p. 31 §7 Representations of $sl(2,F)$ 5/16/17

$L = sl(2,F)$ $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[hx] = 2x$ $[hy] = -2y$ $[xy] = +h$.

7.1

Recall Cor 6.4 (V is an L-module)

$\phi : L \to gl(V)$ rep'n L semisimple

$x = s + n$ abstract Jordan decomp

$\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomp in $End(V)$

h is semi-simple $\Rightarrow$ h acts diagonally on V

Let $V_\lambda = \{v \in V : h.v = \lambda v\}$ $\lambda \in F$

$V = \bigoplus_{\lambda \text{ eigenvale}} V_\lambda$ $V_\lambda = 0$ if λ not an eigenvalue of $v \to h.v$

def if $V_\lambda \neq 0$ call λ a <u>weight</u> of h in V ($\Leftrightarrow$ λ is eigen vector of $v \to h.v$)

call V_λ a <u>weight space</u>

<u>Lemma</u> $x.(V_\lambda) \subset V_{\lambda+2}$, $y.(V_\lambda) \subset V_{\lambda-2}$

Proof $h.(x.v) = [h,x]v + x.(h.v) = 2x.v + x.(\lambda v)$

$= (\lambda + 2)\, x.v$

$h.(y.v) = [h,y]v + y.(h.v) = -2y.v + \lambda y.v$

Remark x and y are represented by nilpotent endomorphisms of V

If $v \in V$ $\quad v = \sum_\lambda v_\lambda \quad\quad x.v = \sum x.v_\lambda \quad\quad x.(x.v) = \sum x.(x.v_\lambda)$

$$\bigoplus_{\lambda \text{ e.v.}} V_\lambda \quad\quad \bigoplus_{\lambda \text{ e.v.}} V_{\lambda+2} \quad\quad \bigoplus_{\lambda \text{ e.v.}} V_{\lambda+4}$$

$$\underbrace{x^n . v}_{} = \phi(x)^n(v) \in \bigoplus_{\lambda \text{ e.v.}} V_{\lambda+2^n} = 0 \quad \text{for suff. large } n.$$

similar for y

Observation $\exists V_\lambda \neq 0$ such that $V_{\lambda+2} = 0$

def. v is a ~~a~~ <u>maximal vector</u> <u>of weight λ</u> if

$v \in V_\lambda$, $v \neq 0$, $x.v = 0$

$\boxed{7.2}$ Assume V is an irreducible L-module.

Let $v_0 \in V_\lambda$ be a maximal vector of weight λ

Set $v_{-1} = 0$, $v_i = \frac{1}{i!} y^i \cdot v_0$ $i = 0, 1, 2, \ldots$

Lemma (a) $h \cdot v_i = (\lambda - 2i) v_i$

(b) $y \cdot v_i = (i+1) v_{i+1}$

(c) $x \cdot v_i = (\lambda - i + 1) v_{i-1}$

$i \geq 0$

proof (a) $h \cdot v_0 = \lambda v_0$

$h \cdot v_i = \frac{1}{i!} h \cdot \underbrace{(y^i \cdot v_0)}_{\in V_{\lambda - 2i}} = \frac{1}{i!} (\lambda - 2i) y^i \cdot v_0$

$= (\lambda - 2i) v_i$

(b) $y \cdot v_i = y \cdot (\frac{1}{i!} y^i \cdot v_0) = \frac{1}{i!} y^{i+1} \cdot v_0 = (i+1) \frac{1}{(i+1)!} y^{i+1} \cdot v_0$

$= (i+1) v_{i+1}$

(c) if $i = 0$ $x \cdot v_0 = 0$ (maximal vector)

& $v_{i-1} = v_{-1} = 0$

assume (c) for $i-1$

$i x \cdot v_i = i x \cdot \left(\frac{y \cdot v_{i-1}}{i} \right) = x \cdot y \cdot v_{i-1} = [x, y] \cdot v_{i-1} + y \cdot x \cdot v_{i-1}$

$= h \cdot v_{i-1} + y \cdot x \cdot v_{i-1} \overset{(a)}{=} (\lambda - 2(i-1)) v_{i-1} + y \cdot (\lambda - i + 2) v_{i-2}$ induction

$\overset{(b)}{=} (\lambda - 2(i-1)) v_{i-1} + (i-1)(\lambda - i + 2) v_{i-1}$

$= (\cancel{\lambda} - 2\cancel{i} + \cancel{2} + i\lambda - i^2 + \cancel{2}i - \cancel{\lambda} + i - \cancel{2}) v_{i-1}$ divide by i.

