$\mathcal{L}$ it has no proper ideal. Since $\mathcal{L}^2 \neq 0$ by assumption, we see that $\mathcal{L}$ is simple. Since the assumptions (i) and (ii) are true in $\mathcal{L}^{\perp}$ we get by an induction argument the decomposition of $\mathfrak{A}$ as a direct sum of simple ideals.

II. Associative Algebras

2.1. Let $\mathfrak{A}$ be an associative algebra over ~~a ring~~ the field $\Phi = \mathbb{R}$ or $\mathbb{C}$ and assume that $\mathfrak{A}$ has a unit element e. An element $a \in \mathfrak{A}$ is called left invertible (resp. right invertible) if there is an element $b \in \mathfrak{A}$ ($b' \in \mathfrak{A}$) such that $ba = e$ (resp. $ab' = e$). a is invertible if a is left and right invertible.

Lemma 1. The following statements are equivalent,

(i) $a \in \mathfrak{A}$ is invertible,

(ii) there is a unique element $a^{-1} \in \mathfrak{A}$ such that $a^{-1}a = aa^{-1} = e$

(iii) $L(a)$ is invertible (in $\mathrm{End}_{\Phi}\mathfrak{A}$).

Proof. Let $b, b' \in \mathfrak{A}$ be such that $ba = ab' = e$. Then $b = be = b(ab') = (ba)b' = eb' = b'$, consequently (i) $\to$ (ii). If $a^{-1}a = aa^{-1} = e$ then $L(a^{-1})L(a) = L(a)L(a^{-1}) = \mathrm{id}$. This shows that $L(a)$ is invertible and $L(a^{-1}) = L(a)^{-1}$, thus (ii) $\to$ (iii). To show (iii) $\to$ (i) assume $L(a)$ invertible, i.e. $L(a)U = UL(a) = \mathrm{id}$ for a unique $U \in \mathrm{End}\,\mathfrak{A}$ (apply (i) $\to$ (ii) to $\mathrm{End}\,\mathfrak{A}$). All terms of this equation acting on $e \in \mathfrak{A}$ gives $au = Ua = e$ for $u = Ue$. But then $L(a)L(u) = \mathrm{id}$

and $L(a)U = \mathrm{id}$, consequently $U = L(u)$ (since the inverse is unique). It follows $au = ua = e$. (Observe that the associative law was used at essential steps).

Lemma 2. If $u \in \mathfrak{A}$ is nilpotent, then $e - u$ is invertible.

Proof. Let $u^k = 0$, then put $v = e + u + \ldots + u^{k-1}$ and check $(e - u)v = v(e - u) = e$.

2.2. Lemma 2 leads to the following definition. Let $\mathfrak{A}$ be an associative algebra (not neccessarily with unit element) and $\hat{\mathfrak{A}} = \Phi 1 \oplus \mathfrak{A}$ be the algebra obtained from $\mathfrak{A}$ by adjoining a unit element (see 1.7.).

$x \in \mathfrak{A}$ is called quasi invertible (q.i.) with quasi inverse y, if $1 - x$ is invertible in $\hat{\mathfrak{A}}$ with inverse $1 + y$. (Remark: If $1 - u$ has left or right inverse $\alpha 1 + v$ in $\hat{\mathfrak{A}}$ then $1 = (\alpha 1 + v)(1 - u) = \alpha 1 + v - \alpha u - vu$ implies $\alpha = 1$.)

Lemma 3. The following statements are equivalent:

(i) $x \in \mathfrak{A}$ is quasi invertible,

(ii) there exists $y \in \mathfrak{A}$ such that $y - x = yx = xy$,

(iii) $\mathrm{id} - L(x)$ is invertible.

In either case the quasi inverse y is uniquely determined by

$$y = (\mathrm{id} - L(x))^{-1}x. \tag{2.1}$$

Proof. (ii) $\rightarrow$ (i). Assume $y - x = yx = xy$, then

$$1 = 1 + y - x - yx = (1 + y)(1 - x) \text{ and}$$

$$1 = 1 + y - x - xy = (1 - x)(1 + y).$$

(i) $\rightarrow$ (iii) If $1 - x$ is invertible in $\hat{\mathfrak{A}}$ then by lemma 1 the left multiplication $L(1 - x)$ of $1 - x$ in $\hat{\mathfrak{A}}$ is invertible and consequently the restriction to $\mathfrak{A}$, $\hat{L}(1 - x)|_{\mathfrak{A}} = id - L(x)$ must be invertible since $\mathfrak{A}$ is an ideal of $\hat{\mathfrak{A}}$. If (iii) holds, set $y := (id - L(x))^{-1}x$ and obtain $y - x = xy = yx$. (For $xy = yx$ use the fact that $L(x)(id - L(x))^{-1} = (id - L(x))^{-1}L(x)$.) Since the inverse of an element is uniquely determined, y is unique and we just saw $y = (id - L(x))^{-1}x$.

