p. 25 § 6 Complete reducibility of representations 5/12/17 II

$\boxed{6.1}$ def. L is a Lie alg. An <u>L-module</u> is a vector space V and a map $L \times V \longrightarrow V$

$(x, v) \longrightarrow x \cdot v$

(M1) $(ax + by) \cdot v = a(x.v) + b(y.v)$
(M2) $x \cdot (av + bw) = a(x.v) + b(x,w)$ } bilinear
(M3) $[xy] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$

L-modules are the same as representations of L

$\phi : L \to gl(V)$ $\longleftrightarrow$ $x \cdot v = \phi(x)(v)$
representation

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V

$x.v = \alpha(x) v$

$\alpha : L \to gl(V)$

W

$x.w = B(x) w$

$B : L \to gl(W)$

def. <u>homomorphism of L-modules</u> is a linear map $\phi : V \to W$

with $\phi(x.v) = x.(\phi(w))$ $x \in L$, $v \in V$

$\phi(\alpha(x) v) = B(x) \phi(w)$

kernel, submodule, homomorphism theorems, isomorphism, equivalent representations

$W \subset V$

$x.W \subset W$

submodule

$$\begin{tikzcd} W \arrow[r, "B(x)"] & W \\ V \arrow[u, "\phi"] \arrow[r, "\alpha(x)"] & V \arrow[u, "\phi"'] \end{tikzcd}$$

equivalent rep'ns

$\phi(V) \simeq V/\ker\phi$

$V_1 \subset V_2 \subset W$

$\left(W/V_1\right) / \left(V_2/V_1\right) \simeq W/V_2$

$V_1, V_2 \subset W$

$(V_1 + V_2)/V_2 \simeq V_1 / V_1 \cap V_2$

def. L-module V is <u>irreducible</u> if the only L-submodules are 0, L
( & $V \neq 0$ )

L-module V is <u>completely reducible</u> if V is a direct sum of irreducible L-modules. ($\Leftarrow\!\!\Rightarrow$ each L-submodule W has a complement $W'$: $L = W \oplus W'$ (Exercise 2) )

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- irreducible representation
- completely reducible representation

$\phi : L \to gl(V)$ is irreducible if $W \subset V$, $\phi(L)\, W \subset W \Rightarrow W = 0, V$

$\phi$ is completely reducible if $\phi = \oplus \phi_i$, $\varphi_i : L \to gl(V_i)$ irreducible rep.n

$\phi(x) = \bigoplus_i \phi_i(x) : \oplus V_i \to \oplus V_i$

$\phi(x)(v_1, \ldots, v_n) = (\phi_1(x) v_1, \ldots, \phi_n(x) v_n)$

<u>Schur's Lemma</u> If $\phi : L \to gl(V)$ is an irreducible representation, then only scalar multiples of the identity commute with elements of $\phi(L)$.

Proof (Erdmann-Wildon p. 62)

- $\lambda I_V$ commutes with $\phi(x) \in gl(V)$ $\forall x$
- let $\theta : V \to V$ satisfy $\theta\phi(x) = \phi(x)\theta$ $\forall x \in L$

let $\lambda$ be an eigenvalue of $\theta$ & $W = \ker(\theta - \lambda I) \subset V$

If $w \in W$, then $\phi(x) w \overset{?}{\in} W$ : $(\theta - \lambda I)\phi(x) w$

$= \phi(x)\theta w - \lambda\phi(x) w = \phi(x) \underbrace{(\theta - \lambda I) w}_{= 0} = 0$

$\therefore W = V$ and $\theta = \lambda I$. ∎

dual of an L-module

If $V$ is an L-module, so is $V^*$ an L-module

via $x.f(v) = -f(x.v)$ $\quad x \in L, v \in V, f \in V^*$

(M1) & (M2) are clear as for (M3)

