The ideal Rad $\mathfrak{A}$ is called the <u>Jacobson radical</u> of $\mathfrak{A}$. $\mathfrak{A}$ is called <u>semi simple</u>, if Rad $\mathfrak{A} = 0$.

A ~~submodule~~ subspace $\mathcal{L}$ of $\mathfrak{A}$ is called a <u>left ideal</u>, if $\mathfrak{A}\mathcal{L} \subset \mathcal{L}$, $\mathcal{L}$ is called <u>quasi invertible</u> (<u>nil</u>) if every element of $\mathcal{L}$ is quasi invertible (resp. nilpotent). Since a nilpotent element is quasi invertible, every nil module is quasi invertible.

<u>Theorem 2.</u> <u>If $\mathcal{L}$ is a quasi invertible left ideal of $\mathfrak{A}$, then $\mathcal{L} \subset$ Rad $\mathfrak{A}$.</u>

<u>Proof</u>. Let $b \in \mathcal{L}$, $x \in \mathfrak{A}$, then $xb \in \mathcal{L}$ and is quasi invertible by assumption, i.e., $q(xb,1)$ ex. From the symmetry principle we get that $q(b,x)$ exists for all $x \in \mathfrak{A}$, hence $b \in$ Rad $\mathfrak{A}$.

<u>Corollary</u>. Rad $\mathfrak{A}$ <u>contains every nil left ideal of</u> $\mathfrak{A}$.

<u>Remark:</u> The sam argument applies to right ideals.

<u>Theorem 3.</u> <u>If $\mathcal{L}$ is an ideal of $\mathfrak{A}$, then</u>

$$\text{Rad}\,\mathcal{L} = \mathcal{L} \cap \text{Rad}\,\mathfrak{A}.$$

<u>Proof.</u> Clearly $\mathcal{L} \cap$ Rad $\mathfrak{A} \subset$ Rad $\mathcal{L}$ since the quasi inverse of an element of $\mathcal{L}$ is in $\mathcal{L}$ (by (2.1)). Conversely let x be an element in Rad $\mathcal{L}$; then $q(x,b)$ exists for all $b \in \mathcal{L}$ and therefore $B(x,b)$ is invertible for all $b \in \mathcal{L}$. Since $B(x,-z)B(x,z) = B(x,z)B(x,-z) = B(x,zxz)$ for all $z \in \mathfrak{A}$ and $zxz \in \mathcal{L}$ $(x \in \mathcal{L})$, we get that $B(x,z)$ has to be invertible for all $z \in \mathfrak{A}$, or $x \in$ Rad $\mathfrak{A}$.

<u>Corollary.</u> <u>Every ideal of a semi simple associative algebra is semi simple</u>.

<u>Exercise</u> 5 If $\alpha : \mathfrak{A} \to \mathfrak{A}$ is an automorphism, then

$$\alpha(\text{Rad}\,\mathfrak{A}) = \text{Rad}\,\mathfrak{A}$$

2.4. Using the notion of quasi invertibility we can introduce a relation on $\mathfrak{A}$ by the following definition

$$R: = \{ (x,y) \in \mathfrak{A} \times \mathfrak{A} \ , \quad x = q(y,w) \text{ for some } w \in \mathfrak{A} \}$$

Using lemma 3 we see $(x,y) \in R$ iff $x - y = xwy = ywx$ for some $w \in \mathfrak{A}$.

Theorem 4. R <u>is</u> <u>an</u> <u>equivalence</u> <u>relation</u> <u>on</u> $\mathfrak{A}$.

Proof. $(x,x) \in R$ for all $x \in \mathfrak{A}$ and

$(x,y) \in R \Rightarrow (y,x) \in R$ are obvious.

If $(x,y),(y,z) \in R$, then $x = q(y,w)$ for some $w \in \mathfrak{A}$ and $y = q(z,u)$ for some u. But then from the addition formula we get $x = q(q(z,u),w) = q(z,u+w)$, in particular $(x,z) \in R$.

