Meyberg pp. 40-46

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1 Lie algebras—Meyberg, Chapter 5

An algebra $\mathcal{L}$ with multiplication $(x, y) \mapsto [x, y]$ is a Lie algebra if

$$[xx] = 0$$

and

$$[[xy]z] + [[yz]x] + [[zx]y] = 0.$$

Left multiplication in a Lie algebra is denoted by $\mathrm{ad}(x)$: $\mathrm{ad}(x)(y) = [x, y]$. An associative algebra A becomes a Lie algebra A^- under the product, $[xy] = xy - yx$.

The first axiom implies that $[xy] = -[yx]$ and the second (called the *Jacobi identity*) implies that $x \mapsto \mathrm{ad}x$ is a homomorphism of $\mathcal{L}$ into the Lie algebra $(\mathrm{End}\,\mathcal{L})^-$, that is, $\mathrm{ad}\,[xy] = [\mathrm{ad}\,x, \mathrm{ad}\,y]$.

Assuming that $\mathcal{L}$ is finite dimensional, the Killing form is defined by $\lambda(x, y) = \mathrm{tr}\,(\mathrm{ad}(x)\mathrm{ad}(y))$.

Theorem 1 (CARTAN criterion—Theorem 1, page 41) *A finite dimensional Lie algebra $\mathcal{L}$ over a field of characteristic 0 is semisimple if and only if the Killing form is nondegenerate.*

Proof. The proof is not given in Meyberg's notes. However, we don't really need this theorem for our purposes since the Killing form will be nondegenerate in the case we are interested in (finite dimensional semisimple Jordan triple systems).

A linear map D is a derivation if $D \cdot \mathrm{ad}(x) = \mathrm{ad}(Dx) + \mathrm{ad}(x) \cdot D$. Each $\mathrm{ad}(x)$ is a derivation, called an inner derivation. Let $\mathcal{D}er(\mathcal{L})$ be the set of all derivations on $\mathcal{L}$.

Theorem 2 (Zassenhaus—Theorem 3, page 42) *If the finite dimensional Lie algebra $\mathcal{L}$ over a field of characteristic 0 is semisimple (that is, its Killing form is nondegenerate), then every derivation is inner.*

Proof. Let D be a derivation of $\mathcal{L}$. Since $x \mapsto \mathrm{tr}\,(D \cdot \mathrm{ad}\,(x))$ is a linear form, there exists $d \in \mathcal{L}$ such that $\mathrm{tr}\,(D \cdot \mathrm{ad}\,(x)) = \lambda(d, x) = \mathrm{tr}\,(\mathrm{ad}\,(d) \cdot \mathrm{ad}\,(x))$. Let E be the derivation $E = D - \mathrm{ad}\,(d)$ so that

$$\mathrm{tr}\,(E \cdot \mathrm{ad}\,(x)) = 0. \tag{1}$$

Note next that

$$\begin{aligned} E \cdot [\mathrm{ad}\,(x), \mathrm{ad}\,(y)] &= E \cdot \mathrm{ad}\,(x) \cdot \mathrm{ad}\,(y) - E \cdot \mathrm{ad}\,(y) \cdot \mathrm{ad}\,(x) \\ &= (\mathrm{ad}\,(x) \cdot E + [E, \mathrm{ad}\,(x)]) \cdot \mathrm{ad}\,(y) - E \cdot \mathrm{ad}\,(y) \cdot \mathrm{ad}\,(x) \end{aligned}$$

so that

$$\begin{aligned} [E, \mathrm{ad}\,(x)] \cdot \mathrm{ad}\,(y) &= E \cdot [\mathrm{ad}\,(x), \mathrm{ad}\,(y)] - \mathrm{ad}\,(x) \cdot E \cdot \mathrm{ad}\,(y) + E \cdot \mathrm{ad}\,(y) \cdot \mathrm{ad}\,(x) \\ &= E \cdot [\mathrm{ad}\,(x), \mathrm{ad}\,(y)] + [E \cdot \mathrm{ad}\,(y), \mathrm{ad}\,(x)] \end{aligned}$$

and

$$\mathrm{tr}\,([E, \mathrm{ad}\,(x)] \cdot \mathrm{ad}\,(y)) = \mathrm{tr}\,(E \cdot [\mathrm{ad}\,(x), \mathrm{ad}\,(y)]).$$

