A semi simple Artinian algebra has a unit element.

Exercise. Let $\mathfrak{A}$ be an Artinian algebra. Show that $\mathfrak{A}$ is semisimple iff any ideal $\mathcal{L}_1$ in $\mathfrak{A}$ has a direct (ideal) complement $\mathcal{L}_2$, i.e., $\mathfrak{A} = \mathcal{L}_1 \oplus \mathcal{L}_2$, $\mathcal{L}_2$ an ideal of $\mathfrak{A}$. If $\mathfrak{A}$ has an involution j and $\mathcal{L}_1$ is j-invariant, then $\mathcal{L}_2$ is j-invariant. (Hint: decompose the unit element e in $\mathfrak{A}$ as $e = e_1 + e_2$, show e_i unit element in $\mathcal{L}_i$ and $j(e_i) = e_i$.).

III. Triple Systems.

3.1. A ~~unital Φ-module~~ vector space $\mathcal{T}$ together with a trilinear map $\mathcal{T} \times \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, $(x,y,z) \mapsto \langle xyz \rangle$ is called a triple system.

$\Phi = \mathbb{R}$ or $\mathbb{C}$

Examples. 1) Let $\mathcal{T} = \Phi^{(p,q)}$ be the ~~Φ-module~~ vector space of rectangular pxq-matrices. If $A,B,C \in \mathcal{T}$, then AB^tC is in $\mathcal{T}$, where B^t denotes the transposed of B. Since $(A,B,C) \mapsto \langle ABC \rangle := AB^tC$ is trilinear, $\mathcal{T}$ together with this "triple product" is a triple system.

2) If $\mathfrak{A}$ is any (non associative) ~~Φ-algebra.~~ real or complex algebra Then $\mathfrak{A}$ together with the map $(x,y,z) \mapsto \langle xyz \rangle := (xy)z$ is a triple system and any ~~submodule~~ vector subspace closed under $(xy)z$ is a triple system.

Note: if $\mathfrak{A}$ has a unit element e then $xz = \langle xez \rangle$ and the structure of $\mathfrak{A}$ as an algebra can be completely recovered from the triple system structure on $\mathfrak{A}$.

3) Most important examples for the situation just described are the following. Let $\mathfrak{A}$ be a ~~Φ-algebra~~ real or complex algebra and $j: \mathfrak{A} \to \mathfrak{A}$ an involutorial automorphism (i.e. $j(ab) = j(a)j(b)$, $j^2 = \mathrm{id}$) then $\mathfrak{A}_\varepsilon = \{x \in \mathfrak{A}, j(x) = \varepsilon x\}$, $\varepsilon = \pm 1$, are closed under $(x,y,z) \mapsto (xy)z$, but in general $\mathfrak{A}_-$ is not a subalgebra.

($\mathfrak{A}_+$ is a subalgebra)

The above examples show that a theory of triple systems of course includes a theory of algebras and "minus spaces" of algebras relative to involutorial automorphisms.

For ~~submodules~~ subspaces $\mathfrak{U}, \mathfrak{V}, \mathfrak{W} \subset \mathfrak{T}$, we denote by $\langle \mathfrak{U}\mathfrak{V}\mathfrak{W} \rangle = \langle UVW \rangle$ the ~~submodule~~ subspace of $\mathfrak{T}$ generated by all "triple products" $\langle uvw \rangle$, $u \in \mathfrak{U}$, $v \in \mathfrak{V}$, $w \in \mathfrak{W}$. A ~~submodule~~ subspace $\mathfrak{U}$ is a subsystem if $\langle \mathfrak{U}\mathfrak{U}\mathfrak{U} \rangle \subset \mathfrak{U}$, it is an ideal, if $\langle \mathfrak{U}\mathfrak{T}\mathfrak{T} \rangle + \langle \mathfrak{T}\mathfrak{U}\mathfrak{T} \rangle + \langle \mathfrak{T}\mathfrak{T}\mathfrak{U} \rangle \subset \mathfrak{U}$. A ~~Φ-linear~~ real or complex linear map $f: \mathfrak{T} \to \mathfrak{T}'$ is a homomorphism of triple systems $\mathfrak{T}$, $\mathfrak{T}'$, if $f(\langle xyz \rangle) = \langle f(x)f(y)f(z) \rangle$ for all $x,y,z \in \mathfrak{T}$. Isomorphisms and automorphisms are defined the usual way and the $\boxed{\text{standard results hold.}}$ (The proofs are the same as for algebras.) If $\mathfrak{U}$ is an ideal in a triple system $\mathfrak{T}$, then $\bar{\mathfrak{T}} = \mathfrak{T}/\mathfrak{U}$ together with

Exercise 1
Prove Theorem 1

$$\langle (x+\mathfrak{U})(y+\mathfrak{U})(z+\mathfrak{U}) \rangle := \langle xyz \rangle + \mathfrak{U}$$

again is a triple system.

