p.21 §5 Killing form

def **Killing form** is $\kappa(x,y) = Tr(\operatorname{ad}x \operatorname{ad}y)$

a symmetric bilinea form on a Lie algebra L.

It is an **associative bilinear form** : $\kappa([xy],z) = \kappa(x,[yz])$

$\kappa([x,y]z) = Tr(\operatorname{ad}[x,y] \operatorname{ad}z) = Tr([\operatorname{ad}x, \operatorname{ad}y] \operatorname{ad}z)$

$\overset{4.3}{=} Tr(\operatorname{ad}x [\operatorname{ad}y, \operatorname{ad}z]) = Tr(\operatorname{ad}x \operatorname{ad}[y,z])$

$= \kappa(x,[y,z])$

$\boxed{(4.3) \text{ says for } x,y,z \in End(V) \quad Tr([x,y]z) = Tr(x[y,z])}$

Lemma If I is an ideal in L then $\kappa_L|_{I\times I} = \kappa_I$

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The **radical** of a bilinear form $\beta : L\times L \to F$ is

$\{x\in L : \beta(x,y)=0 \ \forall y \in L\}$ (in short $\beta(x,L)=0$)

β is **non-degenerate** if it has zero radical.

The radical is an ideal

<u>example</u> the Killing form of $sl(2,F)$

From linear algebra K is non-degenerate $\Leftrightarrow$

$$\det \begin{bmatrix} K(x,x) & K(x,y) & K(x,h) \\ K(y,x) & K(y,y) & K(y,h) \\ K(h,x) & K(h,y) & K(h,h) \end{bmatrix} \neq 0 \qquad = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$\operatorname{ad}x \operatorname{ad}y = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \begin{matrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad Tr = 4$$

(Exercise 1.3 : $\operatorname{ad}x \sim \begin{matrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix}$ $\quad \operatorname{ad}y \sim \begin{matrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{matrix}$ $\quad \operatorname{ad}h \sim \begin{matrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{matrix}$

(def. $\operatorname{Rad} L = 0$)

<u>Theorem</u> Lie algebra L is <u>semi-simple</u> $\Leftrightarrow$ Killing form is non-degenerate.

Note first that L is semisimple $\Leftrightarrow$ it contains no $\neq 0$ ideals.

$\boxed{\Rightarrow}$ I abelian $\Rightarrow I$ solvable $\Rightarrow I \subset \operatorname{Rad} L = 0$

$\boxed{\Leftarrow}$ if L is not semisimple let $J = \operatorname{Rad} L \neq 0$

Then $0 = J^{(n)} = [J^{(n-1)}, J^{(n-1)}]$ so $J^{(n-1)}$ is a $\neq 0$ abelian ideal

Proof of theorem

Suppose L is semi-simple, i.e. $\operatorname{Rad} L = 0$ & let S = radical of K, i.e.

$Tr(\operatorname{ad}x \operatorname{ad}y) = 0 \ \forall x \in S, y \in L$, in particular $\forall x \in [SS], y \in S$

By the Corollary to Cartan's Criterion (Theorem 4.3)

S is solvable, so $S \subset \operatorname{Rad} L = 0$ and K is non-degenerate.

Conversely let $S = 0$. L is ss $\Leftrightarrow$ it contains no $\neq 0$ abelian ideals

let I be an abelian ideal. We show $I \subset S$ so L is ss.

let $x \in I$ and $y \in L$. Then $\operatorname{ad} x \operatorname{ad} y : L \to I$. For $z \in I$,

$$\operatorname{ad} x \operatorname{ad} y\,(z) = [x, [y,z]] \in [I, I] = 0$$

i.e. $((\operatorname{ad} x)(\operatorname{ad} y))^2 = 0$ so $\overbrace{\operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)}^{K(x,y)=} = 0$

$\forall x \in I, y \in L$, so $K(x, L) = 0$ and $x \in S = 0$, $I = 0$.

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$\boxed{5.2}$

Theorem • If L is semi-simple, then $\exists$ ideals $L_1, \dots, L_t$ which are simple lie algebras and $L = L_1 \oplus \dots \oplus L_t$

• Every simple ideal of L coincides with some L_i, and $K_{L_i} = K_L \big|_{L_i \times L_i}$.

