 A is an assoc. algebra with unit e. 10/19/16 ①

Lemma 1 ↺ OR TFAE

(i) $a \in A$ is invertible (i.e. has left & right inverse)

(ii) $\exists 1 \; \bar{a}^{1} \in A \quad a\bar{a}^{-1} = \bar{a}^{-1}a = e$

(iii) $L(a)$ is invertible in End(A) (i.e. has left & right inverse)

Proof $\boxed{(i) \Rightarrow (ii)}$ let $b, b' \in A$, $ba = ab' = e$. Then

$b = be = b(ab') = (ba)b' = eb' = b'$ (This also shows $\bar{a}^{1}$ is unique)

$\boxed{(ii) \Rightarrow (iii)}$ $\quad \bar{a}^{1}a = a\bar{a}^{1} = e \Rightarrow L(\bar{a}^{1})L(a) = L(a)L(\bar{a}^{1}) = L(e) = id$

Thus $L(a)$ is invertible in End(A). (Also $L(a)^{-1} = L(\bar{a}^{1})$)

$\boxed{(iii) \Rightarrow (i)}$ $\quad L(a)$ invertible $\Rightarrow \exists 1\, U \in \mathrm{End}(A)$ $(U = L(a)^{-1})$

$L(a)U = UL(a) = id$ (by (i) ⇒ (ii) applied to End(A))

apply to e. $\quad L(a)(Ue) = U(ae) = e$ i.e. $a(Ue) = \cancel{}$ $U(a) = e$

$\Rightarrow L(a)L(Ue) = L(e) = id$

But $L(a)U = id$

$\Rightarrow U = L(Ue)$ (inverse is unique; or apply $L(a)^{-1}$)

So $L(a)L(Ue) = L(Ue)L(a) = id$

$L(a \cdot Ue) = L(Ue \cdot a) = id$

$a \cdot Ue \cdot x = Ue \cdot a \cdot x = x$ for every $x \in A$

Let $x = e$ $\quad a \cdot Ue = Ue \cdot a = e$ so a is invertible

and $\left(\bar{a}^{1} = Ue = L(a)^{-1}e\right)$

Lemma 2 If $u \in A$ is nilpotent, then $e-u$ is invertible.

proof If $u^k = 0$, set $v = e+u+u^2+\dots+u^{k-1}$,

then
$$(e-u)v = (e-u)(e+u+u^2+\dots+u^{k-1})$$
$$= (e+u+u^2+\dots+u^{k-1}) - (u+u^2+\dots+u^{k-1}+\underbrace{u^k}_{=0})$$
$$= e$$

and $v(e-u) = e$ also $\boxed{\times}$

$\boxed{2.2}$ p. 11 Def. Let A be an assoc. alg (not nec. with unit element) $\hat{A} = \Phi 1 \oplus A$

$x \in A$ is quasi-invertible with quasi-~~inverse~~ inverse $y \in A$

if $1-x$ is invertible in $\hat{A}$ with inverse $1+y$.

Lemma 3 TFAE

(i) $x \in A$ is quasi-invertible

(ii) $\exists\, y \in A$, $y-x = yx = xy$

(iii) $\mathrm{id} - L(x)$ is invertible in End(A) (End A has unit element id)

Moreover $y = (\mathrm{id}-L(x))^{-1}x$ added by me

Proof $\boxed{(ii) \Rightarrow (i)}$ assume $y-x = yx = xy$. Then

$$1 = 1+y-x-yx = (1+y)(1-x)$$
$$1 = 1+y-x-xy = (1-x)(1+y)$$

so $(1-x)$ is invertible in $\hat{A}$ with inverse $1+y$

$\boxed{(i) \Rightarrow (iii)}$ Assume $1-x$ is invertible in $\hat{A}$.

By lemma 1 $L(1-x)$ is invertible in $\text{End}(\hat{A})$

$$(\lambda 1+b)(0+a) = \lambda a + ba$$

A is an ideal in $\hat{A}$ $\quad \Big(\underbrace{(0,a)}_{\in A} \underbrace{(\lambda, b)}_{\in \hat{A}} = (0, \lambda b + ab) \Big)$

$$L(1-x)(\lambda 1+b) = (1-x)(\lambda 1+b) = \lambda 1 - \lambda x + b - xb$$

$$L(1-x)(0+b) = b - xb \in A \quad \text{so } L(1-x)(A) \subseteq A$$

$$\boxed{\text{so} \quad L(1-x)\Big|_A = \text{id} - L(x)}$$

Start over $\quad (1-x)(1+y) = 1 = (1+y)(1-x)$

$$\Rightarrow \quad \underbrace{L(1-x)}\underbrace{L(1+y)}_{\substack{\text{maps}\\ A \text{ to } A}} = \text{id}_{\hat{A}} = L(1+y)\underbrace{L(1-x)}_{\substack{\text{maps}\\ A \text{ to } A}}$$

$$\Rightarrow \quad L(1-x)|_A \, L(1+y)|_A = \text{id}_A \quad L(1+y)|_A \, L(1-x)|_A$$

so $L(1-x)|_A$ is invertible in $\text{End}(A)$

i.e. $\text{id} - L(x)$ is " " " " .

p. 12 $\boxed{(iii) \Rightarrow (ii)}$ suppose $\mathrm{id} - L(x)$ invertible in $\mathrm{End}(A)$

Set $y = (\mathrm{id} - L(x))^{-1} x$ $(\in A)$ Then $x = (\mathrm{id} - L(x))\, y$

$= y - xy$

it remains to show $xy = yx$

claim $L(x)(\mathrm{id} - L(x))^{-1} = (\mathrm{id} - L(x))^{-1} L(x)$

[Same as $(\mathrm{id} - L(x)) L(x) = L(x)(\mathrm{id} - L(x))$

i.e. $L(x) - L(x)^2 = L(x) - L(x)^2$]

Thus $y = (\mathrm{id} - L(x))^{-1} x = (\mathrm{id} - L(x))^{-1} L(x)\, (e)$

$= [L(x)(\mathrm{id} - L(x))^{-1}]\, (e)$ OOPS!

