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INNER DERIVATIONS IN A JORDAN-BERNSTEIN ALGEBRA

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Introduction

The aim of this paper is to study inner derivations in a Jordan - Bernstein algebra. The study of derivations in genetics algebras has been done recently in quite a lot of papers. References [2] , [3] , [6] and [10] are examples of them.

In his paper [12] Schafer studies inner derivations of non-associative algebras. If A is a non-associative algebra over a field K, $\mathfrak{T}(A)$ denotes the Lie transformation algebra, that is, the Lie algebra generated by the right and the left multiplications of A. So, a derivation D of A is an inner derivation if $D \in \mathfrak{T}(A)$.

Schafer also studies in the above mentioned paper, the cases associative, Lie,alternative and Jordan. He gets:

i) Associative case: $\mathfrak{T}(A) = R(A) + L(A)$. If A is an associative algebra without absolute right (left) zero divisors, then D is an inner-derivation if and only if $D = R_d - L_d$.

ii) Lie case: $\mathfrak{T}(A) = R(A)$. D is an inner-derivation of a Lie algebra if and only if $D = R_d$.

iii) Alternative case (char $K \neq 2$): $\mathfrak{T}(A) = R(A) + L(A) + [\, L(A),R(A)\,]$.

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iv) Jordan case (char $K \neq 2$): $\mathfrak{I}(A) = R(A) + [\ R(A),R(A)\ ]$. If A has unit element 1, then D is an inner-derivation if and only if $D = \sum [R_{x_i}, R_{z_i}]$.

Jacobson in [8] proves that if A is a semisimple associative (Lie, alternative or Jordan) algebra over K (char $K \neq 2$), then all the derivations of A are inner.

Schenkman proves (see [9]) that every nilpotent Lie algebra over K has a derivation which is not inner.

We study in this paper inner-derivations of Jordan-Bernstein algebras. These algebras are very closed to be nilpotent Jordan algebras. They have a nilpotent ideal , the kernel of their weight homomorphism, of codimension one. We can see that in many cases the Jordan-Bernstein algebras have a derivation which is not inner. It is naturel to conjecture that this is true in any case.

Previous results

DEFINITION : A commutative algebra over a field K, char $K \neq 2$, is called a Bernstein algebra if A is a baric algebra (that is, there is a non-zero homomorphism $\omega: A \longrightarrow K$, "weight homomorphism") satisfying the identity $(x^2)^2 = \omega(x)^2\ x^2\ \ \forall x \in A$.

It is known (see [13]) that w is unique, Ker w is an ideal of A of codimension one, A has always an idempotent element and if e is a such idempotent ($0 \neq e = e^2$), then $\omega(e) = 1$ and A has a Peirce's decomposition $A = Ke \dot{+} U_e \dot{+} Z_e$, where $U_e = \{\ x \in \operatorname{Ker} \omega\ :\ ex = \frac{1}{2}x\ \}$, $Z_e = \{\ z \in \operatorname{Ker} \omega :\ ze = 0\ \}$. The following identities are verified: $U_e U_e \subset Z_e$, $U_e Z_e \subset U_e$ and $Z_e Z_e \subset U_e$. For other properties of a Bernstein algebra see [1] and [13].

If $D: A \longrightarrow A$ is a derivation, that is a linear application satisfying $D(xy) = D(x)y + xD(y)\ \forall\, x,y \in A$, then it is easily proved that $D(e) \in U_e$ and that there are two linear applications $f: U \longrightarrow U$, $g: Z \longrightarrow Z$ such that

(*) $D(u) = f(u) + 2D(e)u$, $D(z) = -2D(e)z + g(z)\quad \forall\, u \in U ,\ z \in Z$.

Since D is a derivation, the linear applications f,g must verify the following relations:

(I) $g(uu') = f(u)\, u' + u\, f(u') \quad \forall\, u,u' \in U$.

(II) $f(uz) = f(u)\, z + u\, g(z) + 2(\, D(e)u\,)z \quad \forall\, u \in U,\ \forall\, z \in Z$.

(III) $f(zz') = g(z)\, z' + z\, g(z') - 2(\, (zD(e))z' + z(D(e)z')\,) \quad \forall\, z,z' \in Z$.

