§ 2 Revisited

p. 6 — $sl(2,F)$ is a simple Lie algebra — 5/7/17

$$L = sl(2,F) \quad \text{basis} \quad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[x,y] = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}\ \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} - \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}\ \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} = \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} - \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} = h$$

$$[h,x] = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}\ \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} - \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}\ \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} - \begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix} = \begin{matrix} 0 & 2 \\ 0 & 0 \end{matrix} = 2x$$

$$[h,y] = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}\ \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} - \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}\ \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} = \begin{matrix} 0 & 0 \\ -1 & 0 \end{matrix} - \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} = \begin{matrix} 0 & 0 \\ -2 & 0 \end{matrix} = -2y$$

Let $I \neq 0$ be an ideal. of $\underbrace{ax+by+ch}_{w} \in I$

$$[x,w] = b[xy] + c[xh] = \underbrace{bh - 2cx}_{u} \in I$$

$$[x,u] = b[xh] = -2bx \in I$$

$$[y,w] = a[yx] + c[yh] = \underbrace{-ah + 2cy}_{v} \in I$$

$$[y,v] = -a[yh] = -2ay \in I$$

if $a \neq 0$ $\quad y \in I$; $h = [xy] \in I$, $x = \frac{1}{2}[hx] \in I$
so $I = L$

if $b \neq 0$ $\quad x \in I$, $h = [x,y] \in I$, $y = -\frac{1}{2}[hy] \in I$
so $I = L$

if $a=b=0$, $0 \neq ch \in I \Rightarrow h \in I$, $x = \frac{1}{2}[hx] \in I$, $y = -\frac{1}{2}[hy] \in I$
$I = L$ ☐

p. 8

example 1 $\quad L \subset gl(V) \quad g \in gl(V)$ invertible

$$gLg^{-1} = L \implies x \xrightarrow{\phi} gxg^{-1} \text{ is an automorphism of } L$$

$$\phi(x+y) = g(x+y)g^{-1} = gxg^{-1} + gyg^{-1} = \phi(x) + \phi(y)$$

$$\phi([x,y]) = g(xy-yx)g^{-1} = (gxg^{-1})(gyg^{-1}) - (gyg^{-1})(gxg^{-1})$$
$$= [\phi(x), \phi(y)].$$

Proposition If δ is a nilpotent derivation of a Lie algebra L then $\exp \delta := I + \delta + \frac{\delta^2}{2!} + \cdots + \frac{\delta^{k-1}}{(k-1)!}$ ($\delta^k = 0$) is an automorphism. In particular, if $\operatorname{ad} x$ is nilpotent, then $\exp(\operatorname{ad} x)$ is an automorphism.

proof. First verify the Leibniz rule for derivation

$$\frac{\delta^n}{n!}(xy) = \sum_{i=0}^{n} \left(\frac{1}{i!}\right) \delta^i(x) \frac{1}{(n-i)!} \delta^{n-i}(y)$$

Note that, for simplicity of notation, we are using xy for $[x,y]$.

$n=1$ is the definition of derivation

assume true for n.

$$\frac{\delta^{n+1}}{(n+1)!}(xy) = \frac{\delta}{n+1}\left(\frac{\delta^n}{n!}(xy)\right) = \frac{1}{n+1}\left(\sum_{i=0}^{n}\left(\frac{1}{i!}\right)\left(\frac{1}{(n-i)!}\right)\left(\delta^i(x)\,\delta^{n-i+1}(y) + \delta^{i+1}(x)\,\delta^{n-i}(y)\right)\right)$$

$$\stackrel{?}{=} \sum_{i=0}^{n+1}\left(\frac{1}{i!}\right)\delta^i(x)\left(\frac{1}{(n+1-i)!}\right)\delta^{n+1-i}(y)$$

look at $\frac{1}{n+1} \sum_{i=0}^{n} \left(\frac{1}{i!}\right)\left(\frac{1}{(n-i)!}\right)\left(\delta^{i}(x)\,\delta^{n-i+1}(y) + \delta^{i+1}(x)\,\delta^{n-i}(y)\right)$

