Suppose $2^i > n-1$. so $L^{(i+1)} = 0$.

No more levels left.

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p. 11 A simple Lie algebra is semi-simple

That is, a simple Lie algebra has no solvable ideals except 0. Suppose $I \neq 0$, $I \subset L$ is a solvable ideal. Then $I = L$ is solvable so $\text{Rad}\, L = L$ But L is semi-simple so $\text{Rad}\, L = 0$

$L/\text{Rad}\, L$ is semi-simple.

proof. Let $J/\text{Rad}\, L$ be a solvable ideal in $L/\text{Rad}\, L$

$\text{Rad}\, L \subset J \subset L$

(see p. 6 of www.math.uci.edu/~brusso/solvableradical.pdf)

J is an ideal of L $\text{Rad}\, L$ is an ideal in J

We need to prove $J/\text{Rad}\, L = 0$.

Now $J/\text{Rad}\, L$ is solvable & $\text{Rad}\, L$ is solvable

So by Prop. 3.1 J is solvable in L so $J \subset \text{Rad}\, L$

and $J/\text{Rad}\, L = 0$.

§ 3.2

We need to show that $L = \vec{t}(n,F)$ is not nilpotent. We know that $L^1 = L^{(1)} = \vec{n}(n,F)$

and $L^2 = [L, L^1] = [\vec{t}(n,F), \vec{n}(n,F)] \overset{(?)}{=} \vec{n}(n,F) = L^1$

~~$\vec{n}(n,F) = [\vec{t}(n,F), \vec{t}(n,F)]$ (Exercise 1.5 the~~

$\vec{n}(n,F)$ is an ideal so $[\vec{t}(n,F), \vec{n}(n,F)] \subset \vec{n}(n,F)$.

By exercise 1.9 $2e_{ij} = [e_{ii} - e_{jj}, e_{ij}]$

implies $\vec{n}(n,F) \subseteq [\vec{t}(n,F), \vec{n}(n,F)]$

proving (?)

Thus $L^i = L^1$ for all $i \geq 1$ is never 0, so

$L = \vec{t}(n,F)$ is not nilpotent.

Let $M = \vec{n}(n,F)$. Then M is nilpotent.

Similar to the proof that $\vec{t}(n,F)$ is solvable

$M^1 = sp\{e_{ij} : j - i \geq 2\}$

$M^2 = sp\{e_{ij} : j - i \geq 3\}$

$\vdots$

$M^k = sp\{e_{ij} : j - i \geq k+1\}$

But $j - i$ is always $\leq n-1$

$\boxtimes$

Proposition 3.2 (a) similar to Prop 3.1(a)

(b) $L/Z(L)$ nilpotent $\overset{?}{\Longrightarrow}$ L is nilpotent

~~just need to show $Z(L)$ is nilpotent~~

~~let $Z = Z(L)$~~

~~$Z^1 = [Z, Z] = 0 \quad Z^2 = [Z, Z^1] = 0$~~

let $M = L/Z(L)$

$$M^1 = [M, M] = sp\{[x+Z, y+Z] : x, y \in L\}$$

$$= sp\{[x,y] + Z \ : x, y \in L\}$$

$$= [L,L]/Z = L^1/Z$$

$$M^2 = [M, M^1] = \left[L/Z(L), \ L^1/Z(L)\right]$$

$$= L^2/Z(L)$$

$M^n = L^n/Z(L)$ so if $M^n = 0$ $L^n \subset Z(L)$

and $L^{n+1} = [L, L^n] \subseteq [L, Z(L)] = 0.$

(b) is proved

(c) L nilpotent, $L \neq 0 \Rightarrow Z(L) \neq 0$

$$L = L^0 \supset L^1 \supset \cdots \supset L^k \supset L^{k+1} = 0$$

$$0 = L^{k+1} = [L, L^k] \Rightarrow \underset{\underset{0}{\neq}}{L^k} \subset Z(L)$$