Theorem Let V be an irreducible module for $L = sl(2, F)$
dim $V = m+1$

(a)
$$V = \bigoplus_{\substack{\mu \text{ e.v.} \\ \text{for } h}} V_\mu, \quad \mu = m, m-2, \ldots, -(m-2), -m$$
and $\dim V_\mu = 1$.

(b) V has a unique maximal vector (up to scalar multiple) whose weight is m (called the highest weight of V)

(c) The action of L on V with respect to a certain basis $v_0, v_1, \ldots, v_m$ is given by

$$\left.\begin{aligned} h \cdot v_i &= (m-2i)\, v_i \\ y \cdot v_i &= (i+1)\, v_{i+1} \\ x \cdot v_i &= (m-i+1)\, v_{i-1} \end{aligned}\right\} \quad i \geq 0$$

In particular $\exists$ at most one irreducible L-module (up to isomorphism) of dimension $k \geq 1$.

Proof. The non-zero v_i in Lemma 7.2 are linearly independent.

(this is with respect to a fixed maximal vector v_0 of weight λ for some fixed λ)

Since the v_i are eigenvalues for "h" corresponding to $\lambda - 2i$ (121AB text Th 5.5 p. 261) they are linearly independent.

Let $m' = \min\{ i : v_i \neq 0, v_{i+1} = 0 \} = \max\{ i : v_i \neq 0 \}$

$v_0, v_1, v_2, \ldots$ is a finite set since $\dim V < \infty$.

$v_0 \neq 0$, ~~so m exists~~

~~$v_1 \in ky.v_0$~~

If any v_i is 0 $\quad v_{i+1} = \dfrac{y.v_i}{i+1} = 0 = v_{i+2}, \ldots$

~~If $v_1 = 0$ then $v_2 = v_3 = \ldots = 0$~~

~~$m = 0$~~

~~If $v_1 \neq 0$ & $v_2 = 0$ then $v_3 = v_4 = \ldots = 0$~~

~~$m = 1$~~

So m exists

~~If $v_1 \neq 0$ & $v_2 \neq 0$~~

$v_0, v_1, v_2, \ldots, v_{m'}, v_{m'+1} = 0$

~~The~~ $\mathrm{sp}\{v_0, \ldots, v_{m'}\}$ is a subspace of V ~~and~~ invariant under L (an L-submodule of V) (by Lemma 7.2)

V irreducible $\Rightarrow$ it equals V so $m' = m = \dim V - 1$.

Relative to the basis $v_0, v_1, \ldots, v_m$

$$h \sim \begin{bmatrix} \lambda & & & & 0 \\ & \lambda-2 & & & \\ & & \lambda-4 & & \\ & 0 & & \ddots & \\ & & & & \lambda-2m \end{bmatrix}$$

diagonal

$$y \sim \begin{bmatrix} 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ 0 & 2 & 0 & & \\ \vdots & & 3 & & \\ 0 & 0 & 0 & & \end{bmatrix}$$

lower triangular nilpotent

$$x \sim \begin{bmatrix} 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda-1 & 0 \\ \vdots & & \vdots & \lambda-3 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \vdots \end{bmatrix}$$

upper triangular nilpotent

By (c) of Lemma 7.2

$$0 = X \cdot v_m = (\lambda - m)\, v_{m-1} \implies \lambda = m$$

$$h \sim \begin{bmatrix} m & & & & 0 \\ & m-2 & & & \\ & & m-4 & & \\ & & & \ddots & \\ 0 & & & & -m \end{bmatrix}$$

$$m+1 = \dim V = \sum_{\mu \text{ e.v.}} \underbrace{\dim V_\mu}_{\dim \geqslant 1} \geqslant m+1 \quad \text{so} \quad \dim V_\mu = 1 \ \& \ V_\mu \neq 0.$$

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TO BE CONTINUED

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Corollary