$$= \frac{1}{n+1}\left(\sum_{i=0}^{n} \left(\frac{1}{i!}\right)\left(\frac{1}{(n-i)!}\right) \delta^{i}(x)\,\delta^{n-i+1}(y) + \underbrace{\sum_{i=0}^{n} \left(\frac{1}{i!}\right)\left(\frac{1}{(n-i)!}\right) \delta^{\overbrace{i+1}^{j}}(x)\,\delta^{n-i}(y)}_{\displaystyle \| } \right)$$

$$\underbrace{\sum_{j=1}^{n+1} \left(\frac{1}{(j-1)!}\right)\left(\frac{1}{(n-j+1)!}\right) \delta^{j}(x)\,\delta^{n-j+1}(y)}_{\displaystyle \|}$$

$$\sum_{i=1}^{n+1} \left(\frac{1}{(i-1)!}\right)\left(\frac{1}{(n-i+1)!}\right) \delta^{i}(x)\,\delta^{n-i+1}(y)$$

$$= \frac{1}{n+1}\left(\frac{1}{n!} \times \delta^{n+1}(y) + \sum_{i=1}^{n} \left(\frac{1}{i!}\right)\left(\frac{1}{(n-i)!}\right) \delta^{i}(x)\,\delta^{n-i+1}(y) + \frac{1}{(i-1)!}\,\frac{1}{(n-i+1)!}\,\delta^{i}(x)\,\delta^{n-i+1}(y) \right.$$

$$\left. + \frac{1}{n!}\,\delta^{n+1}(x)\,y \right)$$

$$= \frac{1}{n+1}\left(\frac{1}{n!} \times \delta^{n+1}(y) + \sum_{i=1}^{n} \left(\left(\frac{1}{i!}\right)\left(\frac{1}{(n-i)!}\right) + \frac{1}{(i-1)!}\,\frac{1}{(n-i+1)!} \right) \delta^{i}(x)\,\delta^{n-i+1}(y) \right.$$

$$\left. + \frac{1}{n!}\,\delta^{n+1}(x)\,y \right)$$

It remains to prove

$$\frac{1}{n+1}\left(\left(\frac{1}{i!}\right)\left(\frac{1}{(n-i)!}\right) + \frac{1}{(i-1)!}\,\frac{1}{(n-i+1)!} \right) \stackrel{?}{=} \left(\frac{1}{i!}\right)\frac{1}{(n+1-i)!}$$

$$\frac{1}{n+1}\left(\frac{n+1-i \;+\; i}{i!\,(n+1-i)!} \right)$$

IT WORKS !

$\exp\delta$ is obviously a linear transf. Suppose $\delta^k = 0$.

$$\exp\delta x \cdot \exp\delta(y) = \left(\sum_{i=0}^{k-1} \frac{\delta^i(x)}{i!}\right)\left(\sum_{j=0}^{k-1} \frac{\delta^j(y)}{j!}\right) \overset{?}{=} \sum_{n=0}^{2k-2}\left(\sum_{i=0}^{n} \frac{\delta^i(x)}{i!}\frac{\delta^{n-i}(y)}{(n-i)!}\right)$$

in different notation

$$\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{j=1}^{n} y_j\right) = \sum_{1\le i,j\le n} x_i y_j = \sum_{k=2}^{2n}\sum_{i+j=k} x_i y_j$$

$$= \sum_{k=2}^{2n}\sum_{i=1}^{k} x_i y_{k-i}$$

$$\begin{array}{ccccc} x_1y_1 & x_1y_2 & x_1y_3 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & x_2y_3 & & x_2y_n \\ x_3y_1 & x_3y_2 & x_3y_3 & & x_3y_n \\ \vdots & & & & x_{n-1}y_n \\ x_ny_1 & & & x_ny_{n-1} & x_ny_n \end{array}$$

Leibniz

$$= \sum_{n=0}^{2k-2} \frac{\delta^n(xy)}{n!} = \sum_{n=0}^{k-1} \frac{\delta^n(xy)}{n!} + \sum_{n=k}^{2k-2} 0 = \exp\delta(xy)$$

so $\exp\delta$ is a homomorphism.

