

Notes on § 3 (p.10)

5/5/17

$L^{(i)}$ is an ideal. $i=1 \quad [L, [L, L]] \subset [L, L]$

Let $x \in L, [y, z] \in [L, L]$

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \in [L, L]$$

assume induction hypothesis for $L^{(i)}$

$$\begin{aligned} \text{Then } [L, L^{(i+1)}] &= [L, [L^{(i)}, L^{(i)}]] \\ &= [[L, L^{(i)}], L^{(i)}] + [L^{(i)}, [L, L^{(i)}]] \\ &\subseteq [L^{(i)}, L^{(i)}] = L^{(i+1)}. \end{aligned}$$

if L is simple $L \supset L^{(1)} \supset L^{(2)} \dots \supset L^{(R)}$
 $[L, L] \neq 0$ so $[L, L] = L$ i.e. $L^{(1)} = L$

$$L^2 = [L^{(1)}, L^{(1)}] = [L, L] = L = L^{(1)}$$

so $L^k = L$ for all k .

basis for $\mathbb{M}^{(n, F)}$ e.g. $i \leq j$

$$[e_j, e_{kl}] = \delta_{jk} e_{il} - \delta_{kl} e_{kj} \quad \text{for all matrix units}$$

$$\text{find } [e_u, e_v] = \delta_{uv} e_{ll} \quad \text{so } \mathbb{M}^{(n, F)} \subset [L, L] \quad (\text{step 1})$$

(2)

we know (obvious) $\vec{t}(n, F) = \vec{n}(n, F) + \underbrace{\delta^D(n, F)}_{\text{diagonal}} \circ \text{abelian}$

in fact by Exercise 1.5

$$[L] \subset [L, L] = \vec{n}(n, F) \quad (\text{step 2})$$

\vec{n} has basis $e_j \quad i < j$

set $i < j, k < l$

$$[e_i, e_k] = \begin{cases} e_{il} & \text{if } j=k \\ 0 & \text{if } i \neq l \\ 0 & \text{if } j \neq k \\ 0 & \text{if } i=l \end{cases}$$

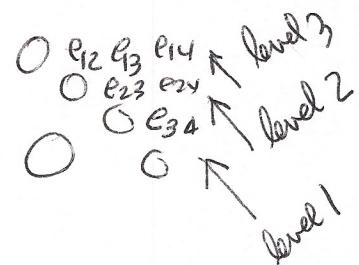
This is
0 unless
 $j=k, i \neq l$

so $[L, L]$ is spanned

by e_{ij}

let $e_{\beta\alpha}$ have "level" $\beta - \alpha$

$$\begin{bmatrix} 0 & \cdots & e_j & \cdots & e_{kl} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_{j=1, l=n}$$



$$\begin{aligned} & \text{if } 2 \leq \beta \leq n \\ & n-1 \geq \alpha \geq 1 \\ & 1-n \leq -\alpha \leq -1 \\ & 3-n \leq \beta - \alpha \leq n-1 \end{aligned}$$

~~$e_{ij} \neq 0$~~

For $i < l$,
 $e_{il} = [e_j, e_{jl}]$

$$[e_{12}, e_{13}]$$

levels
 i
 $l-i$
 $j-i$ $l-j$
add to $l-i$

(3)

Thus $L^{(2)}$ is spanned by $\{[e_{ij}, e_{k\ell}] : i < j, k < \ell\}$
elements of level 2

i.e. by e_{il} where
~~for suppose~~

level of e_{il} = level of $[e_{ij}, e_{k\ell}]$

suppose $l-i = 1$

$$1 = j-i + l-j$$

$L^{(2)}$ is spanned by $[e_{ij}, e_{jl}]$

which has level $\underbrace{(j-i) + (l-j)}_{''} \geq 2$

$$L^{(2)} \subset \begin{bmatrix} 0 & 0 & & \\ 0 & 0 & \ddots & * \\ \vdots & & & \\ 0 & 0 & & 0 \end{bmatrix}^{l-i}$$

~~L~~ $L^{(3)} = [L^{(2)}, L^{(2)}]$

$$= \text{sp}\{[e_{ij}, e_{k\ell}] : j-i \geq 2, l-k \geq 2\}$$

$$= \text{sp}\{[e_{ij}, e_{k\ell}] : \underbrace{\substack{j-i \geq 2 \\ l-k \geq 2}}_{e_{il}}\}$$

$$\cancel{j-i} \geq 2 + 2 = 4 \Rightarrow 2^2$$

(4)

$$L^{(4)} = [L^{(3)}, L^{(3)}] = \text{sp} [e_1, e_{j-e}]$$

e.i.e

$j-i \geq 4$
 $i-j \geq 4$
 $i-i \geq 8 = 2^3$

$$L^{(i+1)} = [L^{(i)}, L^{(i)}] = \text{sp} [e_1, e_{j-e}]$$

$j-i \geq 2^{i-1}$
 $i-j \geq 2^{i-1}$
 $i-i \geq 2 \cdot 2^{i-1} = 2^i$

but $i - i \leq n - 1$

$$\text{so } n-1 \geq i - i \geq 2^i$$

Suppose $2^i > n - 1$.
 No more levels left. so $L^{(i+1)} = 0$.

