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p.35

 $L$  is a semisimple Lie alg $x = x_s + x_n$  is the abstract Jordan decomposition.If  $L$  is not nilpotent it contains  $x$  with $x_s \neq 0$ . So  $L$  contains toral subalgebras

Def. A subalg consisting only of semisimple elements

is called a toral subalgebra. Example  $\text{spf} \{x_3\}$  1-dim!Lemma A toral subalgebra of  $L$  is abelian ( $L$  is semisimple)Proof Need to show  $\text{ad}_T(x) = 0 \quad \forall \begin{cases} x \in T \\ \neq 0 \end{cases} \quad ([T, T] \stackrel{!}{=} 0)$  $\text{ad } x$  is diagonalizable since  $x$  is semisimple in  $L$ let  $a \neq 0$  be a ~~as~~ (supposed) eigenvalue of  $\text{ad } x|_T$ .

$$[xy] = ay \quad \text{for some } y \neq 0, y \in T$$

 $y$  is an eigenvector of  $\text{ad}_T x$  with eigenvalue  $a \neq 0$  ~~$\text{ad}_T y$~~   $\text{ad}_T y (x)$  is an eigenvector of  $\text{ad}_T y$  with eigenvalue 0

$$\boxed{\text{ad}_T y (\text{ad}_T y (x)) = \text{ad}_T y (-ay) = 0}$$

The eigenvectors of  $\text{ad}_T y$  span  $L$  so the eigenvectors of  $\text{ad}_T y$ 

span  $T \quad \therefore x = \sum \lambda_i x_i \quad \text{ad}_T y (x_i) = \lambda_i x_i$

 $\{x_i\}$  linearly indep. wlog

$$0 = \text{ad}_T(y) \text{ad}_T(y)x = \text{ad}_T(y) \left( \sum \lambda_i x_i \right) = \sum \lambda_i^2 x_i \Rightarrow \lambda_i = 0 \quad x=0 \quad \boxed{\leq}$$

(2)

Fix a maximal toral subalgebra  $\mathfrak{h}$  of  $L$

(exists by finite dimensionality)  
not Zorn's lemma!

example  $L = \mathfrak{sl}(n, F)$   $H = \text{diagonal matrices}$  (Exercise 1)  
of trace 0

$H$  is abelian so  $\text{ad}_L H$  is a commuting family of semisimple endomorphisms of  $L$

Remark  $L = \bigoplus_{\alpha \in H^*} L_\alpha$   $L_\alpha = \{x \in L : [h, x] = \alpha(h)x \quad \forall h \in H\}$

$$\begin{bmatrix} h_{11} & & \\ & \ddots & \\ & & h_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} h_{11}x_1 \\ \vdots \\ h_{nn}x_n \end{bmatrix} = \begin{bmatrix} \alpha(\lambda_1) \\ \vdots \\ \alpha(\lambda_n) \end{bmatrix} \quad h_{ii} = \alpha(\lambda_i) \quad \alpha \in H^*$$

Note:  $L_0 = C_L(H) \supset H$   
Lemma

def  $\Phi = \{ \alpha \in H^* : \alpha \neq 0, L_\alpha \neq 0 \}$  roots of  $L$  relative to  $H$

Root space decomposition:  $L = C_L(H) \bigoplus_{\alpha \in \Phi} L_\alpha$

Example If  $L = \mathfrak{sl}(n, F)$ , the root space decomposition corresponds to the decomposition of  $L$  given by the standard basis  $\{e_{ij} : i \neq j\} \cup \{h_i = e_{ii} - e_{i+1, i+1}\}$  (Exercise!)