Corollary L semi-simple $\Rightarrow L = [LL]$ & all homo images and ideals of L are semi-simple. Also each ideal of L is a sum of simple ideals of L.

Proof of Theorem Let I be an ideal $S = I \cap I^{\perp}$ ~~the ideal~~

$I^{\perp} = \{x \in L : K(x, I) = 0\}$ is an ideal ~~since~~

$$\left.K([x,z]\,y) = K(x, \underbrace{[z,y]}_{\in I}) = 0\right\} \Rightarrow I^{\perp} \text{ is an ideal}$$

$x \in I^{\perp}, z \in L$ ~~..~~ ~~..~~ $y \in I$

If $x \in [S,S] \subset S$, $y \in S$ $\quad K_S = K_L | S \times S$

$K_S(x,y) = Tr(ad_S x, ad_S y) = 0$ since $x \in I, y \in I^{\perp}$

Cartan criterion $\Rightarrow S$ is solvable so $S \subset Rad L = 0$

and $I \cap I^{\perp} = 0$ Since K is non-degenerate

$L = I + I^{\perp}$ so $L = I \oplus I^{\perp}$ & $\dim L = \dim I + \dim I^{\perp}$

$\boxed{\text{See Exercise 16.8 in Erdmann-Wildon book p. 207}}$

I is semisimple since $K_I = K_L \big|_{I \times I}$ is non-degenerate.

So is $I^{\perp}$ semisimple.

If L is simple, nothing to prove. Otherwise

$L = I \oplus I^\perp \qquad \dim I < \dim L, \ \dim I^\perp < \dim L$

so by induction $\quad I = I_1 \oplus \cdots \oplus I_r \quad I_i$ simple ideal in I

$\qquad I^\perp = J_1 \oplus \cdots \oplus J_s \quad J_j$ " " " $I^\perp$

I_i is a simple ideal of L

since if $0 \neq I' \subsetneq I_i \quad I'$ an ideal in L

then I' is an ideal in I so $I' = 0$ or $I' = I_i$.

Similarly J_j is a simple ideal of L

and the existence of a decomposition holds.

<u>uniqueness</u>: let I be a simple ideal of L $\quad (\therefore I \neq 0)$

$[IL]$ is an ideal of I, $[IL] \neq 0$ since $Z(L) = 0$

so $[IL] = I \qquad$ But $[IL] = \bigoplus_{i=1}^{t} [IL_i]$

so $I = [IL] = [IL_i]$ for some $i \qquad$ so $I \subset L_i$

and $I = L_i$ as L_i is simple. This proves uniqueness.

The last statement follows from Lemma 5.1

The Corollary is immediate $\quad$ e.g. $[L,L] = \sum_{i,j} [L_i, L_j] = \sum_i [L_i, L_i] = \underbrace{\oplus L_i}_{= L}$

$\boxed{5.3}$

Theorem If L is semi-simple, then all derivations of L are inner.

Proof. L semisimple $\Rightarrow Z(L) = 0$ so $ad : L \to ad\,L$ is an isomorphism of Lie algebras. Let $M = ad\,L \subset gl(V)$

M has a non-degenerate Killing form. M is an ideal in $D = Der(L)$, so $K_M = K_D|_{M \times M}$.

$$M^{\perp} = \{T \in D : K_D(T, M) = 0\}$$

claim $M^{\perp} \cap M = 0$

Let $T \in M^{\perp} \cap M$, $S \in M$ then $K_M(S,T) = 0$ but $K_M = K_D|_{M \times M}$ is non-degenerate so $T = 0$

M & $M^{\perp}$ are ideals of $D \Rightarrow [M^{\perp}, M] = 0$

If $\delta \in M^{\perp}$, $x \in L$: $ad(\delta x) = [\delta, ad\,x] \in [M^{\perp}, M] = 0$

Since ad is one-to-one $\delta x = 0 \ \forall x \in L$

$\delta = 0$ so $M^{\perp} = 0$ & $D = M$. $\blacksquare$

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5.4 Abstract Jordan Decomposition