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$\boxed{2.1}$ L a Lie algebra

ideal $I \subset L$ $[I, L] \subset I$

center of L $Z(L) = \{x \in L : [x, L] = 0\}$

L is abelian $\Longleftrightarrow$ $Z(L) = L$

derived algebra $[L, L] = \underset{\text{Span}}{\{}[x, y] : x, y \in L\} \subseteq L$

(it is an ideal)

L is abelian $\Longleftrightarrow$ $[L, L] = 0$

The classical Lie algs A_ℓ, B_ℓ, C_ℓ, D_ℓ satisfy $[L, L] = L$.

(exercise 9 in Assignment 1)
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L is **simple** if $0, L$ are the only ideals, and $[L, L] \neq 0$

L is simple $\Rightarrow$ $Z(L) = 0$ and $L = [L, L]$

Example $L = sl(2, F)$ is simple

fill in the proof

p.7 If I is an ideal in L, L/I is a Lie algebra

$$[x+I, y+I] = [xy]+I$$

$$x+I + y+I = x+y+I$$

Some definitions for later use

If $K \subset L$ is a subspace, the <u>normalizer of K in L</u> is

$$N_L(K) = \{x \in L : [x, K] \subset K\}$$

$N_L(K)$ is a subalgebra of L, the largest subalgebra of L which contains K as an ideal (if K is a subalgebra to begin with)

K is <u>self-normalizing</u> if $K = N_L(K)$

If $X \subset L$ is a subset, the <u>centralizer of X in L</u> is $C_L(X) = \{x \in L : [x, X] = 0\}$. It is also a subalgebra, and for example $C_L(L) = Z(L)$.

$\boxed{2.2}$ <u>homomorphism</u> $\phi : L \to L'$

<u>monomorphism</u> $\operatorname{Ker} \phi = 0$

<u>epimorphism</u> $\operatorname{Im} \phi = L'$

$\Big\}$ <u>isomorphism</u>

Prop (a) $L/\operatorname{Ker}\phi \simeq \operatorname{Im}\phi$

(b) $I \subset J \subset L \Rightarrow \dfrac{L/I}{J/I} \simeq L/J$

(c) $(I+J)/J \simeq I/I\cap J$

"homomorphism theorems"

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<u>representation</u> of L is a homomorphism $\phi : L \to gl(V)$
(V a vector space)

example: $ad : L \to gl(L)$. $\quad$ Ker ad $= Z(L)$

If L is simple, it is isomorphic to a linear Lie algebra.

$\boxed{2.3}$ Aut(L) is the group of all automorphisms of L

<u>example 1</u> If L is a linear Lie algebra $L \subset gl(V)$
and $g \in gl(V)$ is invertible
and if $gLg^{-1} = L$

($GL(V)$ denotes the group of invertible elements in $gl(V)$)

then $x \to gxg^{-1}$ is an automorphism of L.

(every $g \in GL(V)$ defines an automorphism of $gl(V)$ and of $sl(V)$.) (see Exercise 12)

<u>example 2</u>
Suppose $x \in L$ (a Lie alg) and suppose ad x is nilpotent, i.e. $(\mathrm{ad}\, x)^k = 0$ for some $k > 0$
Then $\exp(\mathrm{ad}\, x) = I + \mathrm{ad}\, x + \frac{(\mathrm{ad}\, x)^2}{2!} + \dots + \frac{(\mathrm{ad}\, x)^{k-1}}{(k-1)!}$

<u>Proposition</u> If δ is a nilpotent derivation, then $\exp \delta$ is an automorphism (proof is on pp 8-9)

fill in the proof of the Proposition

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- ___inner automorphism___ is $\exp(\operatorname{ad} x)$ where $\operatorname{ad} x$ is nilpotent

  $\operatorname{Int} L$ = the set of inner automorphisms is a normal subgroup of $\operatorname{Aut} L$.

  $\phi(\operatorname{ad} x)\phi^{-1} = \operatorname{ad}\phi(x)$ if $\phi \in \operatorname{Aut} L$

___example___ $L = sl(2, F)$ standard basis x, y, h

$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (see Exercise 3 in Assign 1) $\operatorname{ad} x$, $\operatorname{ad} y$ are nilpotent

define $\sigma = \exp(\operatorname{ad} x) \cdot \exp(\operatorname{ad} -y) \cdot \exp(\operatorname{ad} x)$

- $\sigma(x) = -y$, $\sigma(y) = -x$, $\sigma(h) = -h$ (Exercise 10 on p. 10)
- $\sigma \circ \sigma = \mathrm{Id}_L$
- $\exp x$, $\exp(-y) \in SL(2, F)$ (:= the group of 2×2 matrices of determinant 1)
- $s :=$ ~~exp y exp~~ $(\exp x)(\exp(-y))(\exp x)$ satisfies $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and ① $z \to s z s^{-1}$ is an automorphism of L
② $s z s^{-1} = \sigma(z)$

Proposition If L is a linear Lie algebra, $L \subset gl(V)$, and $x \in L$ is nilpotent, then

$$(\exp x)\, y\, (\exp x)^{-1} = \exp(\operatorname{ad} x)(y) \quad \forall y \in L$$

fill in the proof on page 9