2.5. The Peirce decomposition. Let $\mathfrak{A}$ be an associative algebra and $c = c^2$ an idempotent in $\mathfrak{A}$. Clearly

$$x = cxc + (cx - cxc) + (xc - cxc) + (x - cx - xc + cxc) \tag{2.6}$$

Define

$\mathfrak{A}_{11} = c\mathfrak{A}c$, $\mathfrak{A}_{10} = c\mathfrak{A}(1-c)$, $\mathfrak{A}_{01} = (1-c)\mathfrak{A}c$, $\mathfrak{A}_{00} = (1-c)\mathfrak{A}(1-c)$; it is immediately seen $c\mathfrak{A}_{11} = \mathfrak{A}_{11}c = \mathfrak{A}_{11}$, $c\mathfrak{A}_{10} = \mathfrak{A}_{10}$, $\mathfrak{A}_{10}c = 0$, $c\mathfrak{A}_{01} = 0$, $\mathfrak{A}_{01}c = \mathfrak{A}_{01}$, $c\mathfrak{A}_{00} = \mathfrak{A}_{00}c = 0$. This together with (2.6) shows

$$\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{10} \oplus \mathfrak{A}_{01} \oplus \mathfrak{A}_{00} \tag{2.7}$$

The decomposition (2.7) is called the <u>Peirce decomposition</u> of $\mathfrak{A}$ relative to c and $\mathfrak{A}_{ij}$ are the Peirce spaces (resp. modules).

$\boxed{\text{Exercise. } 6}$ $\mathfrak{A}_{ii}\mathfrak{A}_{ii} \subset \mathfrak{A}_{ii}$ $(i = 0,1)$, $\mathfrak{A}_{11}\mathfrak{A}_{10} \subset \mathfrak{A}_{10}$, $\mathfrak{A}_{11}\mathfrak{A}_{01} = 0$, etc.

Lemma 7. If $\mathcal{L} \subset \mathfrak{A}$ is an ideal, then

$$\mathcal{L} = \bigoplus_{i,j=0,1} (\mathfrak{A}_{ij} \cap \mathcal{L})$$

Proof. The decomposition (2.6) (which is unique) shows that the components of $b \in \mathcal{L}$ in the different Peirce spaces are elements of $\mathcal{L}$ since $\mathcal{L}$ is an ideal.

Theorem 5. (i) $\operatorname{Rad} \mathfrak{A} = \oplus(\mathfrak{A}_{ij} \cap \operatorname{Rad} \mathfrak{A})$

(ii) $\operatorname{Rad} \mathfrak{A}_{ii} = \mathfrak{A}_{ii} \cap \operatorname{Rad} \mathfrak{A}$

Proof. Clearly $\mathfrak{A}_{ii} \cap \operatorname{Rad} \mathfrak{A} \subset \operatorname{Rad} \mathfrak{A}_{ii}$. Assume $i = 1$ and $x \in \operatorname{Rad} \mathfrak{A}_{11}$, then $x = cxc$ and $q(x,cyc)$ exist for all $y \in \mathfrak{A}$. But by the symmetry principle this is the case iff $q(cxc,y) = q(x,y)$ exists for all $y \in \mathfrak{A}$, consequently $x \in \operatorname{Rad} \mathfrak{A}$.

Example. Let $\mathfrak{A}$ be an associative Φ-algebra with unit element e. Consider the Φ-algebra $\mathfrak{A}^{(n,n)}$ of all nxn matrices with coefficients in $\mathfrak{A}$. The multiplication is the usual matrix multiplication. $\mathfrak{A}^{(n,n)}$ is associative.

$$\text{The matrix} \quad E = \begin{pmatrix} e & & & & & \\ & e & & & 0 & \\ & & \ddots & & & \\ & & & e & & \\ & & & & 0 & \\ & 0 & & & & \ddots \\ & & & & & & 0 \end{pmatrix} \begin{matrix} \}\, p \\ \\ \}\, q \end{matrix} \quad , \; p + q = n$$

obviously is an idempotent.