Let $\eta = \exp\delta - 1 = \delta + \frac{\delta^2}{2!} + \cdots + \frac{\delta^{k-1}}{(k-1)!}$ Then $\eta^k = 0$

and $\exp\delta\,(1-\eta+\eta^2-\eta^3+\cdots+(\pm1)\eta^{k-2} + (\mp)\eta^{k-1})$

$= (1+\eta)(\text{— — — — — — — — .})$

$= 1-\eta+\eta^2-\eta^3 \cdots\cdots \pm\eta^{k-2} \mp \eta^{k-1}$

$\quad +\eta-\eta^2+\eta^3 \cdots\cdots \pm\eta^{k-1} \mp\eta^k = 1$

This proves the Proposition on p. 2

so $\exp\delta$ has inverse $(1-\eta+\eta^2 - \cdots \pm\eta^{k-2} \mp \eta^{k-1})$

p. 9

<u>def</u> inner automorphism $:=$ exp (inner derivation)

Int L = all inner automorphism is a <u>normal</u> subgroup of Aut(L):

if $\phi \in \mathrm{Aut}(L)$, $x \in L$

$$\phi(\mathrm{ad}\,x)(y) = \phi([x,y]) = [\phi(x), \phi(y)]$$

$$(\mathrm{ad}\,\phi(x))\,\phi(y) = [\phi(x), \phi(y)]$$

so $\phi \cdot \mathrm{ad}\,x \cdot \phi^{-1} = \mathrm{ad}\,\phi(x)$

<u>example</u> $L = sl(2,F)$ standard basis x, y, h

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

let $\sigma = \exp(\mathrm{ad}\,x) \cdot \exp(\mathrm{ad}(-y)) \cdot \exp(\mathrm{ad}\,x) \in \mathrm{Int}\,L$

(Int L is a group!)

$(\mathrm{ad}\,x)^3 = 0 \quad (\mathrm{ad}\,y)^3 = 0$

$\exp \mathrm{ad}\,x \;(x) = x$

$$\exp \mathrm{ad}\,{-y}\;(x) = \left(1 - \mathrm{ad}\,y + \frac{(\mathrm{ad}\,y)^2}{2}\right)x = x - [y,x] + \frac{[y,[y,x]]}{2}$$

$$= x - h - \frac{[y,h]}{2} = x - h - y$$

$$\sigma(x) = \left(1 + \mathrm{ad}\,x + \frac{(\mathrm{ad}\,x)^2}{2}\right)(x - h - y) = x - h - y - [x,h] - [x,y]$$

$$- \tfrac{1}{2}[x,[x,h]] - \tfrac{1}{2}[x,[x,y]] = x - h - y - 2x + h + x$$

$$= -y$$

$\therefore \sigma(x) = -y$. (similarly $\sigma(y) = -x \quad \sigma(h) = -h$)

$$x^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \exp x = 1 + x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$y^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \exp y = 1 + y = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

determinants $= \underline{1}$

$$s = \exp x \exp -y \exp x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

invertible. $s^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$z \longrightarrow s z s^{-1}$ is an automorphism of L

By page 8 need to know $s\, sl(2,F)\, s^{-1} = sl(2,F)$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c & -a \\ -a & -b \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -a & -c \\ -b & a \end{pmatrix} \checkmark$$

$$s x s^{-1} \overset{?}{=} \sigma(x) = -y$$
$$s y s^{-1} \overset{?}{=} \sigma(y) = -x$$
$$s h s^{-1} \overset{?}{=} \sigma(h) = -h$$

$$sxs^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -y \checkmark$$

$$sys^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -x \checkmark$$

$$shs^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -h \checkmark$$

Proposition $L \subset gl(V)$ a linear Lie alg, $x \in L$ nilpotent

$$\Rightarrow (\exp x)\, y\, (\exp x)^{-1} = (\exp \operatorname{ad} x)(y)$$

$$\exp x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{k-1}}{(k-1)!} \qquad x^k = 0$$

$$\exp \operatorname{ad} x = I + \operatorname{ad} x + \frac{(\operatorname{ad} x)^2}{2!} + \cdots$$

better to follow the proof in the book.