$\boxed{\text{Exercise 1 Prove } xy = yx}$

Remarks. 1) Lemma 2 shows that nilpotent elements are quasi invertible.

2) The equivalence (i) <=> (ii) shows that if $\mathfrak{A}$ has a unit element e, then x is q.i. iff $e - x$ is invertible in $\mathfrak{A}$.

2.3. Let $\mathfrak{A}$ be an associative algebra and $u \in \mathfrak{A}$. The map $(x,y) \mapsto xuy$, $x,y \in \mathfrak{A}$ defines another multiplication on $\mathfrak{A}$. The module $\mathfrak{A}$ together with this multiplication is denoted by $\mathfrak{A}_u$ and is called the u-homotope of $\mathfrak{A}$. It is obvious that any homotope of an associative algebra is associative.

Lemma 3 shows that x q.i. in $\mathfrak{A}_u$ with quasi inverse y, iff

$$y - x = xuy = yux. \tag{2.2}$$

We introduce the following notations; we say $q(x,y)$ exists, if x is q.i. in $\mathfrak{A}_y$ with quasi inverse $q(x,y)$; if x is q.i. in $\mathfrak{A}$ we denote the quasi inverse of x by $q(x,1)$. Furthermore, we define

$$B(x,y) := id - L(xy) \tag{2.3}$$

Lemma 4. (Symmetry principle). *The following statements are equivalent,*

(i) $q(x,y)$ *exists,*

(ii) $q(xy,1)$ *exists,*

(iii) $q(y,x)$ *exists,*

(iv) $q(yx,1)$ *exists,*

(v) $B(x,y)$ *invertible,*

(vi) $B(y,x)$ *invertible,*

In either case

$$q(x,y) = B(x,y)^{-1}x \tag{2.4}$$

Exercise. 2 $q(x,x)$ exists $\rightarrow q(x^2,1)$ exists.

Proof. (i) $\rightarrow$ (ii). Let $u = q(x,y)$. Then by (2.2) $u - x = xyu = uyx$. Multiply by y from the right to obtain $uy - xy = xyuy = uyxy$, this means that $q(xy,1)$ exists.

(ii) $\rightarrow$ (iii) Let $w = q(xy, 1)$, then

$$w - xy = wxy = xyw, \text{ hence}$$

$$yw - yxy = ywxy = yxyw. \text{ It follows}$$

$$(yw + y) - y = yw = ywxy + yxy = yxyw + yxy = (yw + y)xy = yx(yw + y).$$

But this means that $q(y,x)$ exists.

(iii) $\rightarrow$ (iv) $\rightarrow$ (i) follows from interchanging x and y in the parts we already proved. (ii) $\Leftrightarrow$ (v) follows from lemma 3.

Then (2.4) follows from (2.1) in the y-homotope.

Remark. Actually we proved a stronger result, namely if

$$u = q(x,y) \text{ then } uy = q(xy, 1) \text{ and}$$

$$q(y,x) = yq(x,y)y + y.$$

Lemma 5. (Shifting principle).

If φ, ψ *are endomorphisms of* $\mathfrak{A}$ *such that*

$$L(\varphi x)R(\varphi y) = \varphi L(x)R(y)\psi$$

and $$L(\psi x)R(\psi y) = \psi L(x)R(y)\varphi \quad \textit{for all } x,y \in \mathfrak{A},$$

then $q(x,\psi y)$ *exists iff* $q(\varphi x,y)$ *exists.*

In either case

$$\varphi q(x, \psi y) = q(\varphi x, y).$$

Proof. Let $u = q(x, \psi y)$, i.e.

$$u - x = u(\psi y)x = x(\psi y)u.$$

Apply φ to obtain (using the assumptions on φ, ψ)

$$\varphi u - \varphi x = \varphi(u\psi yx) = \varphi(x\psi yu)$$
$$= (\varphi u)y(\varphi x) = (\varphi x)y(\varphi u)$$

This shows $\varphi q(x, \psi y) = q(\varphi x, y)$.

Assume $q(\varphi x, y)$ exists, then by the symmetry principle $q(y, \varphi x)$ exists, by the part we already proved we get that $q(\psi y, x)$ exists, again the symmetry principle implies that $q(x, \psi y)$ exists.