$\therefore Z(L) \neq 0$

Correction to p. 5 of 199C: Solvable and nilpotent
Third Meeting (Week 4) April 26, 2017 Humphreys 10–14)
Lie algebras (informal notes; Humphreys 10–14)
(www.math.uci.edu/~brusso/humphreys10-14.pdf)

I stated incorrectly that for nilpotent Lie algebra you can consult 199A

That is not so. The definition of nilpotent for an algebra given there ($a^n = 0$ for some n) is not the same as the definition of nilpotent for Lie algebras (for some n and $x_1, \dots, x_n, y \in L$

$$[x_1, [x_2, [x_3, \cdots [x_{n-1}, [x_n, y]] \cdots] = 0$$

i.e. $\operatorname{ad} x_1 \operatorname{ad} x_2 \cdots \operatorname{ad} x_n = 0$

in particular $(\operatorname{ad} x)^n = 0$ for some n.

Engel's theorem and the following lemma can be used to prove that $\vec{n}(n, F)$ is nilpotent.

<u>Lemma</u> $x \in gl(V)$ nilpotent $\Rightarrow$ $\operatorname{ad} x \in \operatorname{End}(gl(V))$ is nilpotent.

<u>Proof of Lemma</u> suppose $x^n = 0$

$$\operatorname{ad} x\,(y) = xy - yx$$

$$(\operatorname{ad} x)^2(y) = x(xy - yx) - (xy - yx)x$$

$$= x^2y - xyx - xyx + yx^2$$

$$(\operatorname{ad} x)^3(y) = x(x^2y - 2xyx + yx^2) - (x^2y - 2xyx + yx^2)x$$

$$= x^3y - 2x^2yx + xyx^2 - x^2yx + 2xyx^2 - yx^3$$

eventually $(\operatorname{ad} x)^k(y) = 0 \quad \forall y.$ ∎

<u>Proof that $\vec{n}(n, F)$ is nilpotent as a Lie algebra</u>

We already saw that each $x \in \vec{n}(n, F)$ is a nilpotent matrix and hence ad nilpotent by the lemma. Thus by Engel's theorem $\vec{n}(n, F)$ is a nilpotent Lie algebra.

Proof of Theorem 3.3 Let L be a Lie subalgebra of $gl(V)$, V of finite positive dimension, and suppose each element of L is a nilpotent endomorphism.

Then $\exists\ v \neq 0, v \in V$ such that $L.v = 0$, i.e. $x(v) = 0\ \forall x \in L$.

Proof starts here: Suppose $\dim L = 1$

$L = sp\{x\}$, $x^n = 0$, $x^{n-1} \neq 0$

Take a $v \neq 0$ with $x^{n-1}(v) \neq 0$

Then $x^{n-1}(v)$ is an eigenvector for x and hence every element of L, with eigenvalue 0.

Now use induction on $\dim L$, as follows.

Let $K \neq L$ be a subalgebra. For $x \in K$

$\operatorname{ad} x : L \to L$ is a nilpotent endomorphism (by the lemma)

Consider $T_x =$ "$\operatorname{ad} x / K$" $: L/K \to L/K$ (just a vector space)

$y + K \to [x,y] + K = \operatorname{ad} x\,(y) + K$ (well defined since $x \in K$)

T_x is also a nilpotent endomorphism

$\tilde{L} := \{ \, ad\,x/_K : x \in K \}$ is a Lie algebra of nilpotent endomorphisms

on L/K $\qquad \dim \cancel{ad_K} \tilde{L} \underset{\leq \dim ad(K)}{\leq} \dim K < \dim L$

So by induction $\exists$ $z + K \neq K$ $\quad z \in L$

such that $ad\,x/_K \, (z+K) = 0 \quad \forall\, x \in K$

$\qquad \qquad \| $

$\quad [x, z] + K$

i.e. $[x,z] \in K \quad \forall\, x \in K \qquad$ i.e. $z \in N_L(K)$

But $z \notin K \qquad$ But $K \subset N_L(K)$

so $K \subsetneqq N_L(K) \qquad$ Hold that thought.