Theorem 1. (i) $\mathfrak{U} \subset \mathfrak{T}$ is an ideal, iff $\mathfrak{U}$ is the kernel of some homomorphism.

(ii) If $f: \mathfrak{T} \to \mathfrak{T}'$ is a homomorphism, then

$$f(\mathfrak{T}) \cong \mathfrak{T}/\text{kernel } f$$

(iii) If $\mathfrak{U}, \mathfrak{V}$ are ideals of $\mathfrak{T}$, then

$$\mathfrak{U}+\mathfrak{V}/\mathfrak{V} \cong \mathfrak{U}/\mathfrak{U}\cap\mathfrak{V} .$$

A triple system $\mathfrak{T}$ is called simple if $\langle \mathfrak{T}\mathfrak{T}\mathfrak{T} \rangle \neq 0$ and $\mathfrak{T}$ has no proper ideals.

3.2. The derivatives of a triple system $\mathfrak{T}$ are defined recursively

$$\mathfrak{T}^{(0)} = \mathfrak{T}, \quad \mathfrak{T}^{(n+1)} = \langle \mathfrak{T}^{(n)} \mathfrak{T}^{(n)} \mathfrak{T}^{(n)} \rangle .$$

Definition: every non-empty set of ideals has a maximal element (see p. 6)

$\mathcal{T}$ is solvable, if $\mathcal{T}^{(n)} = 0$ for some n.

Exercise. 2 State and prove the corresponding results to 1.5. (follow the proof of Theorem 2 on p. 6)

If $\mathcal{T}$ is Noetherian then there exists a unique maximal solvable ideal Rad $\mathcal{T}$ in $\mathcal{T}$, the solvable radical of $\mathcal{T}$. $\mathrm{Rad}(\mathcal{T}/\mathrm{Rad}\,\mathcal{T}) = 0$, and if $\mathrm{Rad}(\mathcal{T}/\mathfrak{U}) = 0$ then $\mathrm{Rad}\,\mathcal{T} \subset \mathfrak{U}$. Powers of an element $a \in \mathcal{T}$ are defined recursively

$$a^1 := a, \quad a^{2(n+1)+1} := \langle a^{2n+1} a a \rangle$$

Note: Only odd power are defined.

$a \in \mathcal{T}$ is nilpotent, if $a^{2n+1} = 0$ for some n. A subsystem $\mathfrak{U} \subset \mathcal{T}$ is nil, if every element in $\mathfrak{U}$ is nilpotent. If $(a^{2n+1})^{2m+1} = a^{(2n+1)(2m+1)}$ for all $m,n > 0$ and all $a \in \mathcal{T}$, then there exists a unique maximal nil ideal in $\mathcal{T}$, the nilradical of $\mathcal{T}$.

Exercise. 3 Prove existence and uniqueness of the nilradical. (follow the proof on p. 7, using Zorn's Lemma)

3.3. Similar to the definition of left and right multiplication in algebras we define bilinear maps $L,R,P: \mathcal{T} \times \mathcal{T} \to \mathrm{End}\,\mathcal{T}$, $L:(x,y) \mapsto L(x,y)$ $R:(x,y) \mapsto R(x,y)$, $P:(x,y) \mapsto P(x,y)$, by $L(x,y)z = \langle xyz\rangle$, $R(x,y)z = \langle zyx\rangle$, $P(x,y)z = \langle xzy\rangle$. Then

$$\langle xyz\rangle = L(x,y)z = R(z,y)x = P(x,z)y.$$

Caution: Observe the reversed order in $\langle xyz\rangle = R(z,y)x$.

Derivations are defined the obvious way. $D \in \mathrm{End}_\Phi \mathcal{T}$ is a derivation of $\mathcal{T}$, if

(3.1) $\quad D\langle xyz\rangle = \langle (Dx)yz\rangle + \langle x(Dy)z\rangle + \langle xy(Dz)\rangle$

for all $x,y,z \in \mathcal{T}$, or equivalently

(3.2) $\quad [D, L(x,y)] = L(Dx,y) + L(x,Dy)$ for all $x,y \in \mathcal{T}$.