(5)

p. 11 A simple Lie algebra is semi-simple

That is, a simple Lie algebra has no solvable ideals except 0 .

\textcircled{a} Suppose $I \neq 0$,

$I \subset L$ is a solvable ideal. Then $I = L$

is solvable so $\text{Rad } L = L$ But L is semi-simple
so $\text{Rad } L = 0$

$L/\text{Rad } L$ is semi-simple.

proof:

let $J/\text{Rad } L$ be a solvable ideal in $L/\text{Rad } L$
 $\text{Rad } L \subset J \subset L$

(see p. ⑥ of www.math.uci.edu/~brusso/solvableradical.pdf)

J is an ideal of L $\text{Rad } L$ is an ideal in J

We need to prove $J/\text{Rad } L = 0$.

Now $J/\text{Rad } L$ is solvable & $\text{Rad } L$ is solvable

So by Prop. 3.1 J is solvable in L so $J \subset \text{Rad } L$

and $J/\text{Rad } L = 0$.

(6)

§ 3.2

We need to show that ~~$L = \vec{t}(n, F)$~~ is not nilpotent. we know that $L^1 = L^{(1)} = \vec{n}(n, F)$ and $L^2 = [L, L^1] = [\vec{t}(n, F), \vec{n}(n, F)] \stackrel{(7)}{\doteq} \vec{n}(n, F) = L^1$

$\vec{n}(n, F) \subseteq [\vec{t}(n, F), \vec{t}(n, F)]$ *(Exercise 1.5.1)*

$\vec{n}(n, F)$ is an ideal so $[\vec{t}(n, F), \vec{n}(n, F)] \subset \vec{n}(n, F)$.

By exercise 1.9 $2e_{ij} = [e_{ii} - e_{jj}, e_{ij}]$

implies $\vec{n}(n, F) \subseteq [\vec{t}(n, F), \vec{n}(n, F)]$

proving (?)

Thus $L^i = L^1$ for all $i \geq 1$ is never 0, so $L = \vec{t}(n, F)$ is not nilpotent.

Let $M = \vec{n}(n, F)$. Then M is nilpotent.

Similar to the proof that $\vec{t}(n, F)$ is solvable

$$M^1 = \text{sp}\{e_{ij} : j-i \geq 2\}$$

$$M^2 = \text{sp}\{e_{ij} : j-i \geq 3\}$$

$$M^k = \text{sp}\{e_{ij} : j-i \geq k+1\}$$

But $j-i$ is always $\leq n-1$



(7)

Proposition 3.2 (a) similar to Prop 3.1(a)

(b) $L/Z(L)$ nilpotent $\xrightarrow{?} L$ is nilpotent

just need to show $Z(L)$ is nilpotent

let $Z = Z(L)$

$$\underline{Z' = [Z, Z] = 0 \quad Z^2 = [Z, Z'] = 0}$$

let $M = L/Z(L)$

$$\begin{aligned} M' &= [M, M] = \text{sp } \{ [x+Z, y+Z] : x, y \in L \} \\ &= \text{sp } \{ [x, y] + Z : x, y \in L \} \\ &= [L, L]/Z = L'/Z \end{aligned}$$

$$M^2 = [M, M'] = \left[\frac{L}{Z(L)}, \frac{L'}{Z(L)} \right]$$

$$= L^2/Z(L)$$

$$M^n = L^n/Z(L) \quad \text{so if } M^n = 0 \quad L^n \subset Z(L)$$

$$\text{and } L^{n+1} = [L, L^n] \subseteq [L, Z(L)] = 0.$$

(b) is proved

(c) L nilpotent, $L \neq 0 \Rightarrow Z(L) \neq 0$

$$L = L^0 \supset L^1 \supset \dots \supset L^k \supset L^{k+1} = 0$$

$$0 = L^{k+1} = [L, L^k] \Rightarrow \begin{matrix} L^k \subset Z(L) \\ \# \\ 0 \end{matrix}$$

$$\therefore Z(L) \neq 0$$

Correction to p. (5) of

Third Meeting (Week 4) April 26, 2017
Lie algebras (informal notes: Humphreys 10-14)
(www.math.uci.edu/~brusso/Humphreys 10-14.pdf)

I stated incorrectly that for nilpotent Lie algebra you can consult

199C:

Solvable and nilpotent

199A

That is not so. The definition of nilpotent

$(a^n = 0 \text{ for some } n)$

for an algebra given there is not the same as the definition of nilpotent for Lie algebras (for some n and $x_1, \dots, x_n, y \in L$)

$$[x_1, [x_2, [x_3, \dots, [x_{n-1}, [x_n, y]] \dots] = 0$$

$$\text{i.e. } \text{ad } x_1 \text{ ad } x_2 \dots \text{ ad } x_n = 0$$

In particular $(\text{ad } x)^n = 0$ for some n .

