p.15 humphreys 15-20 NOTES.pdf 5/2/17 (1)

$\boxed{4.1}$

F is algebraically closed & char 0.

Theorem If L is a solvable subalgebra of $gl(V)$, V finite dimensional, $V \neq 0$ then V contains a common eigenvector for all elements of L.

Compare this with Theorem 3.3 which states: L consists of nilpotent elements $\Rightarrow$ V contains a common eigenvector for all elements of L corresponding to eigenvalue 0

This is a long proof!

step (1) locate an ideal of codimension 1

step (2) by induction common eigenvectors exist for all of K

step (3) if v is such an eigenvector for K, then each element of $L(v)$ is also an eigenvector for K

step (4) Find an eigenvector for K which is also an eigenvector for some $z \in L$ satisfying $L = K + Fz$.

$\boxed{\text{step (1)}}$ ~~Since~~ L is solvable $\Rightarrow L \supsetneq [L,L]$.

$L/[L,L]$ is abelian $\Rightarrow$ every subspace is an ideal

let $S \subset L/[L,L]$ be a subspace of co-dimension 1

let $\pi : L \to L/[L,L]$ $\quad \pi(x) = x + [L,L]$

let $K = \pi^{-1}(S)$ is an ideal, $L = K + sp\{y_0\}$

$L/[L,L] = S + sp\{y_0 + [L,L]\}$

for step (1).

$\boxed{\text{step 2}}$ K is a subalgebra of a solvable algebra so K is solvable. By induction hypothesis, $\exists\, v \neq 0$, $\exists$ linear function $\lambda : K \to F$ so that $x(v) = \lambda(x)\, v \quad (x \in K)$.

This proves step 2.

$\boxed{\text{step 3}}$ let $W = \{ w \in V : x(w) = \lambda(x)\, w \;\; \forall x \in K \}$

So $v \in W \neq (0)$. Need to prove $L(W) \subset W$.

Assume this and complete step 4, then return to step 3.

$\boxed{\text{step 4}}$ Since K has codimension 1, write $L = K \oplus Fz$ for some $z \in L$. Since F is algebraically closed and $z(W) \subset W$, z has an eigenvector $v_0 \in W$ (corresponding to some eigenvalue α of z, $z(v_0) = \alpha v_0$) Then v_0 is a common eigenvector for L; Indeed if $y = x + \mu z$ $x \in K, \mu \in F$ then $y(v_0) = x(v_0) + \mu z(v_0) = \lambda(x)\, v_0 + \mu \alpha\, v_0 = (\lambda(x) + \mu\alpha)\, v_0$

(we also see that λ extends to a linear functional $\tilde{\lambda}$ on L via $\tilde{\lambda}(x + \mu z) = \lambda(x) + \mu\alpha$)

<u>Return to the proof of step 3</u> This is long. skip for now.

Note Theorem 4.1 $\Rightarrow$ Corollary A $\Rightarrow$ Corollary B $\Rightarrow$ Corollary C

Corollary A (Lie's theorem) If L is a solvable subalgebra of $gl(V)$, $\dim V = n < \infty$ then the matrices of the elements of L relative to a suitable basis of V are upper triangular.

In other words there are subspaces $V_0 \subseteq V_1 \subset V_2 \cdots \subseteq V_n$ with $\dim V_i = i$ and $L(V_i) \subset V_i$. see p. 2'–2'' for the proof, which parallels the proof of the Corollary to Theorem 3.3.

Corollary B L solvable $\Rightarrow \exists$ ideals $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ such that $\dim L_i = i$.

proof. $ad : L \to End(L)$ is a homomorphism, so $ad\, L$ is solvable so $\exists$ subspaces $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ $\dim L_i = i$ and $ad\, x\,(L_i) \subset L_i$ $\forall x \in L$

This says that L_i is an ideal. $\boxed{}$

Corollary C L solvable & $x \in [L, L] \Rightarrow ad_L\, x$ is nilpotent. In particular $[L, L]$ is nilpotent. $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$

proof Choose a basis $x_1, \ldots, x_n$ such that $x_1 \in L_1$, $x_1, x_2 \in L_2, \ldots, x_1, \ldots, x_n \in L_n$

The matrix of $ad\, x$ (for $x \in L$) is upper triangular (w/R this basis!)