$[xy].f(v) = -f([xy].v) = -f(x.y.v - y.x.v)$

$= -f(x.y.v) + f(y.x.v) = x.f(y.v) - y.f(x.v)$

$= -y.x.f(v) + x.y.f(v) = (x.y - y.x).f(v)$

so $[x,y].f = x.y.f - y.x.f$.

tensor product of L-modules $\quad$ If $V, W$ are L-modules

$V \otimes W$ is a vector space with basis $\{v_i \otimes w_j\}_{i,j}$

$v_1, \dots, v_n$ basis for $V$ $\quad w_1, \dots, w_m$ basis for $W$

let $L$ act on $V \otimes W$ as follows

$x.(v \otimes w) = x.v \otimes w + v \otimes x.w$ $\quad v \in V, w \in W, x \in L$

Need to verify (M3) $[xy].(v \otimes w) = x.y.(v \otimes w) - y.x.(v \otimes w)$

(calculate both sides)

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End(V) as an L-module

Exercise 1 If V is a finite dim'l vector space then

$$V^* \otimes V \cong \mathrm{End}(V)$$

via. $f \otimes v \xrightarrow{\phi} \phi(f \otimes v)$

$$\phi(f \otimes v)(w) = f(w)\,v$$

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Exercise 2 If V is an L-module via $x.v$

then $V^*$ is an L-module via $x.f(v) = -f(x.v)$

and $V^* \otimes V$ is an L-module via $x.(f \otimes v) = x.f \otimes v + f \otimes x.v$

Prove that End(V) is an L-module via

$$(x.T)(v) = x.T(v) - T(x.v)$$

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Exercise 3 If V, W are L-modules so is

Hom(V,W) (= all linear maps $V \to W$) via

$$(x.T)(v) = x.(Tv) - T(x.v)$$

## 6.2 The Casimir element of a representation.

Remark: Let $L$ be semisimple. $\phi: L \to gl(V)$ a faithful representation. Define $\beta(x,y) = Tr(\phi(x)\phi(y))$

$\beta$ is a symmetric associative bilinear form (it is the Killing form if $\phi = ad$). associative $\Rightarrow$ the radical $S$ of $\beta$ is an ideal. Of course $S \simeq \phi(S)$

claim $\phi(S)$ is solvable. $\therefore$ $S$ is solvable and $S=0$ since $L$ is semisimple

Thus $\beta$ is non-degenerate

$\beta(x,y) = 0 \quad \forall \; x \in S, y \in L$

$\Rightarrow$

$Tr(\phi(x), \phi(y)) = 0 \quad \forall \quad \phi(x) \in \phi(S), \phi(y) \in \phi(L)$

$\phi(S) \subset gl(V)$ & $Tr(AB) = 0 \quad \forall \; A \in [\phi(S), \phi(S)] \subset \phi(S)$

& $\forall B \in \phi(S)$

By Cartan's criterion $\phi(S)$ is solvable

Switch gears: Let $L$ be semi-simple and $\beta$ any non-degenerate symmetric associative bilinear form.

Fix a basis $x_1, \ldots, x_n$ for $L$ and a dual bases $y_1, \ldots, y_n$ for $L$

This means $\beta(x_i, y_j) = \delta_{ij}$.

Then for $x \in L$

$$[x, x_i] = \sum_{j=1}^{n} a_{ij} x_j$$

$$[x, y_i] = \sum_{j=1}^{n} b_{ij} y_j$$

claim $a_{ik} = - b_{ik}$

$$a_{ik} = \sum_j a_{ij}\,\beta(x_j, y_k) = \beta([x, x_i], y_k) = \beta(-[x_i, x], y_k)$$

$$= \beta(x_i, -[x, y_k]) = -\sum_j b_{kj}\,\beta(x_i, y_j) = -b_{ki}$$

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Given $L, \beta, x_i, y_i$ and a rep. $\phi : L \to gl(V)$

define $c_\phi(\beta) = \sum_{i=1}^{n} \phi(x_i)\phi(y_i') \in \mathrm{End}(V)$

claim $[\phi(x), c_\phi(\beta)] = 0$

Note that in End(V), $[x, yz] = [x,y]z + y[x,z]$

$$\left( x(yz) - yzx \stackrel{?}{=} xyz - \cancel{yxz} + \cancel{yxz} - yzx \right)$$