Remark: $\varphi = L(a)$, $\psi = R(a)$ and $\varphi = R(b)$, $\psi = L(b)$, $a, b \in \hat{\mathfrak{A}}$ satisfy the hypotheses of the lemma.

Corollary. If $a, b \in \hat{\mathfrak{A}}$, $x, y \in \mathfrak{A}$, then

$$q(axb, y) \text{ exists iff } q(x, bya) \text{ exists.}$$

Lemma 6. (Addition formula.) If $q(x,y)$ exists, then

(i) $B(x,y)B(q(x,y),z) = B(x,y+z)$

(ii) $q(q(x,y),z)$ exists iff $q(x,y+z)$ exists. If this is the case then

$$q(q(x,y),z) = q(x,y+z) \tag{2.5}$$

Proof. Put $u = q(x,y)$. Since $u - x = uyx = xyu$ we get

$$(id - L(xy))(id - L(uz)) = id - L(xy) - L(uz) + L(xyuz)$$
$$= id - L(x(y+z)) = B(x, y+z)$$

This is (i). Since $q(a,b)$ exists iff $B(a,b)$ invertible, the first part of (ii) can be read off from (i) since $B(x,y)$ is invertible. Using (2.4) and (i) we get

$$q(x,y+z) = B(q(x,y),z)^{-1}B(x,y)^{-1}x = B(q(x,y),z)^{-1}q(x,y)$$
$$= q(q(x,y),z).$$

Now we define

$$\mathrm{Rad}\,\mathfrak{A} := \{x \in \mathfrak{A},\ q(x,y) \text{ exists for all } y \in \mathfrak{A}\}$$

Note: If $x \in \mathrm{Rad}\,\mathfrak{A}$ then in particular $q(x,1)$ exists (see exercise, p. 13).

Theorem 1. Rad $\mathfrak{A}$ is an ideal in $\mathfrak{A}$ and $\mathrm{Rad}(\mathfrak{A}/\mathrm{Rad}\,\mathfrak{A}) = 0$.

Proof. $x \in \mathrm{Rad}\,\mathfrak{A}$ is equivalent to $B(x,y)$ invertible for all $y \in \mathfrak{A}$, by lemma 4. If $\alpha \in \Phi$, $x \in \mathrm{Rad}\,\mathfrak{A}$ then $\alpha x \in \mathrm{Rad}\,\mathfrak{A}$ follows immediately from $B(\alpha x,y) = B(x,\alpha y)$. If $y,z \in \mathrm{Rad}\,\mathfrak{A}$ then $B(x,y)$ and $B(u,z)$ are invertible for all $x,u \in \mathfrak{A}$ (symmetry principle) in particular $B(q(x,y),z)$ is invertible. The addition formula then shows that $B(x,y+z)$ is invertible for all x, thus $y+z \in \mathrm{Rad}\,\mathfrak{A}$. We proved that Rad $\mathfrak{A}$ is a subspace ~~submodule.~~ If $q(x,y)$ exists for all $y \in \mathfrak{A}$, $q(x,ayb)$ exists for all $a,b \in \hat{\mathfrak{A}}$. But then $q(bxa,y)$ exists (Shifting principle resp. its corollary). Consequently $\hat{\mathfrak{A}}(\mathrm{Rad}\,\mathfrak{A})\hat{\mathfrak{A}} \subset \mathrm{Rad}\,\mathfrak{A}$ and Rad $\mathfrak{A}$ is an ideal. If $\bar{x} \in \mathrm{Rad}\,\bar{\mathfrak{A}}$, $\bar{\mathfrak{A}} = \mathfrak{A}/\mathrm{Rad}\,\mathfrak{A}$, then for every $\bar{y}$ there exists $\bar{u}$ such that $\bar{u} - \bar{x} = \overline{uyx} = \overline{xyu}$ or equivalently $u - x - uyx \in \mathrm{Rad}\,\mathfrak{A}$. But then $B(u-x-uyx,-y) \boxed{= \quad \text{Exercise 3}}$ $B(u,-y)B(x,y)$ is invertible and therefore $B(x,y)$ is left ~~right~~ invertible, $\boxed{\text{similarly we get that } B(x,y) \text{ is also right } \cancel{\text{left}} \text{ invertible,}}$ hence invertible. This is true for all $y \in \mathfrak{A}$, hence $x \in \mathrm{Rad}\,\mathfrak{A}$ and $\bar{x} = 0$.

Exercise 4