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(revised 6/16/17)
§7 Representations of $sl(2, F)$

$$L = sl(2, F) \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[hx] = 2x \quad [hy] = -2y \quad [xy] = +h.$$

7.1

Recall cor 6.4

(V is an L -module)

$\phi: L \rightarrow gl(V)$ rep'n L semisimple

$x = s + n$ abstract Jordan decomp

$\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomp in $End(V)$

h is semi-simple $\Rightarrow h$ acts diagonally on V

$$\text{let } V_\lambda = \{v \in V : h.v = \lambda v\} \quad \lambda \in F$$

$$V = \bigoplus_{\lambda \text{ eigenvalue}} V_\lambda \quad V_\lambda = 0 \text{ if } \lambda \text{ not an eigenvalue of } v \rightarrow h.v$$

def if $V_\lambda \neq 0$ call λ a weight of h in V ($\Leftrightarrow \lambda$ is eigen vector of $v \rightarrow h.v$)
call V_λ a weight space

Lemma $x.(V_\lambda) \subset V_{\lambda+2}, \quad y.(V_\lambda) \subset V_{\lambda-2}$

Proof $h.(x.v) = [h, x]v + x.(h.v) = 2x.v + x.(\lambda v)$
 $= (\lambda + 2)x.v$

$h.(y.v) = [h, y]v + y.(h.v) = -2y.v + \lambda y.v$ □

Remark x and y are represented by nilpotent endomorphisms of V

$$\left[\begin{array}{lll} \forall v \in V & v = \sum_{\lambda} v_{\lambda} & x.v = \sum x.v_{\lambda} \end{array} \right. \quad \begin{array}{c} \Downarrow \\ \bigoplus_{\lambda \text{ e.v.}} V_{\lambda} \end{array} \quad \begin{array}{c} \Downarrow \\ \bigoplus_{\lambda \text{ e.v.}} V_{\lambda+2} \end{array} \quad \begin{array}{c} \Downarrow \\ \bigoplus_{\lambda \text{ e.v.}} V_{\lambda+4} \end{array}$$

$$\underbrace{x^n.v}_{=}$$

$$\phi(x)^n(v) \in \bigoplus_{\lambda \text{ e.v.}} V_{\lambda+2^n} = 0 \text{ for suff. large } n.$$

similar for y

Observation $\exists V_{\lambda} \neq 0$ such that $V_{\lambda+2} = 0$

def. v is a maximal vector of weight λ if

$$v \in V_{\lambda}, v \neq 0, x.v = 0$$

7.2

Assume V is an irreducible L -module.Let $v_0 \in V_\lambda$ be a maximal vector of weight λ

$$\text{Set } v_{-1} = 0, \quad v_i = \frac{1}{i!} y^i \cdot v_0 \quad i = 0, 1, 2, \dots$$

Lemma

(a) $h \cdot v_i = (\lambda - 2i) v_i$

(b) $y \cdot v_i = (i+1) v_{i+1}$

(c) $x \cdot v_i = (\lambda - i + 1) v_{i-1}$

$$\left. \begin{array}{l} \text{(a)} \\ \text{(b)} \\ \text{(c)} \end{array} \right\} i \geq 0$$

Proof (a) ~~$h \cdot v_0 = \lambda v_0$~~ $h \cdot v_0 = \lambda v_0$ ✓

$$h \cdot v_i = \frac{1}{i!} h \cdot (y^i \cdot v_0) = \frac{1}{i!} (\lambda - 2i) y^i \cdot v_0$$

$$\underbrace{\in V_{\lambda-2i}}_{\substack{\text{(a)} \\ \text{induction}}} = (\lambda - 2i) v_i \quad \checkmark$$

$$\begin{aligned} \text{(b)} \quad y \cdot v_i &= y \cdot \left(\frac{1}{i!} y^i \cdot v_0 \right) = \frac{1}{i!} y^{i+1} \cdot v_0 = (i+1) \frac{1}{(i+1)!} y^{i+1} \cdot v_0 \\ &= (i+1) v_{i+1} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \text{if } i=0 \quad x \cdot v_0 &= 0 \quad (\text{maximal vector}) \\ &\neq v_{i-1} = v_{-1} = 0 \end{aligned}$$