So, every derivation D of a Bernstein algebra defines and is defined by a triplet ($u^* = D(e)$, f , g) with D, f and g related by (*) and f, g and u^* satisfying relations (I), (II) and (III).

It was proved in [5] that a Bernstein algebra $A = Ke \dot{+} U_e \dot{+} Z_e$ is Jordan (A is a Jordan-Bernstein algebra) if and only if $z^2 = 0 = (uz)z \quad \forall\, u \in U\ ,\ \forall\, z \in Z$ and if and only if $Z_f^{\,2} = 0 \quad \forall\ f = f^2 \in A$.

Inner Derivations in a Jordan-Bernstein algebra

Using the above mentioned results it is clear that in Jordan-Bernstein algebras condition (III) is always satisfied for every pair (f , g) of linear applications $f: U \longrightarrow U$ and $g: Z \longrightarrow Z$, and consequently only (I) and (II) must be considered. By the above mentioned Schafer's result, $Inn(A) = \mathfrak{D}er(A) \cap \mathfrak{T}(A)$ and $\mathfrak{T}(A) = R(A) + [R(A),R(A)]$. But $[R_x,R_y]$ is always a derivation in a Jordan-Bernstein algebra, so $D \in Inn(A)$ if and only if $D = R_y + \Sigma\, [R_{x_i}\, ,\, R_{x'_i}]$ and R_y is a derivation.

But $R_y \in Der(A)$ implies that $y \in Ker\omega$ (see [3]), that is, $\exists u \in U_e$, $z \in Z_e$ with $y = u + z$. $Ry \in Der(A) \Leftrightarrow uZ_e = 0 = z(U_eZ_e)$.

But $uZ_e = 0$ implies that $R_u = [R_{2e}, R_u]$, so we can writte $Inn(A) = \{R_z + \Sigma\, [R_{x_i},R_{y_i}] \ :\ z \in Z\ ,\ z(UZ) = 0\ ,\ x_i, y_i \in A\ \}$.

Note : R_z is not generally a derivation , however $R_zR_{z'}$ is always a derivation $\forall\, z,z' \in Z$ because $R_zR_{z'} = -R_{z'}R_z = \frac{1}{2}[R_z\, ,\, R_{z'}] \in Inn(A)$.

A Jordan-Bernstein algebra is a Z_2 graduated algebra, $A = A_0 \dot{+} A_1$ with $A_1 = U$ and $A_0 = K(e) + Z$ ($A_1 A_1 \subset A_0$, $A_0 A_0 \subset A_0$, $A_1 A_0 \subset A_1$) . So also is $Der(A) = \mathfrak{D}_0\,(a) + \mathfrak{D}_1(A)$ (direct sum as vector spaces) with $\mathfrak{D}_0(A) = \{\, D \in Der(A) : D(A_0) \subset A_0\ ,\ D(A_1) \subset A_1\, \}$ and $\mathfrak{D}_1(A) = \{\, D \in Der(A) : D(A_0) \subset A_1\ ,\ D(A_1) \subset A_0 \}$.

But if $D \in \mathcal{D}_0(A)$, then $D(e) \in A_0 \cap A_1 = 0$, therefore $D(e) = 0$. Reciprocally, if $D(e) = 0$ then $\frac{1}{2}D(u) = D(eu) = D(e)u + eD(u) = eD(u)$, so $D(u) \in U \ \forall u \in U$ and similarly $0 = D(ez) = eD(z)$, that is, $D(z) \in Z \ \forall z \in Z$.

Consequently $\mathcal{D}_0(A) = \{ D \in Der(A) : D(e) = 0 \}$.

If $D \in \mathcal{D}_1(A)$, then $D(Z) \subset U$ and $D(U) \subset Z$. This implies that $D(u) = 2D(e)u$ and $D(z) = -2D(e)z$, that is, D is defined by D(e). This means that we have an isomorphism, as vector spaces, between U and $\mathcal{D}_1(A)$, because each element u' in U defines a unique derivation by $D(e)= u'$, $D(u) = 2u'u$ and $D(z) = -2u'z$. It is clear that $\dim_K \mathcal{D}_1(A) = \dim_K U = r$. It is also clear that $\mathcal{D}_1(A) \subset Inn(A)$. In fact, if $D_{u'}$ is the derivation in $\mathcal{D}_1(A)$ such that $D(e) = u'$, then $D_{u'} = [R_{4e}, D_u] \in Inn(A)$. So $Inn(A) = \mathcal{D}_1(A) \dot{+} (\mathcal{D}_0(A) \cap Inn(A))$.

