4.3

Observations:

- L is solvable if $[L,L]$ is nilpotent (converse of Corollary 4.1C)

proof

- $[L,L]$ is nilpotent $\Longleftrightarrow$ each $\mathrm{ad}_{[L,L]}x$, $x \in [L,L]$ is nilpotent (This is just Engel's theorem)

suppose $K := [L,L]$ is nilpotent

$K^1 = [K,K] = [L^{(1)}, L^{(1)}] = L^{(2)}$

$K^2 = [K,K^1] = [L^{(1)}, L^{(2)}] \supseteq [L^{(2)}, L^{(2)}] = L^{(3)}$

$\cdots\cdots$

$K^i \supseteq L^{(i+1)}$ so if $K^i = 0$, then $L^{(i+1)} = 0$.

Lemma Let $A \subset B \subset gl(V)$ be subspaces ($\dim V < \infty$)

Let $M = \{x \in gl(V) : [x, B] \subset A\}$. If $x \in M$ satisfies $\mathrm{Tr}(xy) = 0 \ \forall y \in M$, then x is nilpotent.

This is long. skip for now.

Remark If $x,y,z \in \mathrm{End}(V)$, then $\mathrm{Tr}([x,y]z) = \mathrm{Tr}(x[y,z])$

$[x,y]z = xyz - yxz$, $x[y,z] = xyz - xzy$

$\mathrm{Tr}([x,y]z) = \mathrm{Tr}(xyz) - \mathrm{Tr}(y(xz))$ $\quad \mathrm{Tr}(x[y,z]) = \mathrm{Tr}(xyz) - \mathrm{Tr}((xz)y)$

but $\mathrm{Tr}(y(xz)) = \mathrm{Tr}((xz)y)$ $\boxtimes$

<u>Theorem (Cartan's criterion)</u> Let L be a subalgebra of $gl(V)$ $(\dim V < \infty)$. Suppose $Tr(xy) = 0$ $\forall\, x \in [L,L]$ $\forall\, y \in L$. Then L is solvable.

proof By the first observation on page ⑥ it suffices to prove that $[L,L]$ is nilpotent. By Engel's theorem it suffices to prove $ad\, x|_{[L,L]}$ is nilpotent $\forall\, x \in [L,L]$.

By Lemma 3.2 on p.12 of Humphreys (proof is written out on page ⑨ of "§3 Revisited") it suffices to prove x is nilpotent $\forall\, x \in [L,L]$

In the Lemma on p. ⑥ let $A = [L,L]$, $B = L$.

Then $M = \{x \in gl(V) : [x, B] \subset A\} = \{x \in gl(V) : [x, L] \subset [L,L]\}$

Of course $L \subset M$. $\boxed{\text{Our hypothesis is } Tr(xy) = 0 \ \forall\, x \in [L,L] \ \forall\, y \in L}$

The Lemma on p.⑥ requires $Tr(xy) = 0$ $\forall\, y \in M$

Since $x \in [L,L]$ $x = \sum [x_i, y_i]$ $x_i, y_i \in L$. So if $y \in M$

$$Tr(xy) = \sum_i Tr([x_i, y_i]\, y) = \sum_i Tr(\underset{\substack{\uparrow \\ L}}{x_i}\, \underset{\substack{\uparrow \\ [L,L]}}{[y_i, y]}) = 0. \quad ▨$$

<u>Corollary</u> Let L be a Lie algebra such that

$Tr(ad\, x\ ad\, y) = 0$ $\forall\, x \in [L,L]$ $\forall\, y \in L$.

Then L is solvable.

proof Let $\tilde{L} = ad(L) \subset gl(L)$. By the theorem $ad\, L$ is solvable. $L/Z(L) = L/\ker ad \simeq ad\, L$ is solvable, and $Z(L)$, being abelian, is solvable. So L is solvable.