For the computation of the Peirce components A_{ij} of

$$A = \begin{array}{c} \quad\quad\quad q \\ \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right) \begin{matrix} \}p \\ \\ \end{matrix} \end{array}$$

we use (2.6) and get

$A_{11} = EAE = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ as component of A in $\mathfrak{A}_{11}$. Similarly

$$A_{10} = \begin{pmatrix} 0 & A_2 \\ 0 & 0 \end{pmatrix}, \quad A_{01} = \begin{pmatrix} 0 & 0 \\ A_3 & 0 \end{pmatrix}, \quad A_{00} = \begin{pmatrix} 0 & 0 \\ 0 & A_4 \end{pmatrix}.$$

2.6. $\mathfrak{A}$-modules. Let $\mathfrak{A}$ be an associative Φ-algebra and $\mathfrak{M}$ a ~~unital Φ-module~~ vector space. $\mathfrak{M}$ together with a map $\mathfrak{A} \times \mathfrak{M} \to \mathfrak{M}$, $(a,m) \mapsto a \cdot m$ is called a left $\mathfrak{A}$-module, if $(a,m) \to a \cdot m$ is Φ-bilinear and if $x \cdot (y \cdot m) = (xy) \cdot m$ for all $x,y \in \mathfrak{A}$, $m \in \mathfrak{M}$.

Example. Any left ideal in $\mathfrak{A}$ is an $\mathfrak{A}$-module.

Remark: If $\mathfrak{L}$ is a subalgebra of $\mathfrak{A}$, then $\mathfrak{M}$ together with the induced map $\mathfrak{L} \times \mathfrak{M} \to \mathfrak{M}$, $(b,m) \mapsto b \cdot m$, is a $\mathfrak{L}$-module.

Right $\mathfrak{A}$-modules are defined accordingly.

2.7. An associative algebra is called (left) Artinian, if any non-empty set of left ideals has a minimal element.

Exercise. $\mathfrak{A}$ is left Artinian, iff any descending chain of left ideals of $\mathfrak{A}$, $\mathfrak{L}_1 \supset \mathfrak{L}_2 \supset \cdots \supset \mathfrak{L}_k \supset \cdots$, becomes stationary, i.e., $\mathfrak{L}_n = \mathfrak{L}_{n+j}$, $j \geq 1$ for some n. Since we will be mainly concerned with the radical of Artinian algebras, we shall only prove the fundamental result about the radical in an Artinian algebra. We need a definition. If $\mathfrak{A}$ is an associative algebra and $\mathfrak{L}$ a subalgebra, then the powers of $\mathfrak{L}$ are defined recursively by $\mathfrak{L}^1 = \mathfrak{L}$, $\mathfrak{L}^{k+1} = \mathfrak{L}^k \mathfrak{L}$. $\mathfrak{L}$ is called nilpotent if $\mathfrak{L}^n = 0$ for some $n \geq 1$.

Theorem 6. If $\mathfrak{A}$ is Artinian, then Rad $\mathfrak{A}$ is nilpotent.

Proof. We set $\mathfrak{R} := \operatorname{Rad}\mathfrak{A}$ and consider the descending chain

$$\mathfrak{R} \supset \mathfrak{R}^2 \supset \ldots \mathfrak{R}^m \supset \ldots$$

then $\mathcal{R}^k = \mathcal{R}^n$ for some k and all $n \geq k$.

Suppose $\mathcal{R}^k \neq 0$ and let

$S = \{ 0 \neq \mathcal{U}$, $\mathcal{U}$ is left ideal of $\mathcal{A}$ and $\mathcal{R}^k\mathcal{U} \neq 0 \}$.

$S \neq \emptyset$ since $\mathcal{R}^k\mathcal{R} = \mathcal{R}^{k+1} = \mathcal{R}^k \neq 0$ implies $\mathcal{R} \in S$. Let $\mathcal{L}$ be a minimal element of S ($\mathcal{A}$ is Artinian). $\mathcal{R}^k\mathcal{L} \neq 0$ implies that there exists an element $b \in \mathcal{L}$ such that $\mathcal{R}^k b \neq 0$. Clearly $\mathcal{R}^k b \subset \mathcal{L}$. Since $\mathcal{R}^k(\mathcal{R}^k b) = \mathcal{R}^{2k} b = \mathcal{R}^k b \neq 0$ we get that $\mathcal{R}^k b \in S$. Consequently $\mathcal{R}^k b = \mathcal{L}$ since $\mathcal{L}$ is minimal in S. Now we have $b = r.b$ for some $r \in \mathcal{R}^k \subset \mathcal{R}$ or equivalently $(1 - r)\, b = 0$. But $r \in \mathcal{R}$ implies $1 - r$ invertible in $\hat{\mathcal{A}}$ (see remark on p. 15), thus $b = 0$ which is a contradiction to $\mathcal{R}^k b \neq 0$. Hence $\mathcal{R}^k = 0$.