$$\operatorname{ad} x (y) = xy - yx = \lambda_x y - \rho_x y \qquad \lambda_x y = xy \qquad \rho_x y = yx$$

so $\operatorname{ad} x = \lambda_x + \rho_{-x}$ $\quad \lambda_x \rho_y = \rho_y \lambda_x$ and are nilpotent if x is.

$$\exp \operatorname{ad} x = \exp(\lambda_x + \rho_{-x}) \overset{?}{=} \exp(\lambda_x) \exp \rho_{-x} \overset{?}{=} \lambda_{\exp x}\, \rho_{\exp(-x)}$$

$\boxed{\text{first ?}}$ if $TS = ST$ $\qquad$ This proves the proposition!

$$\exp(T+S) = 1 + T + S + \frac{(T+S)^2}{2!} + \cdots + \frac{(T+S)^{k-1}}{(k-1)!}$$

$$= 1 + T + S + \frac{T^2 + 2ST + S^2}{2!} + \cdots + T^{k-1} + \binom{k-1}{k-2} T^{k-2} S + \cdots + \binom{k-1}{1} S^{k-2} + S^{k-1}$$

$$(\exp T)(\exp S) = \left(1 + T + \frac{T^2}{2!} + \cdots \frac{T^{k-1}}{(k-1)!}\right)\left(1 + S + \frac{S^2}{2!} + \cdots\right)$$

$=$

If $xy = yx$

$$e^{x+y} = 1 + x + y + \frac{(x+y)^2}{2!} + \frac{(x+y)^3}{3!} + \cdots$$

$$e^x e^y = \left(1 + x + \frac{x^2}{2!} + \cdots\right)\left(1 + y + \frac{y^2}{2!} + \cdots\right)$$

$$= \cancel{1} + \cancel{y} + \cancel{\frac{y^2}{2!}} + \cancel{\frac{y^3}{3!}} + \cancel{\frac{y^4}{4!}} + \frac{y^5}{5!} + \cdots$$

$$+ \cancel{x} + \cancel{xy} + \cancel{\frac{xy^2}{2!}} + \cancel{\frac{xy^3}{3!}} + \frac{xy^4}{4!} + \frac{xy^5}{5!} + \cdots$$

$$+ \cancel{\frac{x^2}{2!}} + \cancel{\frac{x^2}{2!}y} + \cancel{\frac{x^2}{2!}\frac{y^2}{2!}} + \frac{x^2}{2}\frac{y^3}{3!} + \frac{x^2}{2}\frac{y^4}{4!} + \frac{x^2}{2}\frac{y^5}{5!} + \cdots$$

$$+ \cancel{\frac{x^3}{3!}} + \cancel{\frac{x^3}{3!}y} + \frac{x^3}{3!}\frac{y^2}{2!} + \cancel{\frac{x^3}{3!}\frac{y^3}{3!}} + \frac{x^3}{3}\frac{y^4}{4!} + \cdots$$

$$e^{x+y} = \cancel{1} + \cancel{x} + \cancel{y} + \frac{\cancel{x^2} + \cancel{2xy} + \cancel{y^2}}{2!} + \frac{\cancel{x^3} + 3x^2y + \cancel{3xy^2} + \cancel{y^3}}{3!} + \cdots$$

$$\frac{x^4 + \cancel{\overset{=4}{\binom{4}{1}}x^3y} + \cancel{\overset{=6}{\binom{4}{2}}x^2y^2} + \cancel{\overset{=4}{\binom{4}{3}}xy^3} + \cancel{y^4}}{4!} + \cdots$$

this only works if $xy = yx$

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{24}{4} = 6$$

$4! = 24$

This "proves" the first ? on p. 7

As for the second ?:

$$\exp \lambda x = I + \lambda x + \frac{(\lambda x)^2}{2!} + \frac{(\lambda x)^3}{3!} + \cdots$$

$$= I + \lambda x + \lambda\frac{x^2}{2!} + \lambda\frac{x^3}{3!} + \cdots$$

$$= \lambda \exp x \quad \blacksquare$$