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Proposition •  $[L_\alpha L_\beta] \subset L_{\alpha+\beta} \quad \forall \alpha, \beta \in H^*$

•  $x \in L_\alpha, \alpha \neq 0 \Rightarrow \text{ad } x \text{ is nilpotent}$

•  $\alpha, \beta \in H^*, \alpha + \beta \neq 0 \Rightarrow K(L_\alpha, L_\beta) = 0.$

( $L_\alpha$  is orthogonal to  $L_\beta$  w/r respect to Killing form  $K$ .)

$$\begin{aligned} \bullet \forall x \in L_\alpha, y \in L_\beta, h \in H \Rightarrow \text{ad } h [x, y] &= [[h, x], y] + [x, [h, y]] \\ &= \alpha(h) [x, y] + \beta(h) [x, y] = (\alpha + \beta)(h) [x, y] \end{aligned}$$

•  $y \in L_\alpha, y \in L = \bigoplus_{\beta \in \Phi \cup \{0\}} L_\beta$ , say  $y = \sum_\beta y_\beta \Rightarrow \text{ad } x(y) = [x, y]$

$\Leftrightarrow = \sum_\beta [x, y_\beta] \in \sum_\beta L_{\alpha+\beta}, \dots, (\text{ad } x)^k(y) \in \sum_\beta L_{\alpha+k\beta} = 0$   
for suff. large  $k$ .

• Pick  $h \in H$   $(\alpha + \beta)(h) \neq 0$ , take  $x \in L_\alpha, y \in L_\beta$

$$\begin{aligned} K([hx], y) &= -K([xh], y) \quad \left. \begin{array}{l} \text{i.e. } \alpha(h) K(x, y) = -\beta(h) K(x, y) \\ = -K(x, [\beta h, y]) \end{array} \right\} \\ &\quad (\alpha(h) + \beta(h)) K(x, y) = 0 \quad \boxed{\text{***}} \end{aligned}$$

Corollary  $K|_{L_0}$  is non-degenerate ( $L_0 = C_L(H)$ )

$K$  is non-degenerate since  $L$  is semi-simple.

By the proposition  $K(L_0, L_\alpha) = 0 \quad \forall \alpha \in \Phi \quad (\alpha + 0 = \alpha \neq 0)$

Suppose  $z \in L_0$  and  $K(z, L_0) = 0$  then  $K(z, \bigoplus_{\alpha \in \Phi \cup \{0\}} L_\alpha) = 0$

8.2

Lemma  $x, y$  commuting endomorphisms,  $y$  nilpotent  
 $\Rightarrow xy$  is nilpotent. Hence  $\text{Tr}(xy) = 0$

$$\boxed{(xy)^n = \cancel{xy \dots xy} \dots xy = x^n y^n.}$$

Trace = sum of eigenvalues = 0

Proposition If  $H$  is a maximal toral subalg. then  $H = G(H)$

Corollary  $K|_H$  is non-degenerate

$\phi \in H^*$  corresponds to  $t_\phi \in H$  such that  $\phi(h) = K(t_\phi, h) \quad (h \in H)$

$\Phi \subset H^*$  corresponds to  $\{t_\alpha : \alpha \in \Phi\} \subset H$ .

Proof of Proposition: has 7 steps. (postponed temporarily)

The map  $H \ni t \rightarrow \phi_t \in H^*$  defined by

$\phi_t(h) = K(t, h)$  is linear and is one-to-one,

since if  $\phi_t(h) = 0 \quad \forall h$ , then  $K(t, H) = 0 \Rightarrow t = 0$ .

But  $\dim H^* = \dim H$  (since  $H^* = \text{Hom}(H, F) \cong (\dim H)$  by 1 matrices)

so the map is onto

Proof of Proposition

(41)

Prove If  $H$  a maximal toral subalgebra  $\Rightarrow H = C_L(H)$   
 let  $C = C_L(H)$

(1)  $C$  contains the semisimple and nilpotent parts of its elements

$$\boxed{\forall x \in C \Rightarrow \text{ad } x(H) = 0}$$

Prop 4.2(c)  $(\text{ad } x)_s$  and  $(\text{ad } x)_n$  map  $H$  to 0

section 5.4  $\Rightarrow (\text{ad } x)_s = \text{ad } x_s$  and  $(\text{ad } x)_n = \text{ad } x_n$

$$\text{So } x \in C \Rightarrow x_s, x_n \in C.$$

(2) All semi-simple elements of  $C$  lie in  $H$ .

