p. 10

$\boxed{3.1}$ L is a Lie algebra.

<u>derived series</u> : $L^{(0)} = L$, $L^{(1)} = [L, L]$, ..., $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$

.... $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ $L^{(i)}$ is an ideal
see p. 1 of §3 Revisited

L is <u>solvable</u> if $L^{(n)} = 0$ for some n

<u>examples</u>
1. abelian (are solvable) $L^{(1)} = 0$
2. simple (are not solvable)
(see p. 1 of §3 Revisited)
3. $L = \vec{t}(n, F)$ is solvable

[step 1 $\vec{n}(n,F) \subset [L, L]$ see p. 1 of §3 Revisited

step 2 $\vec{n}(n,F) = [L, L]$ see p. 2 of §3 Revisited

step 3 $L^{(i)} = 0$ if $2^{i-1} > n-1$. see pp. 2-4 of §3 Revisited

fill in the proofs

p. 11

Proposition 3.1 If L is a Lie algebra, then

(a) if L is solvable, so are its subalgebras and homomorphic images

(b) if I is a solvable ideal of L, and L/I is solvable, then L is solvable.

(c) If I and J are solvable ideals of L, so is $I+J$.

(not done by me on §3 Revisited)

fill in the proofs, either from p. 11 or 199A — see

Meyberg notes p. 5-6 (Lemma 1, Lemma 2, Theorem 2 (i))
www.math.uci.edu/~brusso/ch1marked.pdf

and

Third Meeting October 14, 2016 Chapter 1 Exercises 4-6 SOLUTIONS
www.math.uci.edu/~brusso/meybergch1ex4-6sol.pdf

p. 11 (continued)

<u>radical of L</u> : Rad L is ~~the~~ the unique maximal solvable ideal of L

L is <u>semi-simple</u> if Rad $L = 0$.

<u>examples</u>

1. a simple algebra is semi-simple (see p. 5 of §3 Revisited)
2. $L = 0$ is semi-simple (really?)
3. $L/_{\mathrm{Rad}\,L}$ is semi-simple see p. 5 of §3 Revisited

see

Third Meeting October 14, 2016 Solvable Radical (informal notes: Meyberg pages 5-6)

www.math.uci.edu/~brusso/solvableradical.pdf

$\boxed{3.2}$ L is a Lie algebra

<u>descending central series</u> : $L^0 = L$, $L^1 = [L, L]$,
(= lower central series)

$$L^i = [L, L^{i-1}]$$

L is <u>nilpotent</u> if $L^n = 0$ for some n.

<u>examples</u>

1. abelian (are nilpotent) $L^1 = 0$
2. nilpotent $\Longrightarrow$ solvable $(L^{(i)} \subset L^i)$
3. Solvable $\not\Longrightarrow$ nilpotent
   ($\vec{t}(n,F)$ is solvable but not nilpotent)
   proof on top of p. 12 } see p. 6 of § 3 Revisited
4. $\vec{n}(n,F)$ is nilpotent.

p. 12

Proposition 3.2 If L is a Lie algebra

(a) L nilpotent $\Rightarrow$ so are subalgebras and homomorphic images

(b) If $L/Z(L)$ is nilpotent, so is L nilpotent.
(see p. 7 of §3 Revisited)

(c) If L is nilpotent and $L \neq 0$, then $Z(L) \neq 0$.

fill in the proof, either from p. 12 or 199A — see see p. 8 of §3 Revisited

Third Meeting, October 14, 2016, Nilradical (informal notes: Meyberg pages 6-7)
www. math. uci. edu/~brusso/nilradical.pdf
NO GO see p. 8 of §3 Revisited

definition: In a Lie algebra L an element $x \in L$ is called ad-nilpotent if $\mathrm{ad}\, x$ is a nilpotent endomorphism.

NOTE: In a nilpotent Lie algebra, every element is ad-nilpotent. Conversely,

Theorem (Engel) If all elements of L are ad-nilpotent, then L is nilpotent.

p.12 (continued)

Application of Engel's theorem: $\vec{n}(n, F)$ is nilpotent see p. 9 of §3 Revisited
(this was already shown to be nilpotent by calculating the descending central series (on p. 12))

This uses the following lemma.

LEMMA If $x \in gl(V)$ is nilpotent, then $\operatorname{ad} x$ is nilpotent in $\operatorname{End}(gl(V))$ $(gl(V) = \operatorname{End}(V)!)$

fill in the proof of the lemma. (see p. 9 of §3 Revisited)

Engel's Theorem will follow from the following theorem

Theorem 3.3 If L is a subalgebra of $gl(V)$, V finite dimensional, $V \neq 0$, and if L consists of nilpotent endomorphisms, then $\exists$ a non-zero vector $v \in V$ for which $L(v) = 0$, i.e.

$Tv = 0 \quad \forall\, T \in L$.

see p. 14 of § 3 Revisited

fill in the proofs of this theorem[3.3] and of Engel's theorem.

$\boxed{\text{see pp 10 - 13 of § 3 Revisited}}$

p. 13 (continued)

A corollary and an application of Theorems 3.3

<u>Corollary</u> Same hypotheses as in theorem 3.3: L is a subalgebra of $gl(V)$, $V \neq 0$, finite dimensional, and L consists only of nilpotent endomorphisms.

Then there exists a basis of V relative to which the matrices of L are in $\vec{n}(n, F)$.

proof Th. 3.3 tells us that $\exists\ v \neq 0, v \in V$ $x(v) = 0\ \forall x \in L$.

See the proof in § 3 Revisited p. 15

Lemma (application) If L is nilpotent, K an ideal in L, $K \neq 0$, then $K \cap Z(L) \neq 0$.

$\ulcorner$ ad L : $K \to K$ $\quad y \to [x, y]$ $\quad x \in L$, $y \in K$

is a Lie subalg of $gl(K)$ consisting of only nilpotent elements. So by Th 3.3

$\exists\ x_0 \neq 0, x_0 \in K$ s.t. $[L, x_0] = 0$

i.e. $0 \neq x_0 \in K \cap Z(L)$ $\blacksquare$