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$\boxed{4.1}$ 5/2/17

F is algebraically closed & char 0.

<u>Theorem</u> If L is a solvable subalgebra of $gl(V)$, V finite dimensional, $V \neq 0$ then V contains a common eigenvector for all elements of L.

Corollary A (Lie's theorem) If L is a solvable subalgebra of $gl(V)$, $\dim V = n < \infty$ then the matrices of the elements of L relative to a suitable basis of V are upper triangular.

Corollary B L solvable $\Rightarrow$ $\exists$ ideals $0 = L_0 \subset L_1 \subset \cdots \subset L_n = L$ such that $\dim L_i = i$.

Corollary C L solvable & $x \in [L, L] \Rightarrow \mathrm{ad}_L x$ is nilpotent. In particular $[L, L]$ is nilpotent.

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$\boxed{4.2}$

③

Review the Jordan canonical form for an endomorphism over an algebraically closed field.

Text for 121 AB Friedberg, Insel, Spence
Linear Algebra 4th Edition chapter 7 pp 482–524

Call $x \in \mathrm{End}(V)$, (V finite dimensional) <u>semi-simple</u> if the roots of its minimum polynomial are all distinct.

<u>Proposition</u> V finite dim'l vector space, $x \in \mathrm{End}(V)$

(a) $\exists!\ x_s, x_n \in \mathrm{End}(V)$, $x = x_s + x_n$ (<u>Jordan decomposition</u>)

x_s is semi-simple, x_n is nilpotent, $x_s x_n = x_n x_s$.

(b) $\exists$ polynomials p, q without constant term

$x_s = p(x)$, $x_n = q(x)$

(c) If $A \subset B \subset V$ are subspaces & $x(B) \subset A$ then $x_s(B) \subset A$ and $x_n(B) \subset A$.

If $x \in gl(V)$ is nilpotent, so is $\operatorname{ad} x$ (Lemma 3.2 on p. 12)

If x is semi-simple, so is $\operatorname{ad} x$

Lemma A If $x \in \operatorname{End}(V)$ $(\dim V < \infty)$, $x = x_s + x_n$ its Jordan decomposition, then

$\operatorname{ad} x = \operatorname{ad} x_s + \operatorname{ad} x_n$ is the Jordan decomp.

in $\operatorname{End}(\operatorname{End}(V))$.

Lemma B Let A be a finite dim'l algebra. Then for any derivation $\delta \in \operatorname{Der}(A) \subset \operatorname{End}(A)$

if $\delta = \delta_s + \delta_n$ is its Jordan decomposition, then δ_s & δ_n are derivations of A.

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$\boxed{4.3}$

Observations:

- L is solvable if $[L,L]$ is nilpotent (converse of Corollary 4.1C)

- $[L,L]$ is nilpotent $\iff$ each $\mathrm{ad}_{[L,L]}\, x$, $x \in [L,L]$ is nilpotent (Engel's theorem)

Lemma: Let $A \subset B \subset gl(V)$ be subspaces ($\dim V < \infty$). Let $M = \{ x \in gl(V) : [x, B] \subset A \}$. If $x \in M$ satisfies $\mathrm{Tr}(xy) = 0$ $\forall\, y \in M$, then x is nilpotent.

<u>Theorem (Cartan's criterion)</u> Let L be a subalgebra of $gl(V)$ ($\dim V < \infty$). Suppose $\mathrm{Tr}(xy) = 0$ $\forall x \in [L,L]$ $\forall y \in L$. Then L is solvable.

<u>Corollary</u> Let L be a Lie algebra such that $\mathrm{Tr}(\mathrm{ad}x\ \mathrm{ad}y) = 0$ $\forall x \in [L,L]$ $\forall y \in L$. Then L is solvable.