Remarks (p. 12)

- nil potent $\Rightarrow$ quasi-invertible

- if A has a unit e, then x is q.i. $\Leftrightarrow$ $e - x$ is invertible in A

(associative algebra)

$\boxed{2.3}$ Given $u \in A$, the u-homotope of A is the algebra with set A and multiplication xuy (product of x and y)

Denote this algebra by A_u (A_u is associative)

(By Lemma 3 x is q.i. in A_u with quasi-inverse y

$\iff$ $y - x = xuy = yux$)

Def (notation really) $q(x,y)$ exists if x is q.i. in A_y with quasi-inverse $q(x,y)$

Better: if x is q.i. in A_y denote its quasi-inverse in A_y by $q(x,y)$.

(if $q(x,y)$ exists, then $q(x,y) - x = x\,y\,q(x,y) = q(x,y)\,y\,x$)

Lemma 4

TFAE OR ☹

(i) $q(x,y)$ exists
(ii) $q(xy,1)$ exists
(iii) $q(y,x)$ exists
(iv) $q(yx,1)$ exists
(v) $B(x,y)$ is invertible
(vi) $B(y,x)$ is invertible

In these cases:

$$q(x,y) = B(x,y)^{-1}x$$

$(B(x,y) := Id - L(xy))$

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proof. $\boxed{i \Rightarrow ii}$ Let $u = q(x,y)$ so x is q.i. in A_y with q.i. u

By lemma 3 $u - x = xyu = \cancel{u}uyx$

$\Rightarrow uy - xy = xyuy = uyxy \xrightarrow[\text{Lemma 3}]{} xy$ is q.i. in A with q.i. uy

$\boxed{ii \Rightarrow iii}$ Let $w = q(xy,1)$ so xy is q.i. in $A_1 = A$ with q.i. w

By lemma 3 $w - xy = wxy = xyw$

$\Rightarrow yw - yxy = ywxy = yxyw$

$\Rightarrow (yw+y) - y = ywxy + yxy = yxyw + yxy = (yw+y)xy = yx(yw+y)$

$\Rightarrow y$ is q.i. in A_x with q.i. $yw+y$ i.e. $q(y,x)$ exists

$(iii) \Rightarrow (iv) \Rightarrow (i)$ follows by interchanging x & y in $i \Rightarrow ii$ & $ii \Rightarrow iii$

$(ii) \Leftrightarrow (v)$ xy is quasi-invertible $\Leftrightarrow \underbrace{Id - L(xy)}_{B(x,y)}$ invertible (Lemma 3)

$(iv) \Leftrightarrow (vi)$ interchange x, y in $(ii) \Leftrightarrow (v)$

Finally by Lemma 3

$$q(x,y) = B(x,y)^{-1}x$$

∎

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p 15 $\quad \mathrm{Rad}(A) = \{x \in A : \ q(x,y) \text{ exists } \forall \ y \in A\}$

Lemma 4 says $q(x,y)$ exists $\Leftrightarrow$ $B(x,y)$ invertible

Theorem 1 $\quad$ Rad A is an ideal & $\mathrm{Rad}(A/\mathrm{Rad}\,A) = 0$.

Proof let $\alpha \in \Phi$, $x \in \mathrm{Rad}(A)$ $\qquad B(\alpha x, y) = (\mathrm{Id} - L(\alpha x y)) = B(x, \alpha y)$

so $\alpha x \in \mathrm{Rad}(A)$

let $y, z \in \mathrm{Rad}(A)$ Then $B(x,y)$ and $B(u,z)$ are invertible $\forall x, u \in A$

In particular $B(q(x,y), z)$ is invertible

Lemma 6 says $\quad B(x,y)\,B(q(x,y),z) = B(x, y+z)$

$\Rightarrow B(x, y+z)$ invertible $\forall x \Rightarrow y+z \in \mathrm{Rad}(A)$

$\therefore$ Rad A is a linear space

If $x \in \mathrm{Rad}\,A$, $q(x,y)$ exists $\forall y \in A$ so $q(x, ayb)$ exists for all $a, b \in \hat{A}$

Corollary to Lemma 5 says $\quad q(axb, y)$ exists $\Leftrightarrow$ $q(x, bya)$ exists $\forall\ a, b \in \hat{A}$, $x, y \in A$

$\therefore$ $q(bxa, y)$ exists and $\hat{A}(\mathrm{Rad}\,A)\hat{A} \subset \mathrm{Rad}\,A$

Thus $\mathrm{Rad}(A)$ is an ideal.

Let $\bar{A} = A/\mathrm{Rad}\,A$ and $\bar{x} \in \bar{A}$ $\qquad (\bar{x} = x + \mathrm{Rad}(A))$

Then $\forall\ \bar{y} \in \bar{A}$ $\exists\ \bar{u}$ with $\bar{u} - \bar{x} = \bar{u}\bar{y}\bar{x} = \bar{x}\bar{y}\bar{u}$

i.e. $u - x - uyx \in \mathrm{Rad}\,A$ so $\underbrace{B(u - x - uyx, -y)}_{\substack{\| \\ B(u,-y)\,B(x,y)}}$ is invertible (exercise 3)

So $B(x,y)$ is left-invertible

Exercise 4 $\quad B(x,y)$ is right invertible, so $x \in \mathrm{Rad}\,A$ & $\bar{x} = 0$