11/3/16

F vector space

$$F \times F \times F \longrightarrow F$$
$$(x, y, z) \longrightarrow \langle xyz \rangle$$
trilinear

(algebra A vector sp

$$A \times A \to A$$
$$(x, y) \to xy$$
bilinear)

Examples rectangular matrices, non-assoc. algebras A

$\langle ABC \rangle = AB^tC$ $\qquad$ $\langle abc \rangle = (ab)c$

(in complex case AB^*C) $\qquad$ (if A is unital with unit e then $ab = \langle aeb \rangle$

Go back to chapter 1 page 8

If A is an algebra and $j : A \to A$ is a linear transformation satisfying $\left.\begin{array}{l} j(ab) = j(b)j(a) \\ \text{and } j(j(a)) = a \end{array}\right\}, \forall a, b \in A$,

j is called an <u>involution</u> (or <u>anti-automorphism of order 2</u>)

example: $A = \mathbb{C}^{(n \times n)}$ $\quad j(a) = a^t$

(or $j(a) = a^* = \bar{a}^t$ (conjugate transpose))

Remark An involution can be viewed as an isomorphism $j : A \to A^{op}$ where A^{op} is the set A with multiplication $a \circ b := ba$ i.e. (reversed)

DEF An algebra (A, j) with involution is *simple* if $A^2 \neq (0)$ and the only ideals $B \subset A$ with $j(B) \subset B$ are $B = (0)$ and $B = A$.

NOTE: An algebra A (without involution) is *simple* if $A^2 \neq (0)$ and the only ideals $B \subset A$ are $B = (0)$ and $B = A$

Theorem 3 (page 8) Let (A, j) be a *simple* algebra with involution. Then either

(i) A is simple (too), or

(ii) $A \cong B \oplus B^{op}$ for a simple ideal B of A

and if $\theta : A \to B \oplus B^{op}$ is an isomorphism, then

$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[d, "j"'] & B \oplus B^{op} \arrow[d, "j'(b_1, b_2) = (b_2, b_1)"] \\ A \arrow[r, "\theta"] & B \oplus B^{op} \end{tikzcd}$$

$\theta \circ j \circ \theta^{-1}(b_1, b_2)$
$= (b_2, b_1)$

Proof. Assume A is not simple. Then $\exists$ ideal B, $0 \neq B \neq A$. $j(B) \cap B$ and $j(B) + B$ are j-stable ideals so $\boxed{j(B) \cap B = (0) \text{ and } j(B) \oplus B = A \text{ (direct sum)}}$

~~The identity map $B \to B^{op}$~~

~~$j(B)$ is an ideal in A and $j : B^{op} \to j(B)$ is an isomorphism since $j(b_1 \circ b_2) = j(b_2 b_1) = j(b_1) j(b_2)$~~

$$0 \subseteq \underbrace{j(B) \cap B}_{=(0) \text{ or } A} \subseteq \underbrace{j(B) + B}_{=(0) \text{ or } A} \subseteq A$$

If $j(B) \cap B = A$ then $B \supseteq A$ so $B = A$ ⇒⇐

$\therefore j(B) \cap B = (0)$

If $j(B) + B = (0)$ then $B \subset j(B) + B = (0) \Rightarrow B = (0)$ ⇒⇐

$\therefore j(B) + B = A$

$$\cancel{\begin{array}{c} A = B \oplus j(B) \\ \big\downarrow j \\ j(B) \end{array}}$$

$$\boxed{\begin{array}{l} B^{op} \xrightarrow{j'} j(B) \quad \text{is an isomorphism} \\ \qquad (j'^{-1} : j(B) \to B^{op}) = (j : j(B) \to B^{op}) \\ j(b_1 \circ b_2) = j(b_2 b_1) = j(b_1) j(b_2) \end{array}}$$

internal direct sum

$$\begin{array}{ccccc} A & = & B & \oplus & j(B) \\ & & \big\downarrow \mathrm{id}_B & & \big\downarrow j|_{j(B)} \\ A' & := & B & \oplus & B^{op} \end{array}$$

external direct sum

$\theta = \mathrm{id}_B \oplus j|_{j(B)}$ is an isomorphism of A onto $B \oplus B^{op}$ (call this A')

$\theta^{-1} = \mathrm{id}_B \oplus j|_{B^{op}}$

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "\theta"'] & A \arrow[d, "\theta"] \\ B \oplus B^{op} \arrow[r, "j'"] & B \oplus B^{op} \end{tikzcd}$$

$\theta^{-1}(b_1, b_2) = (b_1, j(b_2)) = b_1 + j(b_2)$

$j(\theta^{-1}(b_1, b_2)) = \underbrace{j(b_1)}_{\in j(B)} + \underbrace{b_2}_{\in B} = (b_2, j(b_1))$

$j' = \theta \circ j \circ \theta^{-1}$ $\quad j'(b_1, b_2) = \theta(j(\theta^{-1}(b_1, b_2))) = (b_2, b_1)$

It remains to show that B is simple (in A)

Need to prove $C = (0)$ or $C = B$

let C be an ideal in B, then

$$AC \overset{?}{\subseteq} BC \subseteq C \quad \therefore \quad C \text{ is an ideal in } A$$

and $CA \subseteq CB \subseteq C$

proof.

$$\theta = id_B \oplus j$$

$$A = B \oplus j(B) \cong B \oplus B^{op} = A'$$

$$\begin{array}{ccc} A & & A' \\ \cup & & \cup \\ C & & \theta(C) = C \end{array}$$

