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Peirce Decomposition

A is an assoc. alg c is an idempotent

i.e. $c^2 = c$. Then $\forall\, x$

$$x = cxc + cx(1-c) + (1-c)xc + (1-c)x(1-c)$$

OR

$$x = \underbrace{cx} + \underbrace{(1-c)x}$$

$$= \overbrace{cxc + cx(1-c)} + \overbrace{(1-c)xc + (1-c)x(1-c)}$$

NOTE: A is not necessarily unital i.e. does not have a unit element e

$(1-c)x$ is shorthand for $x - cx$

$$A = \underbrace{cAc} + \underbrace{cA(1-c)} + \underbrace{(1-c)Ac} + \underbrace{(1-c)A(1-c)}$$

$$A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$$

i.e. $A_{11} \cap A_{10} = (0)$ $\quad cxc = cy(1-c)$

$$\Rightarrow cxc^2 = cy(1-c)c = 0$$

etc

Lemma 7 If B is an ideal in A, then

$$B = (A_{11} \cap B) \oplus (A_{10} \cap B) \oplus (A_{01} \cap B) \oplus (A_{00} \cap B)$$

since
$$b = \underbrace{cbc}_{\in B} + (\underbrace{cb - cbc}_{\in B}) + (\underbrace{bc - cbc}_{\in B}) + (\underbrace{b - bc - cb + cbc}_{\in B})$$

Theorem 5 (i) $\displaystyle \operatorname{Rad} A = \bigoplus_{i,j=0,1} (A_{ij} \cap \operatorname{Rad} A)$

(since $\operatorname{Rad} A$ is an ideal)

(ii) $\operatorname{Rad} A_{ii} = A_{ii} \cap \operatorname{Rad} A \quad i = 0,1$

(By Exercise 6 A_{00}, A_{11} are algebras so $\operatorname{Rad} A_{ii}$ is an ideal in A_{ii})

Proof Let $x \in A_{ii} \cap \operatorname{Rad} A$ so x is qi in A_y $\forall y \in A$

and $x = cxc \in A_{11}$ (or $x = (1-c)x(1-c) \in A_{00}$)

Need to show $x \in \operatorname{Rad} A_{ii}$ i.e. x is qi in $(A_{ii})_y$ $\forall y \in A_{ii}$

$\cancel{\exists u \in A, \; u - x = xyu = uyx}$

depends on x and y

$\cancel{cuc - cxc = cxyuc = cuyxc}$

$\cancel{cuc - x = xyuc = cuyx}$

need a c here

$\cancel{\exists v \in (A_{ii})_y \; v - x = xyv = vyx}$

Let $y \in A_{11}$ so $y = cyc$. Since $\cancel{y}\, x \in \operatorname{Rad} A$

$\exists\, u \in A \qquad u - x = xyu = uyx$

$$\underbrace{cuc}_{\text{call this } v} - cxc = cxyuc = cuyxc$$

$$v - x = xyv = vyx$$

as required to show $x \in \operatorname{Rad} A_{11}$

This proves $A_{11} \cap \operatorname{Rad} A \subset \operatorname{Rad} A_{11}$.

Conversely let $x \in \operatorname{Rad} A_{11}$ To show $x \in \operatorname{Rad} A$

~~so q(x,x)~~

so $x = cxc \in A_{11}$ and x is q.i. in $(A_{11})_y$ for all $y \in A_{11}$

~~$q(x, cyc)$ exists $\forall y \in A$~~

~~By symmetry principle (Lemma 4)~~

~~$q(cyc, cxc)$ exists $\forall y \in A$~~

~~By Lemma 4 (symmetry principle) y is q.i. in $(A_{11})_z$ $\forall z \in A_{11}$~~

The Cor to Lemma 5 says

axb is q.i in A_y $\forall y \in A \qquad a, b \in \hat{A}$

$\Longleftrightarrow$ x is q.i. in A_{bya} $\forall y \in A$

Let $a=b=c$, the idempotent in Corollary to Lemma 5

cxc is q.i. in A_y $\forall y \in A$

$\iff$ x is q.i. in A_{cyc} $\forall y \in A$

We know that x is q.i. in $(A_{11})_y$ $\forall y \in A_{11}$

so $\exists\, u \in A_{11} \subseteq A$

$u - x = uyx = xyu$

So x is q.i. in A_y $\forall y \in A_{11}$

by Lemma 3

so x is q.i. in A_{cyc} $\forall y \in A$

So cxc is q.i. in A_y $\forall y \in A$

But $cxc = x$ so x is q.i. in A_y $\forall y \in A$

i.e. $x \in \operatorname{Rad} A$. ////

Example

$A^{(n,n)}$ = all $n\times n$ matrices with entries from A

(here we are assuming A has a unit element)

$$E = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \qquad I_p = \begin{bmatrix} e & & O \\ & e & \\ O & & e \end{bmatrix} \in A^{(p,p)}$$

$$A = \begin{bmatrix} A_{11} & A_{10} \\ A_{01} & A_{00} \end{bmatrix} = \underset{EAE}{\begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}} + \underset{EA(1-E)}{\begin{bmatrix} 0 & A_{10} \\ 0 & 0 \end{bmatrix}} + \underset{(1-E)AE}{\begin{bmatrix} 0 & 0 \\ A_{01} & 0 \end{bmatrix}} + \underset{(1-E)A(1-E)}{\begin{bmatrix} 0 & 0 \\ 0 & A_{00} \end{bmatrix}}$$