$\boxed{\text{If } x \text{ is ss and } x \in C \text{ then } H + Fx \text{ is an abelian subalgebra and is toral (the sum of commuting ss elements is ss)}}$

By maximality of  $H$ ,  $H + Fx = H \quad \forall x \in H.$

(3)  $\kappa|_{H \times H}$  is non degenerate

$\boxed{\text{let } \kappa(h, h) = 0 \text{ for some } h \in H \text{ To prove } h = 0.}$

claim:

$\text{Tr}(\text{ad } x \text{ ad } h) = 0 \quad \forall x \in C, x \text{ nilpotent}, \forall y \in H.$

(since  $\text{ad } x$  is nilpotent and  $[x, H] = 0$  - by the trivial lemma)

By (1) and (2),  $\kappa(h, C) = 0$ , so  $h = 0$  b/c because  $\kappa|_{C \times C}$  is non-degenerate)

steps (4) - (7)

8.3

By Prop 8.1  $K(L_\alpha, L_\beta) = 0$  if  $\alpha, \beta \in H^*$ ,  $\alpha + \beta \neq 0$

In particular  $\beta = 0, \alpha \in \Phi \Rightarrow K(H, L_\alpha) = 0 \quad \forall \alpha \in \Phi$

and  $K|_{H \times H}$  is non-degenerate

Proposition (a)  $\Phi$  spans  $H^*$

$$(b) \alpha \in \Phi \Rightarrow -\alpha \in \Phi$$

$$(c) \alpha \in \Phi, x \in L_\alpha, y \in L_{-\alpha} \Rightarrow$$

$$[xy] = K(x, y)t_\alpha$$

(where  $t_\alpha \in H$  satisfies

$$(d) \alpha \in \Phi \Rightarrow [L_\alpha, L_{-\alpha}] = \text{span}\{t_\alpha\} \text{ is 1-dimensional.} \quad \alpha(h) = K(t_\alpha, h) \quad h \in H.$$

$$(e) \alpha(t_\alpha) (= K(t_\alpha, t_\alpha)) \neq 0 \quad \forall \alpha \in \Phi$$

$$(f) \alpha \in \Phi, 0 \neq x_\alpha \in L_\alpha \Rightarrow \exists y_\alpha \in L_{-\alpha} \text{ such that}$$

with  $h_\alpha = [x_\alpha y_\alpha]$ ,  $x_\alpha, y_\alpha, h_\alpha$  span a 3-dimensional simple subalgebra of  $L$  isomorphic to  $sl(2, \mathbb{F})$

$$\text{via } x_\alpha \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y_\alpha \rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h_\alpha \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(g) h_\alpha = \frac{2t_\alpha}{K(t_\alpha, t_\alpha)} = -h_{-\alpha}.$$

( $\Phi$  spans  $H^*$ ):

Proof (a)  $t \rightarrow \hat{t} : H \rightarrow H^{**}$   $\hat{t}(\phi) = \phi(t)$

is a linear isomorphism onto. If  $\text{sp } \Phi \subsetneq H^*$

pick  $\hat{t}_0 \in H^{**}$   $\hat{t}_0(\Phi) = 0$  &  $\hat{t}_0 \neq 0$

Then  $\alpha(t_0) = 0$  &  $\alpha \in \Phi$ . and  $t_0 \neq 0$

Then  $[t_0, L_\alpha] = 0 \quad \forall \alpha \in \Phi$  But  $H$  is abelian so

$[t_0, H] = 0$  so  $[t_0, L] = 0$  &  $t_0 \in Z(L) = 0$   $\Rightarrow$  since  $L$  is semi-simple.