In conclusion, a Jordan-Bernstein algebra will have a derivation not inner if and only if there is a derivation in $\mathcal{D}_0(A)$ which is not inner.

If A is a Jordan-Bernstein algebra, then $B = Ker\ \omega$ is a nil algebra ($x^3 = 0 \ \forall x \in Ker\omega$). So it is known that $Ker\ \omega$ is a nilpotent Jordan algebra. That is, there is an m in Z such that $B^m = 0$.

Since $D(A) \subset B$ for every $D \in Der(A)$, it is clear that every inner derivation is nilpotent. So, if $D \in \mathcal{D}_0$ is defined by the pair (f, g), it is clear that zero is the only eigenvalue of f and g.

It is also clear that if we define $UZ^1 = UZ$ and recursively $UZ^{i+1} = (UZ^i)Z$, then we get a strict chain $U \supset UZ \supset UZ^2 \supset UZ^3 \supset \dots$ and there is an l such that $UZ^l \neq 0$ and $UZ^{l+1} = 0$.

Also there is always an $0 \neq x \in Ker\ \omega$ such that $xKer\ \omega = 0$. If $x = u + z$, $u \in U$, $z \in Z$, then it is easy to see that $uU = 0 = uZ$ and $zU = 0$. Consequently in a Jordan-Bernstein algebra there is always either an element u in U with $uKer\ \omega = 0$ or an element z in Z with $zKer\ \omega = 0$.

We can notice that if $s = \dim Z$, then $UZ^{s+1} = 0$ always, by using $(uz)z' = -(uz')z$ for every u in U and z,z' in Z.

CASE $UZ = 0$

We can notice in this case that $R_z = 0 \ \forall z \in Z$, so $\mathrm{Inn}(A) = \{ \sum [R_{x_i}, R_{x_j}] : x_i, x_j \in A \}$.

If $x = \alpha e + u + z$, $x' = \alpha' e + u' + z'$ and $D = [R_x, R_{x'}]$, then $eD = \frac{1}{4}(\alpha u' - \alpha' u)$, $u^* D = \frac{1}{2}(\alpha u' - \alpha' u) u^* = 2D(e) u^*$ and $z^* D = 0$, so $D \in \mathcal{D}_1(A)$. That is, in this case $\mathcal{D}_1(A) = \mathrm{Inn}(A)$ and $\mathrm{Inn}(A) \cap \mathcal{D}_0 = 0$.

As we have before seen, every D in $\mathcal{D}_0$ is associated with a pair (f,g), $f : U \longrightarrow U$, $g : Z \longrightarrow Z$ linear applications satisfying (I) and (II). But in this case ($UZ = 0$) (II) is trivially satisfied, so it suffices to look for a pair $(f,g) \neq (0,0)$ satisfying (I) to find a derivation in $\mathcal{D}_0$. It is clear that $(1_U, 2\,1_Z)$ satisfies (I), so defines a derivation in $\mathcal{D}_0$, that clearly is not inner.

COROLLARY :If A is a Jordan-Bernstein algebra with $U_e Z_e = 0$ (for any idempotent element e), then A has a derivation which is not inner.

As we have seen, $\dim_K \mathrm{Inn}(A) = r$ in this case and $\dim_K \mathrm{Der}(A) = r + r^2 + s^2$ if $U^2 = 0$ and $\dim K\, \mathrm{Der}(A) < r + r^2 + s^2$ if $U^2 \neq 0$.

If $s' = \dim_K U^2$ (where U^2 denotes the verctor space spanded by the set of products of two elements in U) then $1 \leq \dim_K \mathcal{D}_0(A) \leq r^2 + (s-s')s$ and both bounds can be reached.

If $\{ u_1, \ldots, u_r \}$ is a basis of U with $\{u_i u_j : i \leq j \}$ $\frac{1}{2}r(r+1)$ linearly independent vectors in Z ($s \geq \frac{1}{2}r(r+1)$), then $\dim \mathcal{D}_0(A) = r^2 + s(s-s')$ ($s' = \frac{1}{2}r(r+1)$) because f can be every linear application of U^2 and g is fixed over U^2 by f and can associate any element of Z to a basis of a supplementary space of U^2 in Z.