The matrix of $[ad\, x, ad\, y]$ ($= ad[x,y]$) is strictly upper triangular.

So $ad[x,y]$ is nilpotent $\forall x, y \in L$ Thus the Lie algebra $[L, L]$ has all elements ad-nilpotent and by Engel's thm $[L, L]$ is nilpotent he alg $\boxed{}$

Let us first prove the Corollary to Theorem 3.3 (on p.13)

<u>Corollary</u> L is a subalgebra of $gl(V)$, $\dim V < \infty$, $V \neq 0$ & L has only nilpotent endomorphisms. Then there are subspaces $0 = V_0 \subset V_1 \subset \dots \subset V_{n-1} \subset V_n = V$ such that $\dim V_i = i$ and $L(V_i) \subset V_{i-1}$

<u>proof</u>. Induction on $n = \dim V$. If $n = 1$ Theorem 3.3 tells us that $\exists\, v_1 \neq 0$ with $L(v_1) = 0$. Thus $0 = V_0 \subseteq V_1 = V = sp\{v_1\}$ satisfies $L(V_1) \subset V_0$.

For the induction hypothesis consider $W = V/V_1$ which has dimension $n-1$. The Lie algebra

$$\tilde{L} = \{\tilde{x} : x \in L\} \subseteq gl(V/V_1) \text{ where } \tilde{x} : V/V_1 \longrightarrow V/V_1$$
$$v + V_1 \longrightarrow x(v) + V_1$$

consists of nilpotent endomorphisms. By the induction hypothesis $\exists_{0 \le i \le n-1}\ 0 = W_0 \subset W_1 \subset W_2 \subset \dots \subset W_{n-2} \subset W_{n-1} = V/V_1$ with $\dim W_i = i$ and $\tilde{L}(W_i) \subset W_{i-1}$. $(1 \le i \le n-1)$

Of course $W_i = V_{i+1}/V_1$ where $V_1 \subset V_{i+1} \subset V$ and $V_n = V$.

and $\dim V_{i+1} = i+1$ & $V_{i+1} \supseteq V_i$

$$\tilde{x}\left(\frac{V_{i+1}}{V_1}\right) \subset \frac{V_i}{V_1}$$

$\tilde{x}(v + V_1) = x(v) + V_1$ so $x(V_{i+1}) \subset V_i$ proving the Corollary to Theorem 3.3

Proof of Corollary A If $\dim V = 1$, $\exists\, v_0 \in V$ $v_0 \neq 0$ and $x(v_0) = \lambda(x) v_0$ $\forall x \in L$. Thus

$$0 = V_0 \subset V_1 = V = sp\{v_0\} \quad \text{and} \quad x(V_1) \subset V_1 \ \forall x \in L$$

For the induction hypothesis ~~con~~ ~~W =~~ ~~Now~~ let $\dim V = n$ and let V_1 be as above $V_1 = sp\{v_0\}$, $x(v_0) = \lambda(x) v_0$ $\forall x \in L$

Consider $W = V/V_1$, $\dim W = n-1$ so as in the proof of the Corollary to Theorem 3.3, the Lie algebra $\tilde{L} = \{\tilde{x} : x \in L\} \subseteq gl(V/V_1)$

$\tilde{x} : V/V_1 \to V/V_1$

$v + V_1 \to x(v) + V_1$

is solvable since $L \to \tilde{L}$, $x \to \tilde{x}$ is a homomorphism

(because
$$[\tilde{x}, \tilde{y}](v+V_1) = (\tilde{x}\tilde{y} - \tilde{y}\tilde{x})(v+V_1)$$
$$= (xy - yx)(v) + V_1$$
$$= \widetilde{[x,y]}(v+V_1))$$

Thus $\exists$
$$0 = W_0 \subset W_1 \subset W_2 \cdots \subset W_{n-2} \subset W_{n-1} = V/V_1$$
with $\dim W_i = i$ & $\tilde{L}(W_i) \subset W_i$. Of course $W_i = V_{i+1}/V_1$ where $V_1 \subset V_{i+1} \subset V$ and $V_n = V$, and $\dim V_{i+1} = i+1$ & $V_{i+1} \supseteq V_i$. Since $\tilde{x}(V_{i+1}/V_1) \subset V_{i+1}/V_1$ we have $x(V_{i+1}) \subset V_{i+1}$ proving Corollary A. ∎

Review the Jordan canonical form for an endomorphism over an algebraically closed field.