Engel's theorem and the following lemma can be used to prove that $\tilde{n}^*(n, F)$ is nilpotent.

Lemma $x \in \text{gl}(V)$ nilpotent $\Rightarrow \text{ad } x \in \text{End}(\text{gl}(V))$ is nilpotent.

Proof of Lemma Suppose $x^n = 0$

$$\text{ad } x(y) = xy - yx$$

$$(\text{ad } x)^2(y) = x(xy - yx) - (xy - yx)x$$

$$= x^2y - xyx - xyx + yx^2$$

$$\begin{aligned} (\text{ad } x)^3(y) &= x(x^2y - 2xyx + yx^2) - (x^2y - 2xyx + yx^2)x \\ &= x^3y - 2x^2yx + xyx^2 - x^2yx + 2xyx^2 - yx^3 \end{aligned}$$

eventually $(\text{ad } x)^R(y) = 0 \quad \forall y. \quad \square$

Proof that $\tilde{n}^*(n, F)$ is nilpotent as a Lie algebra

We already saw that each $x \in \tilde{n}^*(n, F)$ is a nilpotent matrix and hence ad nilpotent by the lemma. Thus by Engel's theorem $\tilde{n}^*(n, F)$ is a nilpotent Lie algebra.

(10)

Proof of Theorem 3.3 let L be a Lie subalgebra of

$gl(V)$, V of finite positive dimension, and suppose

each element of L is a nilpotent endomorphism

Then $\exists v \neq 0, v \in V$ such that $L.v = 0$, i.e.

$$x(v) = 0 \quad \forall x \in L.$$

Proof starts here: Suppose $\dim L = 1$

$$L = \text{sp}\{x\}, \quad x^n = 0, \quad x^{n-1} \neq 0$$

Take a $v \neq 0$ with $x^{n-1}(v) \neq 0$

Then $x^{n-1}(v)$ is an eigenvector for every element of L , with eigenvalue 0. x and hence

Now use induction on $\dim L$, ~~as follows~~ as follows.

Let $K \neq L$ be a subalgebra. For $x \in K$

$\text{ad } x : L \rightarrow L$ is a nilpotent endomorphism (by the lemma)

Consider $T_x = \frac{\text{ad } x}{K} : L_K \rightarrow L_K$ just a vector space

$$y + K \rightarrow [x, y] + K \quad (\text{well defined since } x \in K)$$

$$\cancel{[x, y] + K} = \text{ad } x(y) + K$$

T_x is also a nilpotent endomorphism of ~~L_K~~

~~$\text{ad } (x+K)$~~

$\tilde{L} = \{ \frac{\text{ad } x}{K} : x \in K \}$ is a Lie algebra of endomorphisms
on L/K

$$\dim \tilde{L} \leq \dim L \quad (\leq \dim \text{ad}(K))$$

So by induction $\exists z+K \neq K \quad z \in L$

such that $\frac{\text{ad } x}{K} (z+K) = 0 \quad \forall x \in K$
"

$$[x, z] + K$$

i.e. $[x, z] \in K \quad \forall x \in K \quad$ i.e. $z \in N_L(K)$

But $z \notin K$ But $K \subset N_L(K)$

so $K \subsetneq N_L(K)$ Hold that thought.

~~Suppose~~ $\exists K$, a maximal proper subalgebra of L

(existence? You can use Zorn's lemma for this)

let $S = \{ \text{all subalgebras } K \neq L \}$ let C be a chain in S .

$C = \{K_\alpha\}$ let $K_0 = \bigcup K_\alpha$ K_0 is a subalgebra

and $K_0 \neq L$ (assuming $K_0 = L$ show that a basis
for L must lie in ~~K_α~~ some $K_\alpha \neq L$)

K_0 is an upper bound in S for C so Zorn's lemma applies

Since K is maximal and $K \not\subseteq N_L(K)$ (an algebra!)

it must be that $N_L(K) = L$, i.e. K is an ideal.

and L/K is a Lie algebra.