B is simple in A

$\iff \theta(B) = B$ simple in $B \oplus B^{op}$

$AC \subseteq BC \Longleftarrow A'C \subseteq BC$

$$A'C = \theta(A)\, C = (B \oplus B^{op})(C \oplus (0))$$

$$= BC \oplus (0) = BC$$

C is now an ideal in A. Suppose $(0) \neq C$

($C \neq A$ since $C \subseteq B \neq A$) Since we are assuming A is not simple we have $A \simeq C \oplus j(C) \subseteq B \oplus j(B) = A$

Hence $C = B$ so B is simple $\blacksquare$

we now return to chapter 3 p. 21

p. 21 continued

If A is an algebra with involution j (A is not nec. associative)

Then $A_+ := \{x \in A : j(x) = x\}$ is a subalgebra (and hence a triple system under $(x,y,z) \to (xy)z$).

$A_- = \{x \in A : j(x) = -x\}$ is in general not a subalgebra, but it is a triple system under $(x,y,z) \longrightarrow (xy)z$

NOTE: $A = A_+ \oplus A_-$

because $x = \frac{x + j(x)}{2} + \frac{x - j(x)}{2}$

p. 22 let F be a triple system $U, V, W \subset F$

$$\langle UVW \rangle = \left\{ \sum_{i=1}^{n} \langle a_i b_i c_i \rangle : (a_i, b_i, c_i) \in U \times V \times W,\ n = 1, 2, \ldots \right\}$$

subsystem = ~~subset~~ subspace U with $\langle UUU \rangle \subset U$

ideal = subspace U with $\langle FFU \rangle + \langle FUF \rangle + \langle UFF \rangle \subseteq U$

homomorphism $f: F \to F'$ is linear map $f\langle xyz \rangle = \langle f(x), f(y), f(z) \rangle$

quotient triple system if U is an ideal

$\overline{F} = F/U = \{x + U : x \in F\}$ is triple system under

$$\langle (x+U)(y+U)(v+U) \rangle := \langle xyz \rangle + U$$

Theorem 1 (page 22) let F be a triple system

(i) a subset $U \subset F$ is an ideal $\Leftrightarrow$

$U = \ker f$ for some homomorphism $f: F \to F'$

(ii) For a homomorphism $f: F \to F'$,

$$f(F) \cong F/\ker f$$

(iii) If U, V are ideals of F, then

$$(U+V)/V \cong U/U\cap V$$

Proof Exercise 1

Def triple system F is simple if $\langle FFF\rangle \neq (0)$ and F has no ideals other than (0) & F.

Def derivatives of F: $F^{(0)} = F$, $F^{(1)} = \langle FFF\rangle$

$F^{(2)} = \langle F^{(1)} F^{(1)} F^{(1)}\rangle \ldots F^{(n+1)} = \langle F^{(n)}, F^{(n)}, F^{(n)}\rangle$

Note: $F \supset F^{(1)} \supset F^{(2)} \supset \ldots \supset F^{(n)} \supset F^{(n+1)} \supset \ldots$

Def F is solvable if $F^{(n)} = 0$ for some $n \geq 1$.

p. 23

More detail for $\boxed{\text{Exercise 2 on p. 23 of Meyberg}}$

(copying from page 5

Lemma 1′ subtriples and homomorphic images of solvable triple systems are solvable

Lemma 2′ if U is an ideal of a triple system F, then F is solvable $\Leftrightarrow$ U and F/U are solvable.

Theorem 2′

(i) if U, V are solvable ideals in triple system F, then $U+V$ is a solvable ideal.

(ii) If F is Noetherian (every non-empty set of ideals has a maximal element), then F has a unique maximal solvable ideal $R(F)$ which contains all other solvable ideals, and $R(F/R(F)) = (0)$.

Another part of Exercise 2: If U is an ideal and $\mathrm{Rad}(F/U) = 0$, then $\mathrm{Rad}\,F \subset U$.

Def $R(F)$ is called the solvable radical of F.

What about the "nilradical" of F?

we are guided by page 7

Def If F is a triple system and $a \in F$, then

$$a^1 = a, \quad a^3 = \langle aaa \rangle, \quad a^5 = \langle a^3 a a \rangle$$

$$\ldots \quad a^{2(n+1)+1} = \langle a^{2n+1} a a \rangle \qquad n = 0, 1, 2, \ldots$$

Def $a \in F$ is <u>nilpotent</u> if $a^{2n+1} = 0$ for some $n \geq 1$

U subsystem $\subset F$ is <u>nil</u> if every element of U is nilpotent [Warning: there will in general not exist N such that if $a^{2n+1} = 0$ then $n \leq N$]

| More detail on Exercise 3 on p.23 of Meyberg |

(copy from p. 7)

<u>Lemma 3'</u> If F is a triple system satisfying

$$(a^{2n+1})^{2m+1} = a^{(2n+1)(2m+1)} \qquad \forall\, m, n > 0 \quad a \in F$$

then the sum of two nil ideals is a nil ideal.

<u>New Lemma</u> In any triple system F, there is a maximal nil ideal $N(F)$ (Zorn's lemma)

see my informal notes for pp 6-7 pages (3)-(4)

<u>New theorem</u> If the triple system F satisfies

$$\left(a^{2n+1}\right)^{2m+1} = a^{(2n+1)(2m+1)} \quad \forall\, m,n>0 \quad a \in F$$

then F has a unique maximal nil ideal called the <u>nilradical</u>.

For now Exercise 4 is optional. We shall return to it later (maybe next quarter)