(b) let  $\alpha \in \Phi$ . If  $-\alpha \notin \Phi$ , i.e.  $L_{-\alpha} = 0$ , then  $\alpha + \beta \neq 0$

$K(L_\alpha, L_\beta) = 0 \quad \forall \beta \in \Phi$  and therefore

$K(L_\alpha, L_\beta) = 0 \quad \forall \beta \in H^*$  (all you need is  $\beta = 0$ ) (?)

$K(L_\alpha, L) = 0$  contradicts  $K$  non-degenerate

$\forall \beta \in \Phi$

"misprint"

in book

( $\forall \beta \in H^*$ )

no go

(c) let  $\alpha \in \Phi$ ,  $x \in L_\alpha$ ,  $y \in L_{-\alpha}$ . Then for arbitrary  $h \in H$

$$K(h, [xy]) = K([hx], y) = d(h) K(x, y) = K(t_\alpha h) \underbrace{K(x, y)}$$

$$= K(K(x, y) t_\alpha, h) = K(h, K(x, y) t_\alpha)$$

$$K(h, [xy] - K(x, y) t_\alpha) = 0 \quad \forall h \in H. \quad \text{But } K|_{H \times H}$$

$\in [L_\alpha, L_{-\alpha}] \subset L_0 = H$

is non degenerate.

$$\text{so } [xy] = K(x, y) t_\alpha \quad \mathbb{E}$$

[NOTE: this holds if  $[xy] = 0$  in which case  $K(x, y) = 0$ ]

(5'')

(d) let  $\alpha \in \Phi$  By (c) it suffices to prove that

$[L_\alpha, L_{-\alpha}] \neq 0$ . To see this let  $\underset{\neq 0}{x} \in L_\alpha$

and suppose  $K(x, L_{-\alpha}) = 0$ , By Prop 8.1  $\begin{cases} K(L_\alpha, L_\beta) = 0 \text{ if } \\ \alpha + \beta \neq 0 \\ \alpha, \beta \in H^* \end{cases}$

so  $K(x, L_\beta) = 0 \forall \beta \neq -\alpha$

so  $K(x, L) = 0 \Rightarrow x = 0$  contradiction

so  $K(x, L_{-\alpha}) \neq 0$ ,  $\exists y \in L_{-\alpha}, K(x, y) \neq 0$

and by (c) again,  $[xy] \neq 0$ .

(e) Suppose  $\alpha(t_\alpha) \equiv 0$ . Then  $[tx, x] = 0 = [tx, y]$

$\forall x \in L_\alpha, y \in L_{-\alpha}$ . From proof of (d) choose

$x, y$  such that  $[x, y] \neq 0$ , or  $K(x, y) \neq 0$

and wlog  $K(x, y) = 1$  so  $[x, y] = t_\alpha$  by (c)

let  $S$  be the subspace of  $L$  spanned by  $x, y, t_\alpha$

Then  $S$  is a 3-dimensional solvable Lie algebra

$$\begin{array}{l} \text{Tr}(ad_x ad_y) = \text{Tr}(0) = 0 \quad \text{Tr}(\underbrace{ad}_{\text{''}} t_\alpha \underbrace{ad}_{} x) = 0 = \text{Tr}(ad_x ad_y) \\ \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array} \end{array}$$

Cartan's criterion

$S$  is solvable (overkill?)  
misprint (A)

$S \cong \text{ad}_L(S) \subset \text{gl}(L)$  By Cor 4.1  $\text{ad}_L S$  is nilpotent  $\forall S \in \mathfrak{S}$

(5'')

so  $\text{ad}_L t_d$  is nilpotent. But  $\text{ad}_L t_d$  is semisimple

so  $\text{ad}_L t_d = 0$ ,  $t_d \in Z(L) = 0$  but  $t_d \neq 0$  contradiction.