Text for 121 AB Friedberg, Insel, Spence Linear Algebra 4th Edition Chapter 7 pp 482–524

This is covered at the end of 121 B

See also the ~~notes~~ book by Gonzalez (which is posted). A summary follows on the next 3 pages. However, we can ignore the details of proof & just state the results

Gonzalez, Fulton B. — Lie Algebras

$\boxed{1.6}$ Upper Triangular Matrices

Theorem 1.6.2 If V is a finite dim'l vector space over $\mathbb{C}$, and T is a linear transf. on V, then there is a basis for which the matrix of T is upper triangular. (121AB p. 370 Theorem 6.14)

Theorem 1.6.3 V, T as in Th. 1.6.2, p the characteristic polynomial of T $(p(\lambda) = \det(\lambda I - T))$ $\Rightarrow p(T) = 0$. (121AB p. 317 Theorem 5.23)

NOTE: if T has matrix $\begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ then $p(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j)$

$\boxed{1.7}$ Generalized Eigenspaces

definition For $\lambda \in \mathbb{C}$, the generalized eigenspace of T corresponding to λ is $\{v \in V : (T - \lambda I)^k v = 0$ for some $k \geq 1\}$

(121AB p. 484 & 485 Definition)

Proposition 1.7.4 The generalized eigenspace of T ~~It has dim~~ corresponding to eigenvalue λ_j has dimension the multiplicity of λ_j.

(121AB p. 486 and 487 Theorem 7.2(a) Theorem 7.4(c))

Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of T.

Let m_j be the multiplicity of eigenvalue λ_j

So that $p(\lambda) = \prod_{j=1}^{k} (\lambda - \lambda_j)^{m_j}$

Theorem 1.7.7 V is the direct sum of the generalized eigenspaces of the eigenvalues of T.

$$V = \bigoplus_{j=1}^{k} \underbrace{\ker (T - \lambda_j I)^{m_j}}_{\text{has dimension } m_j}$$

(121AB p. 486 Theorem 7.3)

$\boxed{1.8}$ Jordan canonical form

Proposition 1.8.2 There is a basis for V such that T has the matrix

$$\begin{bmatrix} A_1 & 0 & 0 & \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 \ddots & \\ & 0 & & A_k \end{bmatrix}$$

A_j is $m_j \times m_j$ matrix

and each A_j is upper triangular of the form

$$A_j = \begin{bmatrix} \lambda_j & & * \\ & \ddots & \\ 0 & & \lambda_j \end{bmatrix}. \qquad A_j = \begin{bmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{bmatrix}$$

Theorem 1.8.5 There is a basis such that $\uparrow$

$\uparrow$ called a Jordan basis

(121AB p. 491 corollary 1)

1.9 The Jordan Chevalley Decomposition

Prop 1.9.1 If T is a linear transf. on V and $p(z)$ is a monic polynomial of smallest positive degree such that $p(T)=0$ then $p(z)$ divides any polynomial $s(z)$ with $s(T)=0$.

This polynomial is unique and is called the minimal polynomial of T. 121 AB Theorem 7.12 p. 516

def an operator T is semi-simple if its minimal polynomial of its minimal polynomial has only roots of multiplicity 1, i.e. $p(z)=\prod_{i=1}^{k}(z-\lambda_i)$ where $\lambda_1,\dots,\lambda_k$ are distinct.

Prop 1.9.8 T is semi-simple $\Leftrightarrow$ it is diagonalizable, i.e. there is a basis of V such that the matrix of T w/R to this basis is diagonal. 121 AB Theorem 7.16 p.520

I could not find this theorem in the 121AB text. However, it is in Gonzalez

Theorem 1.9.14 (Jordan-Chevalley Decomposition)

T a linear operator on complex vector space V

$\Rightarrow$ $\exists$ polynomial $p(z)$ such that, with $q(z)=z-p(z)$

1. $S:=p(T)$ is semi-simple & $N=q(T)$ is nilpotent
2. If an operator commutes with T, then it commutes with S and N
3. The decomposition $T=S+N$ is unique if S commutes with N
4. If $A\subset B\subset V$ are subspaces & $T(B)\subset A$, then $S(B)\subset A$ & $N(B)\subset A$. $\square$

Call $x \in \mathrm{End}(V)$, (V finite dimensional) <u>semi-simple</u> if the roots of its minimum polynomial are all distinct.