However, since E is a derivation

$$\begin{aligned} [E, \mathrm{ad}\,(x)] \cdot \mathrm{ad}\,(y) &= E \cdot \mathrm{ad}\,(x) \cdot \mathrm{ad}\,(y) - \mathrm{ad}\,(x) \cdot E \cdot \mathrm{ad}\,(y) \\ &= (\mathrm{ad}\,(Ex) + \mathrm{ad}\,(x) \cdot E) \cdot \mathrm{ad}\,(y) - \mathrm{ad}\,(x) \cdot E \cdot \mathrm{ad}\,(y) \\ &= \mathrm{ad}\,(Ex) \cdot \mathrm{ad}\,(y). \end{aligned}$$

Thus

$$\begin{aligned} \lambda(Ex, y) &= \mathrm{tr}\,(\mathrm{ad}\,(Ex) \cdot \mathrm{ad}\,(y)) \\ &= \mathrm{tr}\,([E, \mathrm{ad}\,(x)] \cdot \mathrm{ad}\,(y)) \\ &= \mathrm{tr}\,(E \cdot [\mathrm{ad}\,(x), \mathrm{ad}\,(y)]) = 0 \text{ by (1)}). \end{aligned}$$

Since x and y are arbitrary, $E = 0$ and so $D - \mathrm{ad}\,(d) = 0$.

Appendix: Trace of a matrix

Reference: (text for 121AB) Linear Algebra, by Friedberg,Insel, Spence

- (Example 4, p. 18) The **trace** of and $n \times n$ matrix M is the sum of the diagonal entries of M: $\mathrm{tr}(M) = \mathrm{trace}(M) = \sum_{i=1}^{n} m_{ii}$, if $M = [m_{ij}]$.
- (Exercise 6, p. 20) Prove that $\mathrm{tr}(aA + bB) = a\,\mathrm{tr}(A) + b\,\mathrm{tr}(B)$.
- (Exercise 13, p. 97) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
- (Exercise 10, p. 118) Prove that if A is similar to B, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.
- (Exercise 16(b), p. 259) How would you define the trace of a linear operator on a finite dimensional vector space?
- (Example 5, p. 331) Let $V = M_{n\times n}(F)$, the vector space of all $n \times n$ matrices over a field F. Show that $\langle A, B\rangle := \mathrm{tr}(B^* A)$ defines an inner product on V.

2 Lie triple systems—Meyberg, Chapter 6

A Lie triple system is a vector space $\mathcal{F}$ together with a triple product $[\cdot,\cdot,\cdot]$ which satisfies

1. $[xxz] = 0$ (implies $[xyz] = -[yxz]$)
2. $[xyz] + [yzx] + [zxy] = 0$ (Jacobi identity)
3. $[uv[xyz]] = [[uvx]yz] + [x[uvy]z] + [xy[uvz]]$

Examples:

- A Lie algebra $(\mathcal{L}, [xy])$ under $[xyz] := [[xy]z]$
- A subspace of a Lie algebra, closed under $[[xy]z]$
- An associative triple system $(\mathcal{F}, < xyz >)$ under $[xyz] :=< xyz > - < yxz > - < zxy > + < zyx >$
- A Jordan algebra under $[xyz] := [L(x), L(y)]z$

Define $L'(x,y), R'(z,y), P'(x,z) \in \mathrm{End}\,(\mathcal{F})$ by

$$[xyz] = L'(x,y)z = R'(z,y)x = P'(x,z)y,$$

where we are using L' (etc.) instead of L to avoid confusion with the operator $L(x,y)$ of a Jordan triple system ($L'(x,y)z = [xyz]$ in Lie triple systems, $L(x,y)z = \{xyz\}$ in Jordan triple systems).

The axioms for a Lie triple system become

1. $L'(x,x) = 0$ (implies $L'(x,y) = -L'(y,x)$)
2. $L'(x,y) = R'(x,y) - R'(y,x)$
3. $[L'(x,y), L'(u,v)] = L'([xyu],v) + L'(u,[xyv])$

A derivation is a linear map $D : \mathcal{F} \to \mathcal{F}$ satisfying $[D, L'(x,y)] = L'(Dx,y) + L'(x,Dy)$, equivalently, $D[xyz] = [Dx,y,z] + [x,Dy,z] + [x,y,Dz]$. From this it follows that if D, D' are derivations, then so is $[D,D']$, so that the set $\mathcal{D}er(\mathcal{F})$ of all derivations on $\mathcal{F}$ is a Lie algebra of operators. Let $\mathcal{H}(\mathcal{F})$ = the span of all $L'(x,y)$.

The following lemma is immediate from the definition of derivation.

