As in the case of algebras (see 1.9.), associative bilinear-forms might be useful.

There are more possibilities to define associative bilinear forms on $\mathcal{T}$. One possible definition is as follows: λ is called associative, if

(3.5) (i) $\lambda(\langle xyz\rangle,u) = \lambda(x,\langle uzy\rangle) = \lambda(z,\langle yxu\rangle)$ for all $x,y,z,u \in \mathcal{T}$. Assume λ non degenerate, symmetric and associative. Then (3.5) is equivalent to

(3.5') $L(x,y)^* = L(y,x)$; $R(z,y)^* = R(y,z)$. If $A \in \operatorname{End}\mathcal{T}$ then there exists a unique $S(A) \in \operatorname{End}\mathcal{T}$ such that

(3.6) $\operatorname{trace} AL(x,y) = \lambda(S(A)x,y)$ (since λ is non degenerate).

Next we show

$$S(uv^*) = L(u,v) \tag{3.7}$$

where $uv^*z = \lambda(z,v)u$ (see (3.3)).

$$\begin{aligned}\lambda(S(uv^*)x,y) &= \operatorname{trace} uv^*L(x,y) = \operatorname{tr} L(x,y)uv^* \\ &= \lambda(\langle xyu\rangle,v) = \lambda(x,\langle vuy\rangle) \\ &= \lambda(\langle uvx\rangle,y) = \lambda(L(u,v)x,y)\,.\end{aligned}$$

(3.6) and (3.7) imply $\operatorname{trace} AS(xy^*)$ $\operatorname{trace} S(A)xy^*$, consequently

(3.8) $\operatorname{trace} S(A)B = \operatorname{trace} AS(B)$.

<u>Exercise:</u> Define $S'(A)$ by $\operatorname{trace} AR(x,y) = \lambda(S'(A)x,y)$ and show $\operatorname{tr} S'(A)B = \operatorname{tr} AS'(B)$.

IV. Associative Triple Systems.

4.1. As we have seen in example 2) of the previous chapter, one can associate to any class of algebras a corresponding class of triple systems by considering the triple composition $(a,b,c) \to \langle abc\rangle = (ab)c$, where $(a,b) \to ab$ is the product in the algebra. Starting with associative algebras we come to the definition:

A triple system $\mathcal{T}$ is <u>associative</u> (of the first kind), if

(4.1) $\quad \langle xy\langle uvw\rangle\rangle = \langle\langle xyu\rangle vw\rangle = \langle x\langle yuv\rangle w\rangle$ for all $x,y,u,v,w \in \mathcal{T}$.

In terms of left and right multiplications (4.1) is equivalent to either

(4.2) $\quad L(x,y)L(u,v) = L(\langle xyu\rangle,v) = L(x,\langle yuv\rangle)$

(4.3) $\quad R(w,v)R(u,y) = R(\langle uvw\rangle,y) = R(w,\langle yuv\rangle)$

$\quad L(x,y)R(w,v) = R(w,v)L(x,y) = P(x,w)P(y,v)$

<u>Example.</u> Any associative algebra $\mathfrak{A}$ together with $(x,y,z) \mapsto (xy)z$ is an associative triple system of the first kind, and so is any ~~submodule~~ subspace of $\mathfrak{A}$ closed under $(xy)z$.

Let $\mathcal{E} := \mathrm{End}_\Phi \mathcal{T} \oplus (\mathrm{End}_\Phi \mathcal{T})^{op}$ the direct sum of the algebra of endomorphisms of $\mathcal{T}$ with its opposite algebra. Consider $\mathcal{L}_0$ the ~~submodule~~ subspace of $\mathcal{E}$ generated by all $\lambda(x,y) := (L(x,y),R(y,x))$ then (4.2) and (4.3) show

$$\begin{aligned}\lambda(x,y)\lambda(u,v) &= (L(x,y),R(y,x))(L(u,v),R(v,u)) \\ &= (L(x,y)L(u,v),R(v,u)R(y,x)) \\ &= (L(\langle xyu\rangle,v),R(v,\langle xyu\rangle)) \\ &= (L(x,\langle yuv\rangle),R(\langle yuv\rangle,x)),\ \text{i.e.}\end{aligned}$$

(4.5) $\quad \lambda(x,y)\lambda(u,v) = \lambda(x,\langle yuv\rangle) = \lambda(\langle xyu\rangle,u)$

consequently

$\mathcal{L}_0$ is a subalgebra of $\mathcal{E}$. Let E denote the unit element of $\mathcal{E}$, then

$\Phi = \mathbb{R}$ or $\mathbb{C}$

$$\mathcal{L} := \Phi E + \mathcal{L}_0$$

is a subalgebra of $\mathcal{E}$, too.