Again $\mathcal{D}(\mathcal{T})$ the Φ-module of all derivations of $\mathcal{T}$ is a subalgebra of $(\mathrm{End}_{\Phi}\mathcal{T})^{-}$

Exercise. 4 If $\mathfrak{A}$ is a triple system coming from an algebra (see example 2) then any algebra derivation or homomorphism is a derivation or homomorphism of the triple system. Skip the rest of this chapter for now

3.4. There is still another aspect of triple systems we want to mention. Let $\mathcal{T}$ be an arbitrary triple system over $\Phi, L(x,y)z = \langle xyz\rangle$. Then by definition $(x,y) \mapsto L(x,y)$ is a bilinear map of $\mathcal{T} \times \mathcal{T}$ into $\mathrm{End}_{\Phi}\mathcal{T}$. But from the definition of the tensor product of Φ-modules, we get a unique linear map

$$S: \mathcal{T} \otimes \mathcal{T} \to \mathrm{End}\,\mathcal{T}, \text{ such that}$$

$$S(x \otimes y) = L(x,y).$$

And obviously any linear map of $\mathcal{T} \otimes \mathcal{T} \to \mathrm{End}\,\mathcal{T}$ defines a triple system structure on $\mathcal{T}$.

Now we restrict to a special case. Assume $\mathcal{T}$ is finite dimensional over a field F. Then $\mathcal{T} \otimes \mathcal{T} \cong \mathrm{End}\,\mathcal{T}$, but there are many ways to obtain this isomorphism. We assume, that λ is a non degenerate symmetric bilinear form on $\mathcal{T}$. We define $xy^* \in \mathrm{End}\,\mathcal{T}$ by

$$(xy^*)z := \lambda(z,y)x$$

It is easy to prove and is left as an exercise, $x \otimes y \to xy^*$ defines an isomorphism (of vector spaces) $\mathcal{T} \otimes \mathcal{T}$ and $\mathrm{End}\,\mathcal{T}$, in particular

(i) $\{xy^*, x,y \in \mathcal{T}\}$ generates $\mathrm{End}\,\mathcal{T}$. Furthermore

(3.3) (ii) $\mathrm{trace}\; xy^* = \lambda(x,y)$

(iii) $(xy^*)^* = yx^*$

(iv) $A(xy^*)B^* = Ax(By)^*$ for all $x,y \in \mathcal{T}$, $A,B \in \mathrm{End}\,\mathcal{T}$

where A^* denotes the adjoint of A relative to λ.

As in the case of algebras (see 1.9.), associative bilinear-forms might be useful.

There are more possibilities to define associative bilinear forms on $\mathcal{T}$. One possible definition is as follows: λ is called associative, if

(3.5) (i) $\lambda(\langle xyz\rangle,u) = \lambda(x,\langle uzy\rangle) = \lambda(z,\langle yxu\rangle)$ for all $x,y,z,u \in \mathcal{T}$. Assume λ non degenerate, symmetric and associative. Then (3.5) is equivalent to

(3.5') $L(x,y)^* = L(y,x); R(z,y)^* = R(y,z)$. If $A \in \operatorname{End}\mathcal{T}$ then there exists a unique $S(A) \in \operatorname{End}\mathcal{T}$ such that

(3.6) $\operatorname{trace} AL(x,y) = \lambda(S(A)x,y)$ (since λ is non degenerate).

Next we show

(3.7) $$S(uv^*) = L(u,v)$$

where $uv^*z = \lambda(z,v)u$ (see (3.3)).

$$\begin{aligned}\lambda(S(uv^*)x,y) &= \operatorname{trace} uv^*L(x,y) = \operatorname{tr} L(x,y)uv^* \\ &= \lambda(\langle xyu\rangle,v) = \lambda(x,\langle vuy\rangle) \\ &= \lambda(\langle uvx\rangle,y) = \lambda(L(u,v)x,y)\,.\end{aligned}$$

(3.6) and (3.7) imply $\operatorname{trace} AS(xy^*)$ $\operatorname{trace} S(A)xy^*$, consequently

(3.8) $\operatorname{trace} S(A)B = \operatorname{trace} AS(B)$.

Exercise: Define $S'(A)$ by $\operatorname{trace} AR(x,y) = \lambda(S'(A)x,y)$ and show $\operatorname{tr} S'(A)B = \operatorname{tr} AS'(B)$.

IV. Associative Triple Systems.

4.1. As we have seen in example 2) of the previous chapter, one can associate to any class of algebras a corresponding class of triple systems by considering the triple composition $(a,b,c) \to \langle abc\rangle = (ab)c$, where $(a,b) \to ab$ is the product in