Lemma 2.1 (Lemma 2, page 44) *$L'(x,y)$ is a derivation of $\mathcal{F}$ and $\mathcal{H}(\mathcal{F})$ is an ideal in $\mathcal{D}er(\mathcal{F})$*

Theorem 3 (Theorem 1, page 45) *Let $\mathcal{F}$ be a Lie triple system, let $\mathcal{G}$ be a subalgebra of $\mathcal{D}er(\mathcal{F})$, and suppose that $\mathcal{H} \subset \mathcal{G}$.*

(i) $\mathcal{L}(\mathcal{G}, \mathcal{F}) := \mathcal{G} \oplus \mathcal{F}$ *is a Lie algebra under the product*

$$[H_1 \oplus x_1, H_2 \oplus x_2] = ([H_1, H_2] + L'(x_1, x_2)) \oplus (H_1 x_2 - H_2 x_1) \tag{2}$$

(ii) $\theta'(H \oplus x) = (-H) \oplus x$ *is an involution of $\mathcal{L}$, that is, $\theta'^2 = Id$ and $\theta'[X, Y] = [\theta' Y, \theta' X]$.*

(iii) $\mathcal{L}(\mathcal{H}, \mathcal{F})$ *is an ideal in* $\mathcal{L}(\mathcal{G}, \mathcal{F})$

(iv) $[xyz] = [[x, y], z]$ *for* $x, y, z \in \mathcal{F}$.

(v) $\mathcal{F} = \{X \in \mathcal{L}(\mathcal{G}, \mathcal{F}) : \theta' X = X\}$

Proof. If $X = H \oplus x$, then $[X, X] = ([H, H] + L'(x, x)) \oplus (Hx - Hx) = 0$. We have to show that $J(Y, Y, Z) := [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ for alll $X, Y, Z \in \mathcal{L}$. It is sufficient to show this for $X, Y, Z \in \mathcal{G} \cup \mathcal{F}$.

Since $\mathcal{G}$ is a subalgebra of $\mathcal{D}er(\mathcal{F})$, $J(\mathcal{G}, \mathcal{G}, \mathcal{G}) = 0$.

If $H_i \in \mathcal{G}, x \in \mathcal{F}$ we get $[[H_1, H_2], x] = [H_1, H_2]x = H_1(H_2 x) - H_2(H_1(x) = [H_1, [H_2, x]] - [H_2, [H_1, x]$.[1] This shows $J(\mathcal{G}, \mathcal{G}, \mathcal{F}) = 0$. Since $J(H_1, x, H_2) = J(H_2, H_1, x)$ and $J(x, H_1, H_2) = J(H_1, H_2, x)$, we have $J(\mathcal{G}, \mathcal{F}, \mathcal{G}) = 0$ and $J(\mathcal{F}, \mathcal{G}, \mathcal{G}) = 0$.

By (2), $[[H, x], y] + [[x, y], H] + [[y, H], x] = L'(Hx, y) + [L'(x, y), H] + L'(x, Hy) = 0$ since H is a derivation. As above, $J(x, H, y) = J(H, y, x)$ and $J(x, y, H) = J(H, x, y)$ so that $J(\mathcal{F}, \mathcal{G}, \mathcal{F}) = J(\mathcal{F}, \mathcal{F}, \mathcal{G}) = 0$. Finally, by the Jacobi identity, $J(\mathcal{F}, \mathcal{F}, \mathcal{F}) = 0$.

This proves (i) and the other statements are left as an informal exercise.

Our goal for the rest of this quarter is to prove the following theorem.

Theorem 4 (Theorem 10, page 57) *If F is a semisimple Lie triple system over a field of characteristic zero, then every derivation of F is inner.*

[1] Longer version: $[[H_1, H_2], x] = [[H_1 \oplus 0, H_2 \oplus 0], 0 \oplus x] = [[H_1, H_2] \oplus 0, 0 \oplus x] = 0 \oplus [H_1, H_2]x = 0 \oplus (H_1(H_2 x) - H_2(H_1 x)) = 0 \oplus H_1(H_2 x) - 0 \oplus H_2(H_1 x) = [H_1 \oplus 0, 0 \oplus H_2 x] - [H_2 \oplus 0, 0 \oplus H_1 x] = [H_1 \oplus 0, [H_2 \oplus 0, 0 \oplus x]] - [H_2 \oplus 0, [H_1 \oplus 0, 0 \oplus x]] = [H_1, [H_2, x]] - [H_2, [H_1, x]]$