It is clear that $\dim \mathcal{D}_0(A) = 1$ if $\dim U = \dim Z = 1$.

If $A = K(e) + K(u_1, u_2, u_3) + K(z_1, z_2)$ with $UZ = Z^2 = 0$, $u_1^2 = u_2^2 = u_3^2 = 0$, $u_1 u_2 = z_1$, $u_1 u_3 = z_2$, $u_2 u_3 = z_1 + z_2$, then it can be seen that $\dim K\, \mathcal{D}_0(A) = 1$ with basis the derivation defined by the pair $(1_U, 2\,1_Z)$.

Notice that clearly $\dim \mathcal{D}_0(A) = 1$ implies that $Z = U^2$, because if $Z \neq U^2$ there is always a pair $(0, g)$ defining another derivation in $\mathcal{D}_0(A)$.

In [4] the homotope algebra of a Jordan-Bernstein algebra is studied and it is seen that the homotope of a Jordan-Bernstein algebra is again a Jordan-Bernstein algebra. It is also seen that there is always an homotope algebra in which $U_e Z_e = 0$. So we have :

COROLLARY : Every Jordan algebra has an homotope algebra which has derivations not inner.

CASE $U^2 = 0$

We can suppose $UZ \neq 0$, because in other case it is known by the previous one. In this case, if $x = \alpha e + u + z$, $x' = \alpha' e + u' + z'$, then $D = [R_x, R_{x'}]$ acts in the following way: $D(e) = \frac{1}{4}(\alpha u' - \alpha' u) + \frac{1}{2}(z'u - zu')$, $D(u^*) = 2(zu^*)z'$ and $D(z^*) = -\frac{1}{2}(\alpha u' - \alpha' u)z^* - (z'u - zu')z^*$. It is clear now that $D = R_{u\#} + 2\, R_{z'} R_z$ where $u\# = 2D(e)$, and clearly the pair $(1_U, 0_Z)$ defines a derivation in A which is not inner.

COROLLARY : If A is a Jordan-Bernstein algebra with $U_e = 0$ (for any idempotent element e) then A has a derivation D which is not inner. $\mathrm{Inn}(A) = R_U(A) + \{ R_z + \Sigma R_{z_i} R_{z_j}$ with $(Uz)Z = 0 \}$, $\mathcal{D}_1(A) = R_U(A)$ and $\mathcal{D}_0(A) \cap \mathrm{Inn}(A) = \{ R_z + \Sigma R_{z_i} R_{z_j} \}$.

If $UZ^i \neq 0 = UZ^{i+1}$, it is clear that $\dim UZ^{i-1} \geq 2$ and $D(u) = 0\ \forall\, u \in UZ^{i-1}$ and $\forall\, D \in \mathcal{D}_0(A) \cap \mathrm{Inn}(A)$, $D(Z) = 0$ and $D(U) \subset UZ$. Consequently $\dim \mathcal{D}_0(A) \cap \mathrm{Inn}(A) \leq (r-1)(r-2)$. Sumarizing:

i) If $UZ = 0$, $\dim \mathrm{Inn}(A) = \dim \mathcal{D}_1(A) = r$.

ii) If $UZ \neq 0 = UZ^2$, then $\dim \mathcal{D}_0(A) \cap \mathrm{Inn}(A) \leq (r-r')r'$, where $r' = \dim UZ < r$ and $\dim \mathcal{D}_0(A) \cap \mathrm{Inn}(A) = \dim \{ R_z : z \in Z \} \leq s$, that is, $r < \dim \mathrm{Inn}(A) \leq r + \min(s, (r-r')r')$.

iii) If $UZ^i \neq 0 = UZ^{i+1}$ with $i \geq 2$, then $r \leq \dim \mathrm{Inn}(A) \leq r + (r-1)(r-2)$.

GENERAL CASE

In the general case, if $UZ^i \neq 0 = UZ^{i+1}$, a pair (f, g) defining a derivation $D \in \mathcal{D}_0(A) \cap \mathrm{Inn}(A)$ satisfies $f(U) \subset UZ$, $f(UZ^i) = 0$ and $g(Z) \subset U^2$.