<u>Proposition</u> V finite dim'l vector space, $x \in \mathrm{End}(V)$

(a) $\exists !\ x_s, x_n \in \mathrm{End}(V)$, $x = x_s + x_n$ (<u>Jordan</u>-Chevalley <u>decomposition</u>)

x_s is semi-simple, x_n is nilpotent, $x_s x_n = x_n x_s$.

(b) $\exists$ polynomials p, q without constant term

$x_s = p(x)$, $x_n = q(x)$

(c) If $A \subset B \subset V$ are subspaces & $x(B) \subset A$ then $x_s(B) \subset A$ and $x_n(B) \subset A$.

Let us defer the proof of this theorem. It uses the Chinese Remainder Theorem for the ring of polynomials!

If $x \in gl(V)$ is nilpotent, so is $\text{ad}\,x$ (Lemma 3.2 on p. 12)

If x is semi-simple, so is $\text{ad}\,x$

proof:

Let $v_1, \dots, v_n$ be a basis for V for which $x \sim \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}$

i.e. $x(v_i) = a_i v_i$. Define $e_{ij} \in gl(V)$

by $e_{ij}(v_k) = \delta_{jk} v_i$. Then $\{e_{ij}\}$ is a basis for $gl(V)$

~~Recall~~ ~~check that~~ $\cancel{[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}}$

and $\text{ad}\,x\,(e_{ij})(v_k) = [x, e_{ij}](v_k) = x e_{ij}(v_k) - e_{ij} x(v_k)$

$= x\,\delta_{jk} v_i - e_{ij}\, a_k v_k = (a_i - a_j)\,\delta_{jk}(v_i) = (a_i - a_j)\, e_{ij}(v_k)$

so $\text{ad}\,x\,(e_{ij}) = (a_i - a_j)\, e_{ij}$ is a diagonal matrix, so semi-simple.

Lemma A. If $x \in \text{End}(V)$ $(\dim V < \infty)$, $x = x_s + x_n$

its Jordan decomposition, then

$\text{ad}\,x = \text{ad}\,x_s + \text{ad}\,x_n$ is the Jordan decomp.

in $\text{End}(\text{End}(V))$.

proof: $\text{ad}\,x_s$ is semisimple and $\text{ad}\,x_n$ is nilpotent

and they commute: $[\text{ad}\,x_s, \text{ad}\,x_n] = \text{ad}\,[x_s, x_n] = 0$

Since $\text{ad}\,x = \text{ad}\,x_s + \text{ad}\,x_n$, the uniqueness in Proposition 4.2 (a)

shows it is the Jordan-Chevalley decomp. of $\text{ad}\,x$.

Lemma B. Let A be a finite dim'l algebra. Then

for any derivation $\delta \in \text{Der}(A) \subset \text{End}(A)$

if $\delta = \delta_s + \delta_n$ is its Jordan decomposition,

then δ_s & δ_n are derivations of A.

TO BE CONTINUED

Proof of lemma B.

$\boxed{4.3}$

Observations:

- L is solvable if $[L,L]$ is nilpotent (Converse of Corollary 4.1C)

- $[L,L]$ is nilpotent $\Longleftrightarrow$ each $\mathrm{ad}_{[L,L]}\, x$, $x \in [L,L]$ is nilpotent (Engel's theorem)

Lemma Let $A \subset B \subset gl(V)$ be subspaces ($\dim V < \infty$). Let $M = \{x \in gl(V) : [x, B] \subset A\}$. If $x \in M$ satisfies $\mathrm{Tr}(xy) = 0$ $\forall\, y \in M$, then x is nilpotent.

<u>Theorem (Cartan's criterion)</u> Let L be a subalgebra of $\mathfrak{gl}(V)$ $(\dim V < \infty)$. Suppose $Tr(xy) = 0$ $\forall\, x \in [L, L]$ $\forall\, y \in L$. Then L is solvable.

<u>Corollary</u> Let L be a Lie algebra such that $Tr(\operatorname{ad}x \operatorname{ad}y) = 0$ $\forall\, x \in [L, L]$ $\forall\, y \in L$. Then L is solvable.