Claim L/K has dimension 1

$$\pi(x) = x + K$$

consider $L \xrightarrow{\pi} L/K$ and suppose $\dim L/K > 1$

(a) let S be a 1-dimensional ~~subspace~~ of L/K
subalgebra

say $S = \text{sp}\{x_0 + K\} \subset L/K$

$$\pi^{-1}(S) = \{y \in L : y + K \in S\}$$

$$\{y \in L : y + K = \lambda^{(y)}(x_0 + K) = \lambda^{(y)}x_0 + K \text{ for some } \lambda^{(y)} \in F\}$$

$$y \in \lambda^{(y)}x_0 + K$$

$\pi^{-1}(S)$ is a proper subalgebra containing K

$K \subset \pi^{-1}(S)$ is clear

~~proper~~ suppose K' subalgebra

$$\pi^{-1}(S) \subset K' \subset L$$

$$\pi(\pi^{-1}(S)) \subset \pi(K') \subset L/K$$

$$\left. \begin{array}{l} \text{if } y_1, y_2 \in \pi^{-1}(S) \\ y_1 + K, y_2 + K \in S \\ [y_1, y_2] + K \in S \end{array} \right\}$$

proper, since $\pi^{-1}(S) = L$

$$\text{then } S = \pi(\pi^{-1}(S)) = \pi(L) = L/K$$

So K has codimension one: $L = K + Fz$ ~~$\{z\}$~~
 $z \in L - K$

Since $\dim K < \dim L$, by induction

$$W = \{v \in V : K(v) = 0\} \neq 0.$$

Let $x \in L$, $y \in K$, $w \in W$. Then

$$yx(w) = \cancel{(xy - [x,y])w} = \underbrace{xy}_{\parallel}^{(w)} - \underbrace{[x,y]w}_{\in K} = 0$$

i.e. $L(w) \in W$

z is a nilpotent automorphism and $z: W \rightarrow W$

Thus z has an eigenvector $v \neq 0$ in W

corresponding to eigenvalue 0

$$z(v) = 0$$

Finally if $x \in L$, $x = y + \lambda z$ $y \in K$, $\lambda \in F$

$$\text{and } x(v) = \underbrace{y(v)}_{\parallel} + \lambda \underbrace{z(v)}_{\parallel} = 0.$$

This proves Theorem 3.3

We now prove Engel's Theorem

Theorem: If all elements of a

lie algebra L

are ad-nilpotent, then L is a nilpotent lie algebra.

Proof Let L be a Lie alg. with all elements ad-nilpotent. The Lie algebra $\text{ad } L \subset \text{End}(L)$ consists only of nilpotent endomorphisms. By

Theorem 3.3 \exists a "vector" $x \in L$, $x \neq 0$
and $[x, L] = 0$. Thus $Z(L) \neq 0$

$L/Z(L)$ has dimension $< \dim L$ and
consists of ad-nilpotent elements. By
induction on $\dim L$, $L/Z(L)$ is nilpotent.

By Prop 3.2 (b) L is nilpotent. 

Corollary Under the hypotheses of Theorem 3.3 : L

is a subalgebra of $gl(V)$ consisting only of nilpotent endomorphisms; there exists a chain of subspaces $0 = V_0 \subset V_1 \subset \dots \subset V_n = V$ with $\dim V_i = i$ and $x(V_i) \subset V_{i-1}$.

Proof. Th 3.3 tells us that $\exists v \neq 0, v \in V \quad x(v) = 0$

$\forall x \in L$. Let $V_1 = Fx$ and $W = V/V_1$. $\dim W = n-1$

For $x \in L$, $\tilde{x} \in \text{End}(W) \quad \tilde{x}(a + V_1) = x(a) + V_1$

well defined since $a \in V_1 \Rightarrow x(a) = 0 \in V_1$

$\tilde{L} = \{\tilde{x} : x \in L\} \subset gl(W)$ is a Lie subalgebra.

$$\begin{aligned} [\tilde{x}, \tilde{y}] (a + V_1) &= (\tilde{x}\tilde{y} - \tilde{y}\tilde{x})(a + V_1) \\ &= (xy - yx)(a) + V_1 = \underline{[\tilde{x}, \tilde{y}]} (a + V_1) \end{aligned}$$

and $\tilde{x}^k = (x^k)^\sim = 0$ for some k depending on x .

(induction)
So \exists subspaces $W_0 = W_0 \subset W_1 \subset \dots \subset W_{n-1} = V/V_1$

$$\dim W_i = i \quad W_i = V_{i+1}/V_1 \quad \dim V_{i+1} = i+1$$

$$\tilde{x}(W_i) \subset W_{i-1} \quad \stackrel{a_{i+1} \in V_{i+1}}{\Rightarrow} \tilde{x}(a_{i+1} + V_1) = x(a_{i+1}) + V_1 \in W_{i-1},$$

$$\Rightarrow x(a_{i+1}) \in V_i$$