Corollary. A simple Artinian associative algebra $\mathcal{A}$ is semi simple.

Proof. If Rad $\mathcal{A}$ is not trivial then $\mathcal{A} =$ Rad $\mathcal{A}$, since Rad $\mathcal{A}$ is an ideal and $\mathcal{A}$ is simple. By the preceding theorem we get that $\mathcal{A}$ is nilpotent. Then $\mathcal{A}\mathcal{A} \subsetneqq \mathcal{A}$ (otherwise $\mathcal{A}^k = \mathcal{A}$ for all k and then $\mathcal{A} = 0$). Since $\mathcal{A}\mathcal{A}$ is an ideal in $\mathcal{A}$ it has to be zero. This is a contradiction to $\mathcal{A}\mathcal{A} \neq 0$.

We state without proof the main results on semi simple associative Artinian algebras.

Theorem 7: An Artinian algebra is semi simple, iff it is the direct sum of a finite number of simple Artinian algebras.

An associative algebra is a division algebra, if every element $\neq 0$ is invertible.

Theorem 8. If $\mathcal{A}$ is a simple Artinian algebra over a field K, then $\mathcal{A}$ is isomorphic to the K-algebra of all nxn matrices over a K-division algebra (for some n).

A semi simple Artinian algebra has a unit element.

Exercise. Let $\mathfrak{A}$ be an Artinian algebra. Show that $\mathfrak{A}$ is semisimple iff any ideal $\mathcal{L}_1$ in $\mathfrak{A}$ has a direct (ideal) complement $\mathcal{L}_2$, i.e., $\mathfrak{A} = \mathcal{L}_1 \oplus \mathcal{L}_2$, $\mathcal{L}_2$ an ideal of $\mathfrak{A}$. If $\mathfrak{A}$ has an involution j and $\mathcal{L}_1$ is j-invariant, then $\mathcal{L}_2$ is j-invariant. (Hint: decompose the unit element e in $\mathfrak{A}$ as $e = e_1 + e_2$, show e_i unit element in $\mathcal{L}_i$ and $j(e_i) = e_i$.).

III. Triple Systems.

3.1. A unital Φ-module $\mathcal{T}$ together with a trilinear map $\mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, $(x,y,z) \mapsto \langle xyz \rangle$ is called a triple system.

Examples. 1) Let $\mathcal{T} = \Phi^{(p,q)}$ be the Φ-module of rectangular pxq-matrices. If $A,B,C \in \mathcal{T}$, then AB^tC is in $\mathcal{T}$, where B^t denotes the transposed of B. Since $(A,B,C) \mapsto \langle ABC \rangle := AB^tC$ is trilinear, $\mathcal{T}$ together with this "triple product" is a triple system.

2) If $\mathfrak{A}$ is any (non associative) Φ-algebra. Then $\mathfrak{A}$ together with the map $(x,y,z) \mapsto \langle xyz \rangle := (xy)z$ is a triple system and any submodule closed under $(xy)z$ is a triple system. Note: if $\mathfrak{A}$ has a unit element e then $xz = \langle xez \rangle$ and the structure of $\mathfrak{A}$ as an algebra can be completely recovered from the triple system structure on $\mathfrak{A}$.

3) Most important examples for the situation just described are the following. Let $\mathfrak{A}$ be a Φ-algebra and $j: \mathfrak{A} \to \mathfrak{A}$ an involutorial automorphism (i.e. $j(ab) = j(a)j(b)$, $j^2 =$ id) then $\mathfrak{A}_\varepsilon = \{x \in \mathfrak{A}, j(x) = \varepsilon x\}$, $\varepsilon = \pm 1$, are closed under $(x,y,z) \mapsto (xy)z$, but in general $\mathfrak{A}_-$ is not a subalgebra.