assume (c) for $i-1$

$$\begin{aligned} i x \cdot v_i &= i x \cdot \left(\frac{y \cdot v_{i-1}}{i} \right) = x \cdot y \cdot v_{i-1} = [x, y] \cdot v_{i-1} + y \cdot x \cdot v_{i-1} \\ &\stackrel{\text{(a)}}{=} h \cdot v_{i-1} + y \cdot x \cdot v_{i-1} = (\lambda - 2(i-1)) v_{i-1} + y \cdot (\lambda - i + 2) v_{i-2} \\ &\stackrel{\text{(b)}}{=} (\lambda - 2(i-1)) v_{i-1} + (i-1) (\lambda - i + 2) v_{i-1} \\ &= (\cancel{\lambda - 2i + 2} + i\lambda - i^2 + \cancel{2i} - \cancel{i} + i - \cancel{2}) v_{i-1} \end{aligned}$$

divide by i . ✓

Theorem let V be an irreducible module for $L = \mathfrak{sl}(3, F)$
 $\dim V = m+1$

(a)

$$V = \bigoplus_{\substack{\mu \text{ e.v.} \\ \text{for } h}} V_{\mu}, \quad \mu = m, m-2, \dots, -(m-2), -m$$

and $\dim V_{\mu} = 1$.

(b) V has a unique maximal vector (up to scalar multiple) whose weight is m (called the highest weight of V)

(c) The action of L on V with respect to a certain basis $v_0, v_1, \dots, v_m$ is given by

$$\left. \begin{aligned} h \cdot v_i &= (m-2i)v_i \\ y \cdot v_i &= (i+1)v_{i+1} \\ x \cdot v_i &= (m-i+1)v_{i-1} \end{aligned} \right\} i \geq 0$$

In particular $\exists$ at most one irreducible L -module

(up to isomorphism) of dimension ~~k~~ $k \geq 1$.

Proof. The non-zero v_i are linearly independent. in Lemma 7.2

(this is with respect to a fixed maximal vector v_0 of weight λ for some fixed λ)

since the v_i are eigenvectors for " h " corresponding to $\lambda - 2i$ (121AB text Th 5.5 p. 261) they are linearly independent.

$$\text{let } m' = \min \{ i : v_i \neq 0, v_{i+1} = 0 \} = \max \{ i : v_i \neq 0 \}$$

$\{v_0, v_1, v_2, \dots\}$ is a finite set since $\dim V < \infty$.
 ~~$v_0 \neq 0$, ~~there exists~~ ~~some~~ ~~exists~~~~

$$v_i = y \cdot v_{i-1}$$

$$\text{if any } v_i = 0 \quad v_{i+1} = \frac{y \cdot v_i}{i+1} = 0 = v_{i+2}, \dots$$

$$\text{if } v_1 = 0 \quad \text{then } v_2 = v_3 = \dots = 0$$

$$m = 0$$

$$\text{if } v_1 \neq 0 \text{ \& } v_2 = 0 \quad \text{then } v_3 = v_4 = \dots = 0$$

$$m = 1$$

so m' exists

$$\text{if } v_1 \neq 0 \text{ \& } v_2 \neq 0$$

$$v_0, v_1, v_2, \dots, v_{m'}, v_{m'+1} = 0$$

(6)

The $\text{sp } \{v_0, \dots, v_m\}$ is a subspace of V ~~and~~ invariant under L (an L -submodule of V) (by lemma 7.2)

V irreducible $\Rightarrow$ it equals V so $m' = m = \dim V - 1$.