Example 1 : If there is an element $u^* \in U - UZ$ with $u^*U = 0 = u^*Z$, then we can complete $\{ u_1,...,u_{r'} \}$ a basis of UZ to a basis of U with u^* and the needed elements : $\{u_1,...,u_{r'}, u^*,...\}$ and define the linear application $f : U \longrightarrow U$ with $f(u^*) = u^*$ and $f(u_i) = 0$ $\forall u_i \neq u^*$ in the above basis. Then $\mathrm{Im}\, f \subset K(u^*)$, so $f(u)Z = f(u)U = 0$ for every u in U and $f(UZ) = 0$. Clarly the pair $(f, g = 0_Z)$ defines a derivation D in $\mathcal{D}_0(A)$ which is not inner.

COROLLARY : Let A be a Jordan- Bernstein algebra. Then there is a Jordan-Bernstein algebra A^* containing A as ideal of codimension one and having derivations not inner.

Proof:

Let $A = Ke + U + Z$ and $A^* = A \oplus K(u^*)$ with the product given by: $eu^* = \frac{1}{2}u^*$, $u^*U = 0 = u^*Z = u^{*2}$ and the product of elements of A the same as the product in A . So $A^* = K(e) + U' + Z'$ with $Z' = Z$, $U' = U + K(u^*)$, is again a Jordan-Bernstein algebra, and $u^*U' = u^*Z' = 0$, $u^* \notin U'Z'$. By using the above example, A^* has derivations which are not inner.

Example 2 : If $U^3 = 0$ (that is, $u_1(u_2u_3) = 0$ for every u_1,u_2,u_3 in U) then $U^2 \not\subset Z$ or $UZ = 0$. In the second case we know that the Jordan-Bernstein algebra has a derivation not inner. In the first one, let us suppose $\{ z_1, ... , z_t \}$ is a basis of U^2. It can be extended to a basis of Z : $\{ z_1,...,z_t, ..., z_s \}$. We define $f = \frac{1}{2}1_U$ and $g : Z \longrightarrow Z$ the linear application given by $g(z_i) = z_i$ $i = 1,...,t$ and $g(z_j) = 0$ $j = t+1,...,s$. Then (f,g) defines a derivation in $\mathcal{D}_0(A)$ which is not inner.

Other bounds for the derivation algebra

LEMMA: $\dim \mathrm{Inn}(A) \leq r^2 + s^2 - s$.

Proof : We have seen that a pair (f , g) defining a derivation in $\mathcal{D}_0(A) \cap \mathrm{Inn}(A)$ satisfies $f(U) \subset UZ$ and $g(Z) \subset U^2$. So, if $U^2 \neq Z$, then dim $(\mathcal{D}_0(A) \cap \mathrm{Inn}(A)) \leq r(r-1) + s(s-1)$ and dim $\mathcal{D}_1(A) = r$, that is dim $\mathrm{Inn}(A) \leq r^2 + s^2 - s$. If $U^2 = Z$, D is defined by (f, g) , then g is fixed by f because $g(uu') = f(u)u' + uf(u')$. So dim $\mathrm{Inn}(A) \leq r + r(r-1) = r^2 \leq r^2 + s^2 - s$.

Using the form of Inn(A), it is clear that if $\{ u_1, \cdots , u_r \}$ is a basis of U , and $\{z_1, \cdots , z_s \}$ is a basis of Z, then the following derivations generate Inn(A) :

a) The r derivations $D_i = [R_e , R_{u_i}]$, $i = 1,\dots,r$. Clearly $D_i (e) = \frac{1}{4}u_i$, $D_i(u) = \frac{1}{2}uu_i$ and $D_i(z) = - zu_i$.

b) The $\frac{1}{2}r(r-1)$ derivations $D_{ij} = [R_{u_i} , R_{u_j}]$ with $1 \leq i < j \leq r$. In this case $D_{ij} (e) = 0$, $D_{ij} (u) = (uu_i)u_j - (uu_j)u_i = 2(uu_i)u_j + (u_iu_j)u$, $D_{ij} (z) = 2 (zu_i)u_j$.

c) The $\frac{1}{2}s(s-1)$ derivations $D_{ij}{}^* = [R_{z_i}, R_{z_j}]$ with $1 \leq i < j \leq s$, and now $D_{ij}{}^* (e) = 0 = D_{ij}{}^* (z)$ and $D_{ij}{}^* (u) = 2(uz_i)z_j$.