The ~~Φ-module~~ vector space $\mathcal{T}$ is in a natural way an $\mathcal{E}$ left and an $\mathcal{E}$ right module according to the following definitions. If $A = (A_1, A_2) \in \mathcal{E}$, define

(4.6) $\qquad A \cdot x := A_1 x, \quad x \cdot A := A_2 x$

and it is obvious that $(A,x) \to A \cdot x$ makes $\mathcal{T}$ a left $\mathcal{E}$ module and $(A,x) \to x \cdot A$ makes $\mathcal{T}$ a right $\mathcal{E}$ module. Since $\mathcal{L}$ is a subalgebra of $\mathcal{E}$ we have the following result:

Lemma 1. $\mathcal{T}$ together with the maps $\mathcal{L} \times \mathcal{T} \to \mathcal{T}$, $(A,x) \mapsto A \cdot x$, $\mathcal{T} \times \mathcal{L} \to \mathcal{T}$, $(x,A) \mapsto x \cdot A$ is a left and a right $\mathcal{E}$ module. (even an $\mathcal{E}$-bimodule)

Consider the ~~Φ-module~~ vector space

$$\mathfrak{A} := \mathcal{L} \oplus \mathcal{T}$$

and define a product in $\mathfrak{A}$ by the formula

(4.7) $\quad (A \oplus x)(B \oplus y) := AB + \lambda(x,y) \oplus A \cdot y + x \cdot B$

Theorem 1. If $\mathcal{T}$ is an associative triple system of the first kind, then $\mathfrak{A} = \mathcal{L} \oplus \mathcal{T}$ with multiplication as defined in (4.7) is an associative algebra with unit element containing $\mathcal{T}$ (isomorphically imbedded) such that $\langle xyz \rangle = (xy)z$ for all $x,y,z \in \mathcal{T}$.

The proof is left as an $\boxed{\text{exercise.}^{1}}$

4.2. Since for later applications we need a classification of a very similar type of triple systems we do not present a structure theory for associative triple systems of the first kind. We leave it as an exercise to use the methods and arguments we shall develop below to build up parts of a structure theory of associative triple systems of the first kind.

Very similar to the definition in 4.1 is the following: A triple system $\mathfrak{M}$ is called <u>associative</u> (<u>of the second kind</u>), if

(4.8) $\langle\langle xyz\rangle uv\rangle = \langle xy\langle zuv\rangle\rangle = \langle x\langle uzy\rangle v\rangle$

<u>Note:</u> The right hand side equations of (4.1) and (4.8) are different. In the sequel "associative triple system' (= a.t.s) always means "associative triple system of the second kind".

(4.8) is equivalent to either

(4.9) $L(x,y)L(z,u) = L(\langle xyz\rangle,u) = L(x,\langle uzy\rangle)$

(4.10) $R(v,u)R(z,y) = R(\langle zuv\rangle,y) = R(v,\langle uzy\rangle)$

(4.11) $R(v,u)L(x,y) = L(x,y)R(v,u) = P(x,v)P(u,y)$

<u>Example</u>. Let $\mathfrak{A}$ be an associative algebra with involution $x \mapsto \bar{x}$, then $\mathfrak{A}$ together with the map $(x,y,z) \mapsto x\bar{y}z$ is an associative triple system, and so is any subspace ~~submodule~~ of $\mathfrak{A}$ which is closed under $x\bar{y}z$. In particular the Vector space ~~Φ-module~~ of all pxq-matrices over $\mathbb{R}$ or $\mathbb{C}$ ~~Φ~~ together with $(A,B,C) \to AB^tC$ is an a.t.s. (see example 1) in 3.1.).

Let $\mathfrak{M}$ be an a.t.s. We set $\mathcal{L} := \mathrm{End}_\Phi \mathfrak{M} \oplus (\mathrm{End}_\Phi \mathfrak{M})^{op}$ we define

$$l(x,y) := (L(x,y), L(y,x))$$

$$r(x,y) := (R(y,x), R(x,y)).$$

Let $\mathcal{L}_0$ be the submodule of $\mathcal{L}$ spanned by all $l(x,y), x,y \in \mathfrak{M}$ and $\mathcal{R}_0$ be the submodule of $\mathcal{L}^{op}$ spanned by all $r(x,y), x,y \in \mathfrak{M}$.

(4.9) and (4.10) imply (do the computations) Exercise 2

(4.12) $l(x,y)l(u,v) = l(\langle xyu\rangle,v) = l(x,\langle vuy\rangle)$

(4.13) $r(x,y)r(u,v) = r(x,\langle yuv\rangle) = r(\langle uyx\rangle,v)$

(<u>Note:</u> the product on the left hand side of (4.13) is taken in $\mathcal{L}^{op}$.)