Relative to the basis $v_0, v_1, \dots, v_m$
 $m \times m$ by $m+1$ matrix

$$h \sim \begin{bmatrix} \lambda & & & & \\ & \lambda-2 & & & \\ & & \lambda-4 & & \\ & & & \ddots & \\ & & & & \lambda-2m \end{bmatrix}$$

where we call $v_0 \in V_\lambda$
 diagonal

$$g \sim \begin{bmatrix} 0 & 0 & 0 & & \\ 1 & 0 & 0 & & \\ 0 & 2 & 0 & & \\ \vdots & & 3 & & \\ 0 & 0 & 0 & & \\ & & & \ddots & \\ & & & & m \end{bmatrix}$$

lower triangular
 nilpotent

$$x \sim \begin{bmatrix} & v_0 & v_1 & v_2 & & \\ 0 & \lambda & 0 & 0 & & \\ 0 & 0 & \lambda-1 & 0 & & \\ \vdots & & \vdots & \lambda-2 & & \\ \vdots & & \vdots & & \ddots & \\ 0 & 0 & 0 & & & \lambda-m+1 \\ & & & & & 0 \end{bmatrix}$$

upper triangular
 nilpotent

By (c) of Lemma 7.2 (c)

(7)

$$0 = X \cdot v_m = (\lambda - m) v_{m-1} \Rightarrow \lambda = m$$

$$h \sim \begin{bmatrix} m & & & & \\ & m-2 & & & \\ & & m-4 & & \\ & & & \ddots & \\ & 0 & & & -m \end{bmatrix}$$

$$x \sim \begin{bmatrix} 0 & m & 0 & \cdots & 0 \\ 0 & 0 & m-1 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \vdots \\ 0 & 0 & & & 2 & 0 \\ & & & & 0 & 1 & 0 \end{bmatrix}$$

$$m+1 = \dim V = \sum_{\substack{\mu \text{ e.v.} \\ \dim \geq 1}} \dim V_\mu \geq m+1$$

$$\text{so } \dim V_\mu = 1 \quad \forall V_\mu \neq 0.$$

This proves (a) and (c)

Since $\dim V_m = 1$ the maximal vector is

unique up to scalar multiples, proving (b)
for uniqueness of V of fixed dimension.

Finally if V' is an irreducible L -module of dimension $m+1$

Then by (b) it has a maximal vector v'_0 of weight m ^{unique up to scalar mult.}

~~which is a scalar multiple of v_0 . Define v'_i by $v'_i = \alpha_i v_i$ for $v_i \neq 0$ as in Lemma 7.2, so $v'_i = \alpha_i v_i$ for $v_i \neq 0$.~~

So v'_i defined as in Lemma 7.2 yields a basis

$$v'_0, v'_1, \dots, v'_m$$

s.t. (c) holds with v_i replaced by v'_i

Let $\phi: V \rightarrow V'$ $\phi(v_i) = v'_i$ ϕ is a lin. isomorphism of V onto V'

Need to prove $\phi(a \cdot v) = a \cdot \phi(v) \quad \forall v \in V, a \in L$.

It suffices to prove this for $a \in \{h, x, y\}$ and $v \in \{v_0, v_1, \dots, v_m\}$

$$\text{for example } \phi(h \cdot v_i) = \phi((m-2i)v_i) = (m-2i)v'_i = h \cdot v'_i = h \cdot \phi(v_i)$$

Similarly for x, y in place of h .



Corollary Let V be any finite dimensional L -module
 $L = \mathfrak{sl}(2, F)$. Then the eigenvalues of h
 on V are all integers and each occurs with
 its negative (an equal number of times).

If $V = \bigoplus_i M_i$ where M_i is ^{an} irreducible L -module
 then the number of summands is $\dim V_0 + \dim V_1$.

Proof. If $V = M_1 \oplus M_2 \oplus \dots \oplus M_n$

then $M_i = V(m_i)$ ~~for some~~ where $m_i + 1 = \dim M_i$

and $h = \bigoplus h_i$ where h_i is the action of h on M_i

Eigenvalues of $h =$ the union of the eigenvalues of h_i
 (an easy property of direct sums).

But eigenvalues of $h_i = \{m_i, m_i - 2, \dots, -m_i + 2, -m_i\}$

This clearly implies the first statement.

For the second statement, each ~~set~~ set of eigenvalues

$\{m_i, m_i - 2, \dots, -m_i + 2, -m_i\}$ contains either 0 or 1

but not both. So the number of summands

is $\dim V_0 + \dim V_1$