d) The rs derivations $D'_{ij} = [R_{u_i} , R_{z_j}]$, $1 \leq i \leq r$, $1 \leq j \leq s$ where $D'_{ij} (e) = \frac{1}{2}u_iz_j$, $D'_{ij} (u) = (u_iz_j)u$ and $D'_{ij} (z) = -(u_iz_j)z$.

e) $R_{z_1} , \dots , R_{z_t}$ where $\{ z_1, \dots , z_t \}$ is a basis of $Z_0 = \{ z \in Z : (UZ)z = 0 \}$.

Clearly the derivations D'_{ij} are linear combinations of $D_1, \dots , D_r$ which generates $\mathcal{D}_1(A)$. The derivations $R_{z_1} , \dots , R_{z_t}$, D_{ij} and $D_{ij}{}^*$ generate $\mathcal{D}_0(A) \cap \mathrm{Inn}(A)$. So dim $(\mathcal{D}_0(A) \cap \mathrm{Inn}(A)) \leq s_0 + \frac{1}{2}(r(r-1) + s(s-1))$.

If $UZ = 0$, then $D_{ij} = 0 = D_{ij}{}^*$, $t = s$ and $R_Z = 0$.

If $U^2 = 0$, then $D_{ij} = 0$, so dim $\mathrm{Inn}(A) \leq r + t + \frac{1}{2}s(s-1) \leq r + \frac{1}{2}s(s+1)$.

If $U^2 \neq 0 \neq UZ$, $(UZ)Z = 0$, so $t = s$, $D_{ij}{}^* = 0$ and dim $\mathrm{Inn}(A) \leq r + s + \frac{1}{2}r(r-1) = s + \frac{1}{2}r(r+1)$.

Note: Every derivation D in the above generator system satisfies $D^2 = 0$.

PARTICULAR CASE

We will consider now a Jordan-Bernstein algebra $A = Ke + U + Z$ with $UZ \neq 0 \neq U^2$ and $(UZ)Z = 0$.

A) Let us suppose that $\dim U = 2$. Then $\dim UZ = 1$. Let $\{ u_1 \}$ a basis of UZ and $\{u_1, u_2\}$ a basis of U. The above means that $u_1Z = 0$ and there is z_1 in Z with $u_1 = u_2z_1$, and $\forall\, z \in Z \quad u_2\, z = \lambda_z\, u_1$. Consequently $u_1{}^2 = 0 = u_1u_2$ and $u_2{}^2 = z_2 \neq 0$ because $\neq 0$. Since $u_2z_2 = 0 \neq u_2z_1$, the set $\{ z_1, z_2 \}$ are linearly independent and $\dim Z \geq 2$.

It is easy to see that $D_{12} = 0$, $Z_0 = Z$ and $R_z = \lambda_z\, R_{z_1}$. So $\{ R_{z_1}\}$ is a basis of and $\{ R_{u_1}, R_{u_2}, R_{z_1} \}$ is a basis of Inn(A).

If we consider $\{ z_1, z_2, ..., z_s \}$ a basis of Z and $g : Z \longrightarrow Z$ the linear application given by $g(z_2) = 2z_2$, $g(z_i) = 0$ if $i \neq 2$, it can be proved that the pair $(1_U, g)$ defines a derivation in $\mathcal{D}_0(A)$ (that is, they satisfy (I) and (II)). Clearly it is not inner.

Summarizing:

a) $\dim \mathrm{Inn}(A) = 3$ independently of dimZ.

b) There are derivations which are not inner.

c) If $\dim Z = 2$ there is, up to isomorphism, a unique Jordan-Bernstein algebra satisfying these conditions. It is $A = Ke + K(u_1,u_2) + K(z_1,z_2)$ with $Z^2 = 0$, $u_1{}^2 = u_1u_2 = 0$, $u_2{}^2 = z_2$, $u_1z_1 = u_1z_2 = 0$, $u_2\, z_1 = u_1$, $u_2\, z_2 = 0$.

d) Every derivation in $\mathcal{D}_0(A) \cap \mathrm{Inn}(A)$ is defined by a pair (f,g) of linear applications $f: U \longrightarrow UZ$, $g : Z \longrightarrow U^2$ satisfying (I) and (II) and $f\,(UZ) = 0$. The converse is not true, because $f = 0_U$ and g defined by $g(z_1) = z_2$, $g(z_2) = 0$ define a derivation D_2 which is not inner because is linearly independent with $D_1 = R_{z_1}$. It can be proved that $\{D_1, D_2\}$ is a basis of $\mathcal{D}_0(A)$.