The last two equations show that $\mathcal{L}_0$ resp. $\mathcal{R}_0$ are subalgebras of $\mathcal{E}$ resp. $\mathcal{E}^{op}$. The algebras $\mathcal{E}$ and $\mathcal{E}^{op}$ have a natural involution, namely $(A,B) \mapsto \overline{(A,B)} = (B,A)$. Obviously $\overline{l(x,y)} = l(y,x)$, $\overline{r(x,y)} = r(y,x)$. Let E_1 resp. E_2 be the unit element in $\mathcal{E}$ resp. $\mathcal{E}^{op}$. We define

$$\mathcal{L} := \Phi E_1 + \mathcal{L}_0 \quad , \mathcal{R} = \Phi E_2 + \mathcal{R}_0 \qquad (\Phi = \mathbb{R} \text{ or } \mathbb{C})$$

From the preceding discussion it follows

<u>Lemma 2.</u> <u>$\mathcal{L}$ and $\mathcal{R}$ are subalgebras of $\mathcal{E}$ resp. $\mathcal{E}^{op}$ invariant under the canonical involution. $\mathcal{L}_0$ (resp $\mathcal{R}_0$) is an ideal in $\mathcal{L}$ (resp. $\mathcal{R}$).</u>

The ~~Φ-module~~ a.t.s $\mathcal{M}$ is in a natural way a left $\mathcal{E}$-module and a right $\mathcal{E}^{op}$-module, according to the following compositions. If $A = (A_1,A_2) \in \mathcal{E}$, $B = (B_1,B_2) \in \mathcal{E}^{op}$ and $x \in \mathcal{M}$ we set

$$(4.14) \qquad A\cdot x := A_1 x \quad , \quad x\cdot B := B_1 x$$

We take an isomorphic copy of $\mathcal{M}$, denoted by $\overline{\mathcal{M}}$. By the definitions

$$(4.15) \qquad \bar{x}\cdot A := \overline{A_2 x} \quad , \quad B\cdot\bar{x} := \overline{B_2 x}$$

if $\bar{x} \in \overline{\mathcal{M}}$, $A = (A_1,A_2) \in \mathcal{E}$ and $B = (B_1,B_2) \in \mathcal{E}^{op}$ it is obvious that $\overline{\mathcal{M}}$ becomes a right $\mathcal{E}$-module and a left $\mathcal{E}^{op}$-module.

Since $\mathcal{L}$ and $\mathcal{R}$ are subalgebras of $\mathcal{E}$ resp $\mathcal{E}^{op}$ we have the following result

<u>Lemma 3.</u> (i) <u>$\mathcal{M}$ together with the mappings defined by</u> (4.14) <u>is a left $\mathcal{L}$-module and a right $\mathcal{R}$-module.</u>

(ii) <u>$\overline{\mathcal{M}}$ together with the mappings defined by</u> (4.15) <u>is a right $\mathcal{L}$-module and a left $\mathcal{R}$-module.</u>

<u>Exercise</u> 3 Show that $\mathcal{M}$ (resp. $\overline{\mathcal{M}}$) is an $(\mathcal{L},\mathcal{R})$-bimodule (resp. $(\mathcal{R},\mathcal{L})$-bimodule), i.e. it is not only a left $\mathcal{L}$-module and a

right $\mathcal{R}$-module, but furthermore $(A\cdot x)\cdot B = A\cdot(x\cdot B)$ holds for all $A\in\mathcal{L}$, $B\in\mathcal{R}$, $x\in\mathfrak{M}$.

Now we consider the module

$$\mathfrak{A} := \mathcal{L}\oplus\mathfrak{M}\oplus\overline{\mathfrak{M}}\oplus\mathcal{R}$$

For the convenience of notation we write the elements of $\mathfrak{A}$ in matrix form

$$\begin{pmatrix} A & x \\ \bar{y} & B \end{pmatrix}, \qquad A\in\mathcal{L},\ B\in\mathcal{R},\ x\in\mathfrak{M},\ \bar{y}\in\overline{\mathfrak{M}}.$$

and by means of the module isomorphisms $A\mapsto\begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix}$, $x\mapsto\begin{pmatrix} 0 & x \\ 0 & 0\end{pmatrix}$ etc., we identify $\mathcal{L}, \mathcal{R}, \mathfrak{M}, \overline{\mathfrak{M}}$ with its image. We define a multiplication on $\mathfrak{A}$ by

$$(4.16)\qquad \begin{pmatrix} A & x \\ \bar{y} & B \end{pmatrix} * \begin{pmatrix} A', & x' \\ \bar{y}', & B' \end{pmatrix} := \begin{pmatrix} AA' + l(x,y'), & A\cdot x' + x\cdot B' \\ \bar{y}\cdot A' + B\cdot\bar{y}', & r(y,x') + BB' \end{pmatrix}$$

The following result is fundamental:

<u>Theorem 2. If $\mathfrak{M}$ is an a.t.s. then</u>

(i) $\mathfrak{A} = \mathcal{L}\oplus\mathfrak{M}\oplus\overline{\mathfrak{M}}\oplus\mathcal{R}$ <u>together with the product defined by</u> (4.16) <u>is an associative Φ-algebra with unit element</u> $e = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$

(ii) $\mathfrak{A}_0 = \mathcal{L}_0\oplus\mathfrak{M}\oplus\overline{\mathfrak{M}}\oplus\mathcal{R}_0$ <u>is an ideal in</u> $\mathfrak{A}$

(iii) <u>The map</u> $j:\ u = \begin{pmatrix} A & x \\ \bar{y} & B\end{pmatrix} \mapsto \bar{u} = \begin{pmatrix} \bar{A}, y \\ \bar{x}, \bar{B}\end{pmatrix}$ <u>is an involution of</u> $\mathfrak{A}$.

(iv) <u>If</u> $x,y,z\in\mathfrak{M}$, <u>then</u> $\langle xyz\rangle = x*\bar{y}*z$. $= \begin{pmatrix} 0 & x \\ 0 & 0\end{pmatrix} * \begin{pmatrix} 0 & y \\ 0 & 0\end{pmatrix} * \begin{pmatrix} 0 & z \\ 0 & 0\end{pmatrix}$

(v) <u>The Peirce components of</u> $\mathfrak{A}$ <u>relative to the idempotent</u> E_1, <u>are</u>

$$\mathfrak{A}_{11} = \mathcal{L},\ \mathfrak{A}_{10} = \mathfrak{M},\ \mathfrak{A}_{01} = \overline{\mathfrak{M}},\ \mathfrak{A}_{00} = \mathcal{R}$$

(For the notations concerning Peirce decomposition see 2.5.).

Proof. The only difficulties related to this theorem are in finding the given construction. The verification of the above statements is done by straightforward computations using (4.9)-(4.16) and the previous 2 lemmas and is left as an $\boxed{\text{exercise.}4}$

$$\mathfrak{A} = \mathfrak{A}(\mathfrak{M}) \text{ is called the standard imbedding of } \mathfrak{M}.$$

Lemma 4. Let $\mathfrak{M}$ be an a.t.s. and $\mathfrak{A}$ its standard imbedding. If $\mathcal{J}$ is an ideal in $\mathfrak{A}$ then

(i) $\mathcal{J} = (\mathcal{J} \cap \mathcal{L}) \oplus (\mathcal{J} \cap \mathfrak{M}) \oplus (\mathcal{J} \cap \overline{\mathfrak{M}}) \oplus (\mathcal{J} \cap \mathcal{R})$

(ii) If $\mathcal{J}$ is j-stable then $\mathcal{J} \cap \mathfrak{M}$ is an ideal in $\mathfrak{M}$

(iii) If $\mathcal{J} \subset \mathcal{L} \oplus \mathcal{R}$, then $\mathcal{J} = 0$.

Proof. (i) follows from part (v) of the above theorem and II, Lemma 7.

(ii) If $b \in \mathcal{J} \cap \mathfrak{M}$ and $x, y \in \mathfrak{M}$ then

$$\langle xyb \rangle = x * \bar{y} * b \quad \text{and} \quad \langle bxy \rangle = b * \bar{x} * y$$

are in $\mathfrak{M} \cap \mathcal{J}$. Since $\mathcal{J}$ is j-invariant $\bar{b}$ is in $\mathcal{J}$ and consequently $\langle xby \rangle = x * \bar{b} * y \in \mathcal{J} \cap \mathfrak{M}$.

(iii) If $\mathcal{J} \subset \mathcal{L} \oplus \mathcal{R}$ then $\mathcal{J} = (\mathcal{L} \cap \mathcal{J}) \oplus (\mathcal{R} \cap \mathcal{J})$ and $\mathcal{J} \cap \mathfrak{M} = \mathcal{J} \cap \overline{\mathfrak{M}} = 0$, by part (i). If $A = (A_1, A_2) \in \mathcal{L} \cap \mathcal{J}$ then $A * \mathfrak{M} = A \cdot \mathfrak{M} = A_1 \mathfrak{M} \subset \mathcal{J} \cap \mathfrak{M} = 0$, thus $A_1 = 0$ similarly $A_2 = 0$ and also $\mathcal{L} \cap \mathcal{J} = \mathcal{R} \cap \mathcal{J} = 0$.

4.3. Let $\mathfrak{M}$ be an a.t.s. For fixed $u \in \mathfrak{M}$ we consider the map $(x, y) \mapsto \langle xuy \rangle = x \cdot y$. The resulting algebra is denoted by $\mathfrak{M}_u$. It is immediately seen from the equation (4.8) (put $y = u$) that $\mathfrak{M}_u$ is an associative algebra with left multiplication