Therefore: $[\phi(x), c_\phi(\beta)] = \sum_i [\phi(x), \phi(x_i)\phi(y_i)]$

$$= \sum_i \left( [\phi(x), \phi(x_i)]\phi(y_i) + \phi(x_i)[\phi(x), \phi(y_i)] \right)$$

$$= \sum_i \left( \phi[x, x_i]\phi(y_i) + \phi(x_i)\phi[x, y_i] \right)$$

$$= \sum_i \left( \sum_j a_{ij}\phi(x_j)\phi(y_i) + \phi(x_i)\sum_j b_{ij}\phi(y_j) \right)$$

$$= \sum_{i,j} a_{ij}\phi(x_j)\phi(y_i) + \sum_{i,j} \underbrace{b_{ij}}_{= -a_{ji}}\phi(x_i)\phi(y_j)$$

$$= \sum_{ij} a_{ij}\phi(x_j)\phi(y_i) - \sum a_{ji}\phi(x_i)\phi(y_j) = 0$$

proving the claim

def. If $\phi : L \to gl(V)$ is a faithful rep'n with non-degenerate ! trace form $\beta(x,y) = Tr(\phi(x)\phi(y))$ and $x_1, \dots, x_n$ is a basis for $L$

$$c_\phi(\beta) = \sum_{i=1}^{n} \phi(x_i)\phi(y_i)$$ is the <u>Casimir element</u> of $\phi$.
(relative to a fixed basis)

<u>Note</u>: $Tr(c_\phi(\beta)) = \sum_i Tr(\phi(x_i)\phi(y_i)) = \sum \beta(x_i, y_i) = \dim L$

If $\phi$ is irreducible, $c_\phi(\beta) = \dfrac{\dim L}{\dim V} \cdot I_V$

by Schur's lemma (so independent of the choice of a basis) ( $c_\phi(\beta) = \lambda I_V \overset{\text{take trace}}{\Rightarrow} \dim L = \lambda \cdot \dim V$ )

<u>Example</u> $L = sl(2,F)$, $V = F^2$, $\phi(x) = x \in gl(F^2)$

$$c_\phi = \begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$$ Basis $x, h, y$

$x = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$ $h = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$ $y = \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$

dual basis: (relative to the trace form $\beta(x,y) = Tr(xy)$)

is $y, h/2, x$ so $c_\phi = xy + \frac{h^2}{2} + yx = \begin{matrix} 3/2 & 0 \\ 0 & 3/2 \end{matrix}$

$3 = \dim L$
$2 = \dim V$

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$\boxed{6.3}$

Lemma If $\phi : L \to \mathfrak{gl}(V)$ is a repn. of a ss Lie alg $L$, then $\phi(L) \subset \mathfrak{sl}(V)$. Hence $L$ acts trivially if $\dim V = 1$

$$L = [L L] \Rightarrow \phi(L) = [\phi(L), \phi(L)]$$

$$\text{if } T \in \phi(L) \quad T = \sum_i [U_i, V_i] \Rightarrow Tr(T) = 0.$$

Theorem (Weyl) A finite dimensional representation $\phi : L \to \mathfrak{gl}(V)$ of a semi-simple Lie algebra is completely reducible.

Proof ~~special case There is an submodule W of codimension 1.~~

For the proof see Humphreys pp 28–29

Better Yet go to Erdmann-Wildon pp 212–214

(stay tuned)

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## $\boxed{6.4}$ Preservation of Jordan Decomposition.

( postponed )