B) Let us suppose $\dim U = 3$. Then $1 \leq \dim UZ \leq 2$.

B$_1$) If $\dim UZ = 2$, let $\{ u_1, u_2 \}$ a basis of UZ and $\{u_1, u_2, u_3\}$ a basis of U. So $u_1Z = 0 = u_2\, Z$ and there are z_1, z_2 in Z such that $u_1 = u_3\, z_1$ and $u_2 = u_3z_2$. This assures that $\{z_1, z_2\}$ is linearly independent.

Also $u_1 = u_2 = 0 = u_1u_2 = u_1u_3 = u_2 u_3$, so $u_3 = z_3 \neq 0$ because $U \neq 0$. Since $u_3z_3 = 0$, the set $\{z_1 , z_2 , z_3 \}$ is linearly independent and so $\dim Z \geq 3$. Also $Uz_3 = 0$.

For every z in Z, $zu_3 = \alpha_z u_1 + \beta_z u_2$ and $u_1z = u_2 z = 0$, what implies that $R_z = \alpha_z R_{z_1} + \beta_z R_{z_2}$. It can be proved that $D_{12} = 0 = D_{13} = D_{23}$. So $\{ R_{z_1} , R_{z_2} \}$ is a basis of $\mathcal{D}_0(A) \cap \mathrm{Inn}(A)$ and $\{R_{u_1} , R_{u_2} , R_{u_3} \}$ is a basis of $\mathcal{D}_1(A)$. Consequently $\dim \mathrm{Inn}(A) = 2+3 = 5$ independently of the dimension of Z (allways ≥ 3). Let us consider $1_U : U \ \ U$ and $g : Z \ \ Z$ the linear application given by $g(z_3) = 2z_3$, $g(z_i) = 0$ if $i \neq 3$. Then $(1_U, g)$ defines a derivation which is not inner.

$B_2)$ If $\dim UZ = 1$, let $\{ u_1\}$ be a basis of UZ. Then $u_1Z = 0$ and there is $u_2 \in U$, $z_1 \in Z$ with $u_1 = u_2 z_1$. Let $\{ u_1 , u_2 , u_3 \}$ be a basis of U. Then $\forall z \in Z$ $u_2z = \alpha_z u_1$, $u_3 z = \beta_z u_1$. So $u_1 = u_1u_2 = 0$. We will distinguish two cases:

Case 1: $\beta_z = 0$ $\forall z \in Z$, that is, $u_3 z = 0$ for every z in Z. In this case $u_1u_3 = (u_2 z_1)u_3 = -(u_3 z_1)u_2 = 0$. Then $u_1u_2 = u_1u_3 = 0 = u_1(u_2u_3)$, $u_2 u_1 = u_2 u_3 = u_2 (u_1 u_3) = 0 = -2 u_3 (u_2u_3)$ and $u_3 (u_1 u_2) = u_3 u_1 = u_3 u_2 = 0$. So $U^3 = 0$.

We have seen that there are derivations of A which are not inner derivations. We can also notice that $R_z = \alpha_z R_{z_1}$, $D_{12} = 0 = D_{13} = D_{23}$, so $\dim (\mathcal{D}_0(A) \cap \mathrm{Inn}(A)) = 1$ and $\dim \mathrm{Inn}(A) = 3+1 = 4$.

Case 2: There is z with $\beta_z \neq 0$. Then $u_1 U = 0 = u_1 Z$.

If $\alpha_z = 0$ if and only if $\beta_z = 0$, then $\{ R_{z_1}\}$ is a basis of the set $\{ R_z : z \in Z_0 \}$ and $u_3u_3{}^2 = 0$ implies that $u_2u_3{}^2 = 0$ and consequently $(u_2u_3)u_3 = 0$. Similarly $u_2u_2{}^2 = 0$ implies that $u_3u_2{}^2 = 0$ and $(u_2u_3)u_2 = 0$. In this case $U^3 = 0$ and we already know that A has derivations which are not inner. Again $D_{12} = 0 = D_{13} = D_{23}$ and so $\dim (\mathcal{D}_0(A) \cap \mathrm{Inn}(A)) = 1$ and $\dim \mathrm{Inn}(A) = 3+1 = 4$.