A-modules p. 19

$A \times M \to M$ left A-module

$(a,m) \to a \cdot m$ $\quad x \cdot (y \cdot m) = (xy) \cdot m$

bilinear

right-module $A \times M \to M$

$(a,m) \to m \cdot a$ $\quad (m \cdot x) \cdot y = m \cdot (xy)$

bilinear

Example left ideal, right ideal

Def Algebra A is <u>left Artinian</u> if $A \supset \underbrace{B_1 \supset B_2 \supset \cdots}_{\text{left ideals in } A}$

$\Rightarrow \exists\, k_0 \quad B_{k_0} \subset B_k \quad \forall k = 1, 2, \ldots$

(compare with non-associative powers)

Powers of a subalgebra B of an associative algebra A.

$B^1 = B, \quad B^2 = BB, \quad B^3 = B^2 B, \ldots \quad B^{k+1} = B^k B$

$B^2 = \{ \sum_{i=1}^{n} b_i c_i \quad b_i, c_i \in B, \; n = 1, 2, \ldots \}$

$B^3 = \{ \sum_{i=1}^{n} b_i c_i : \; b_i \in B^2, \; c_i \in B \quad n = 1, 2, \ldots \}$

$= \{ \sum_{i=1}^{n} \left(\sum_{j=1}^{m} d_j e_j \right) c_i \; : c_i, d_j, e_j \in B \;\; n = 1, 2, \ldots \}$

$= \{ \sum_{i,j} d_j e_j c_i \; : \; \ldots \}$

$= \{ \sum_{i=1}^{n} a_i b_i c_i \; : \; a_i, b_i, c_i \in B \quad n = 1, 2, \ldots$

$\vdots$

$B^m = \{ \sum_{i=1}^{n} a_{i1} a_{i2} \cdots a_{im} \; : \; a_{ij} \in B, \; n = 1, 2, \ldots \}$

Theorem 6 (p 19) A Artinian $\Rightarrow$ Rad A nilpotent
i.e. $\exists\, n > 0, (\text{Rad } A)^n = 0$

~~Proof~~

Def A is left Art $\Rightarrow$ if $A \supset B_1 \supset \cdots$ left ideals

then $B_{k_0+n} = B_{k_0} \quad \forall n$

Conversely if $B_n = B_{n+j}$ then B_n is a minimal element

$B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots \supset B_{n+j}$

(This is the exercise on p. 19)

Th 6 A (left) Artinian $\Rightarrow$ Rad A nilpotent

Proof $\mathcal{R} = \text{rad } A$ is a left ideal

$\mathcal{R} \supset \mathcal{R}^2 \supset \mathcal{R}^3 \supset \cdots \mathcal{R}^m \supset \cdots$

$\mathcal{R}^2 = \{ \sum x_i y_i : x_i, y_i \in \mathcal{R} \}$

$\mathcal{R}$ if I, J are ideals then IJ is an ideal

so R^k is an ideal and in particular a left ideal.

$\exists k$

$\mathcal{R}^k = \mathcal{R}^n \quad \forall n \geq k$

suppose $R^k \neq 0$ let $\mathcal{S} = \{ U \neq 0, U \text{ left ideal}, R^k U \neq 0 \}$

let B be a min elt of $\mathcal{S}$

$\mathcal{S}$ is not empty since $R^k R = R^{k+1} = R^k \neq 0$ so $R \in \mathcal{S}$.

~~Theorem 6 (p. 19) A Artinian ⇒ Rad A nilpotent i.e. $(\text{Rad } A)^n = 0$ for some $n > 0$.~~

~~Proof. Let $R = \text{Rad } A$. Then $R \supseteq R^2 \supseteq R^3 \ldots$ so $R^k = R^n$ for some k & all $n \geq k$~~

$R^k B \neq 0$

~~Suppose $R^k \neq 0$~~

$B \supset R^k b \neq 0$ for some b

$R^k b \in S$

$R^k b = B$

$\therefore b = rb \quad \exists\, r \in R^k \subset R$

$(1-r)\, b = 0$

$r \in R \Rightarrow (1-r)$ invertible in $\hat{A}$

$\Rightarrow b = 0 \qquad \Rightarrow\Leftarrow$ contradiction

$\therefore \quad R^k = 0$

see p. 8

Def A is simple if the only ideals are 0 & A and $A^2 \neq 0$

Cor A simple Artinian associative algebra A is semi-simple i.e. $\text{Rad } A = 0$

Proof If $\text{Rad } A \neq 0$ then $\text{Rad } A = A$ since A is simple.

So A is nilpotent, say $A^n = 0$. If $AA = A$

then $A^3 = A^2 A = AA = A, \ldots, A^k = A$ for every k

but $A^n = 0$ so $A^2 = AA \subsetneq A$

But A^2 is an ideal so $A^2 = 0$ a contradiction.