(F) Given  $x_d \in L_d$  and  $y_d \in L_{-d}$  with  $K(x_d, y_d) = \frac{2}{\alpha(t_d, t_d)}$

$$\begin{aligned} \text{Let } h_d &= \frac{2t_d}{K(t_d, t_d)} = \frac{2t_d}{\alpha(t_d)}. \text{ Then } [x_d, y_d] = K(x_d, y_d)t_d \\ &= \frac{2}{\alpha(t_d, t_d)} t_d = h_d. \quad \text{Moreover} \end{aligned}$$

$$[h_d, x_d] = \frac{2}{\alpha(t_d)} [t_d, x_d] = \frac{2}{\alpha(t_d)} \alpha(t_d) x_d = 2x_d$$

$$[h_d, y_d] = \frac{2}{\alpha(t_d)} [t_d, y_d] = \frac{2}{\alpha(t_d)} (-\alpha(t_d)) y_d = -2y_d$$

$$\text{so } \text{sp}\{x_d, y_d, h_d\} \cong \text{sl}(2, \mathbb{F})$$

(g)  $t_d$  is defined by  $K(t_d, h) = \alpha(h) \quad \forall h \in \mathfrak{h}$

$$t_{-d} \dots \dots K(t_{-d}, h) = -\alpha(h)$$

$$K(t_d + t_{-d}, h) = 0 \quad \forall h \Rightarrow t_d + t_{-d} = 0$$

$$h_d = \frac{2t_{-d}}{K(t_d, t_{-d})} = \frac{-2t_d}{K(t_d, t_d)} = -h_d$$



8.4

Proposition (a)  $\alpha \in \Phi \Rightarrow \dim L_\alpha = 1$ . In particular,

~~$$S_\alpha = L_\alpha + L_{-\alpha} + H_\alpha \quad (H_\alpha = [L_\alpha, L_{-\alpha}])$$~~

and  $0 \neq x_\alpha \in L_\alpha \Rightarrow \exists y_\alpha \in L_{-\alpha}$  with  $[x_\alpha, y_\alpha] = h_\alpha$

(b)  ~~$\alpha \in \Phi$~~  &  $\lambda\alpha \in \Phi$  for  $\lambda \in \mathbb{F} \Rightarrow \lambda = \pm 1$ .

(c)  $\alpha, \beta \in \Phi \Rightarrow \beta(h_\alpha) \in \mathbb{Z}$  &  $\beta - \beta(h_\alpha)\alpha \in \Phi$   
 $\uparrow$  def. Cartan integer

(d)  $\alpha, \beta, \alpha + \beta \in \Phi \Rightarrow [L_\alpha, L_\beta] = L_{\alpha+\beta}$

(e)  $\alpha, \beta \in \Phi$ ,  $\beta \neq \pm \alpha$ ,  $r, q$  the largest integers

for which  $\beta - r\alpha, \beta + q\alpha \in \Phi \implies$

$$\beta + i\alpha \in \Phi \quad (-r \leq i \leq q) \quad \& \quad \beta(h_\alpha) = r - q$$

(f)  $L$  is generated as a Lie algebra by  $\{L_\alpha : \alpha \in \Phi\}$

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$L$  is a semisimple Lie alg over  $F$  alg. closed char 0.

$H$  is a maximal toral subalgebra

$\Phi \subset H^*$  the set of roots of  $L$  relative to  $H$

$$E_{\mathbb{Q}} = \text{sp}_{\mathbb{Q}} \Phi \subset H^*$$

$$E = R \otimes_{\mathbb{Q}} E_{\mathbb{Q}} \quad (\stackrel{?}{=} \text{sp}_R \Phi)$$

Theorem (a)  $\Phi$  spans  $E$

$$(b) \alpha \in \Phi \Rightarrow -\alpha \in \Phi \quad \& \quad \lambda \alpha \in \Phi \Rightarrow \lambda = \pm 1$$

$$(c) \alpha, \beta \in \Phi \Rightarrow \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha \in \Phi$$

$$(d) \alpha, \beta \in \Phi \Rightarrow \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}.$$