In other case, there are z_1, z_2 in Z linearly independent such that $u_1 = u_2 z_1$, $u_2 z_2 = 0$, $u_3 z_1 = 0$ and $u_1 = u_3 z_2$. So $\{ R_{z_1}, R_{z_2} \}$ is a basis of the set $\{ R_z : z \in Z_0 \}$.

Clearly $D_{12} = 0 = D_{13}$ and $D_{23} = 0$ if and only if $U^3 = 0$.

The case $U^3 = 0$ is known (in this case dim InnA = 3+2 = 5) so we can consider $U^3 \neq 0$. In this case we can suppose that $u_2^{\,2} = z_2$ and $u_2 u_3 \neq 0$. We know that now $D_{23} \neq 0$, but $u_1 U = 0$, that is, (UZ)U = 0. So $D_{23}(e) = 0$, $D_{23}(z) = 0 \ \forall z \in Z$, $D_{23}(u_2) = \alpha u_1$, $D_{23}(u_3) = \beta u_1$, with $\alpha, \beta \in K$. This means that $D_{23} = \alpha R_{z_1} + \beta R_{z_2}$ and consequently dim ($\mathcal{D}_0(A) \cap \mathrm{Inn}(A)$) = 2 and dim Inn(A) = 5.

We need distinguish the following cases:

i) $u_2 u_3$, $u_3^{\,2} \in K(z_1, z_2)$,

ii) $u_2 u_3 \in K(z_1, z_2)$ and $u_3^{\,2} = z_3$ with $\{ z_1, z_2, z_3 \}$ linearly independent ,

iii) $u_2 u_3 \in K(z_1, z_2, z_3)$ and $u_3^{\,2} = z_3$ " " " ",

iv) $u_2 u_3 = z_4$ and $u_3^{\,2} = z_3$ with $\{ z_1, z_2, z_3, z_4 \}$ linearly independent.

In all the above cases it can be proved that dim $\mathcal{D}_0(A) \geq 3$, so there are always derivations which are not inner.

References

[1] Cortés T. "Bernstein algebras. Determination by the lattice of subalgebras and Modularity". Doctoral Disertation. University of Zaragoza. 1990.

[2] Gonshor : "Derivations in genetic algebras" , Comm. in Algebra 16, nº 8, 1525-1542 , 1988.

[3] González S. : "Inner Derivations in Bernstein Algebras" , To appear in Linear Algebra and its Applications.

[4] González S. : "Homotope Algebra of a Bernstein Algebra' International Workshop on Hadronic Mechanics and Nonpotential Interaction. Cedar Falls, Iowa , 1990.

[5] González S. - Martínez C. : "Idempotent elements in a Bernstein algebra" , To appear in J. London Math. Soc.

[6] Holgate P. : "The Interpretation of Derivations in Genetic Algebras" , Linear Algebra and its applications 85 , 75-79 , 1987.

[7] Jacobson N. : "Abstract derivation and Lie algebras" Trans. Amer. Math. Soc. 42 , 206-224 , 1937.

[8] Jacobson N. : "Derivation algebras and multiplication algebras of semisimple Jordan-algebras" , Ann. of Math. (2) 50 , 866-874 , 1949.

[9] Jacobson N. : "A note on authomorphisms and derivations of Lie algebras", Proc. Amer. Math. Soc. 6 , nº 1 , 281-283 , 1955.

[10] Micali A. , Campos M. T. , Costa e Silva M. C. , Ferreira S. M. and Costa R. C. : "Derivations dans les algëbres gamétiques" , Comm. in Algebra 16, nº 8 , 1525-1542, 1988.

[11] Schafer : "A theorem on the derivations of Jordan algebras" , Proc. Amer. Math. Soc. 2, 290-294 , 1951.

[12] Schafer : "Inner derivations of non-associative algebras" , Bull. Amer. Math. Soc. 55, 769-776 , 1949.

[13] Wörz-Busekros A.: "Bernstein Algebras" , Arch. Math. 48, 388-398 , 1987.