~~Suppose~~ $\exists$ K, a maximal proper subalgebra of L

(existence? You can use Zorn's lemma for this)

Let $\mathcal{S} = \{$all subalgebras $K \neq L\}$ let $\mathcal{C}$ be a chain in $\mathcal{S}$.

$\mathcal{C} = \{K_\alpha\}$ $\quad$ let $K_0 = \bigcup_\alpha K_\alpha$ $\quad$ K_0 is a subalgebra

and $K_0 \neq L$ (assume $K_0 = L$ show that a basis for L must lie in ~~K_0~~ some $K_\alpha \neq L$)

K_0 is an upper bound in $\mathcal{S}$ for $\mathcal{C}$ so Zorn's lemma applies

Since K is maximal and $K \subsetneq N_L(K)$ (an algebra!)

it must be that $N_L(K) = L$, i.e. K is an ideal.

and L/K is a Lie algebra.

<u>Claim</u> L/K has dimension $\underline{1}$

$\pi(x) = x + K$

consider $L \xrightarrow{\pi} L/K$ and suppose $\dim L/K > 1$

Let S be a 1-dimensional ~~subspace~~ subalgebra of L/K.

say $S = sp\{x_0 + K\} \subset L/K$

$\pi^{-1}(S) = \{ y \in L : y + K \in S \}$

$= \{ y \in L : y + K = \lambda^{(y)}(x_0 + K) = \lambda^{(y)} x_0 + K$ for some $\lambda(y) \in F \}$

$y - \lambda x_0 \in K$

$\pi^{-1}(S)$ is a proper subalgebra containing K

$$\boxed{\begin{array}{l} \text{if } y_1, y_2 \in \pi^{-1}(S) \\ y_1 + K,\ y_2 + K \in S \\ [y_1, y_2] + K \in S \end{array}}$$

$K \subset \pi^{-1}(S)$ is clear

~~proper~~ ~~suppose K' subalgebra~~

~~$\pi^{-1}(S) \subset K' \subsetneq L$~~

~~$\pi(\pi^{-1}(S)) \subset \pi(K') \subsetneq L/K$~~

proper, since if $\pi^{-1}(S) = L$

then $S = \pi(\pi^{-1}(S)) = \pi(L) = L/K$

So K has codimension one : $L = K + Fz$ ~~[illegible]~~
$z \in L - K$

Since $\dim K < \dim L$, by induction
$W = \{v \in V : K(v) = 0\} \neq 0$.

let $x \in L$, $y \in K$, $w \in W$. Then

$$yx(w) = (xy - [x,y])w = \underbrace{xy(w)}_{\substack{\shortparallel \\ 0}} - \underbrace{[x,y]}_{\in K}(w) = 0$$

i.e. $L(W) \in W$

z is a nilpotent automorphism and $z : W \to W$
Thus z has an eigenvector $v \neq 0$ in W
corresponding to eigenvalue 0 $\quad z(v) = 0$
Finally if $x \in L$, $x = y + \lambda z$ $\quad y \in K$, $\lambda \in F$

and $$x(v) = \underbrace{y(v)}_{\substack{\shortparallel \\ 0}} + \lambda \underbrace{z(v)}_{\substack{\shortparallel \\ 0}} = 0.$$

This proves Theorem 3.3

We now prove Engel's Theorem

Theorem: If all elements of a Lie algebra L are ad-nilpotent, then L is a nilpotent Lie algebra.

<u>Proof</u> Let L be a Lie alg. with all elements ad-nilpotent. The Lie algebra $\operatorname{ad} L \subset \operatorname{End}(L)$ consists only of nilpotent endomorphisms. By Theorem 3.3 $\exists$ a "vector" $x \in L$, $x \neq 0$ and $[x, L] = 0$. Thus $Z(L) \neq 0$

$L/Z(L)$ has dimension $<$ dim L and consists of ad-nilpotent elements. By induction on dim L, $L/Z(L)$ is nilpotent.

By Prop 3.2 (b) L is nilpotent. $\blacksquare$