$\boxed{6.3}$ Meyberg p 47

$\mathfrak{F}_1, \mathfrak{F}_2$ Lie triple system

$\mathcal{L}_1 = \mathfrak{H}_1 \oplus \mathfrak{F}_1$ $\mathcal{L}_2 = \mathfrak{H}_2 \oplus \mathfrak{F}_2$ the standard embeddings

If $\phi : \mathfrak{H}_1 \to \mathfrak{H}_2$ and $\eta : \mathfrak{F}_1 \to \mathfrak{F}_2$ are linear maps

then $\lambda : \mathcal{L}_1 \to \mathcal{L}_2$ defined by $\lambda(H \oplus x) = \phi(H) \oplus \eta(x)$

is also a linear map.

<u>Lemma 3</u> Assume $\phi : \mathfrak{H}_1 \to \mathfrak{H}_2$ is a Lie algebra homomorphism

and (i) $\phi(L_1(x,y)) = L_2(\eta x, \eta y)$ $\forall\, x, y \in \mathfrak{F}_1$

(ii) $\eta \circ H = \phi(H) \circ \eta$ $\forall\, H \in \mathfrak{H}_1$

Proof = Exercise 4 of chapter 6

$\mathfrak{H}_i = \{ \sum_{i=1}^{n} L(x_i, y_i) : x_i, y_i \in \mathfrak{F}_i, n = 1, 2, .. \}$

$L(x,y)z = [xyz]$

$$\begin{tikzcd} \mathfrak{F}_1 \arrow[r, "H"] \arrow[d, "\eta"'] & \mathfrak{F}_1 \arrow[d, "\eta"] \\ \mathfrak{F}_2 \arrow[r, "\phi(H)"'] & \mathfrak{F}_2 \end{tikzcd}$$

<u>def.</u> $\eta : \mathfrak{F}_1 \to \mathfrak{F}_2$ is a <u>Lie triple homomorphism</u>

if η is linear and $\eta[xyz] = [\eta x, \eta y, \eta z]$

$$\begin{tikzcd} \mathfrak{F}_2 \arrow[d, "L_2(\eta x, \eta y)"'] & \mathfrak{F}_1 \arrow[l, "\eta"'] \arrow[d, "L_1(x,y)"] \\ \mathfrak{F}_2 & \mathfrak{F}_1 \arrow[l, "\eta"] \end{tikzcd}$$

$\eta \circ L_1(x,y) = L_2(\eta x, \eta y) \circ \eta$

"Hint" for

Exercise 5 of chapter 6 (see Meyberg p. 47)

If $\eta : \mathcal{T}_1 \to \mathcal{T}_2$ is a Lie triple homo then $\eta \circ H \circ \eta^{-1} \in \mathcal{H}_2$ for every $H \in \mathcal{H}_1$ and $\phi(H) = \eta \circ H \circ \eta^{-1}$ defines a Lie algebra isomorphism ϕ of $\mathcal{H}_1$ onto $\mathcal{H}_2$ satisfying (i) & (ii) of Lemma 3.

Meyberg p. 48 Given Lie triple system $\mathcal{T}$, $\mathcal{L} = \mathcal{H} \oplus \mathcal{T}$

$\boxed{6.4}$ Let $\mathcal{K}$ be θ-invariant subspace of $\mathcal{L}$.

Then $\mathcal{K} = (\underbrace{\mathcal{H} \cap \mathcal{K}}_{\mathcal{H}\oplus 0}) \oplus (\underbrace{\mathcal{T} \cap \mathcal{K}}_{0 \oplus \mathcal{T}})$

Always $(\mathcal{H}\cap\mathcal{K}) \oplus (\mathcal{T}\cap\mathcal{K}) \subseteq \mathcal{K}$ for any subspace $\mathcal{K}$ (not always $= \mathcal{K}$)

If $H \oplus x \in \mathcal{K}$ then $\theta(H \oplus x) = (-H) \oplus x \in \mathcal{K}$

$H \oplus x - (-H) \oplus x \in \mathcal{K}$, $2H \oplus 0 \in \mathcal{K}$

so $H \oplus 0 \in \mathcal{K}$. Similarly $H \oplus x + (-H) \oplus x$

$= 0 \oplus 2x \in \mathcal{K}$, so $0 \oplus x \in \mathcal{K}$

and $H \oplus x = H \oplus 0 + 0 \oplus x$

Thus $\mathcal{K} \subseteq (\mathcal{H} \cap \mathcal{K}) + (\mathcal{T} \cap \mathcal{K})$ so equality holds

Conversely if $\mathcal{K}$ is a subspace of $\mathcal{L}$ and $\mathcal{K} = (\mathcal{H} \cap \mathcal{K}) \oplus (\mathcal{T} \cap \mathcal{K})$, then $\mathcal{K}$ is θ-invariant.

then $H \in \mathcal{H}$ and $x \in \mathcal{J}$ are unique (direct sum!) but $H \oplus x = (H_1 \oplus 0) + (0 \oplus x_1)$ where $H_1 \oplus 0 \in \mathcal{H} \cap \mathcal{K}$, $0 \oplus x_1 \in \mathcal{J} \cap \mathcal{K}$ so $H \oplus 0 = H_1 \oplus 0 \in \mathcal{K}$

let $H \oplus x \in \mathcal{K}$ then $\boxed{H \oplus 0 \in \mathcal{K}}$ and $0 \oplus x \in \mathcal{K}$

so $-(H \oplus 0) = (-H) \oplus 0 \in \mathcal{K}$ i.e. $\boxed{\theta(H \oplus x) \in \mathcal{K}}$.

$$\boxed{\theta(H \oplus x) = (-H) \oplus x = (-H) \oplus 0 + 0 \oplus x \in \mathcal{K}}$$

Thus every θ-invariant subspace $\mathcal{K}$ of $\mathcal{L}$ has the form $\mathcal{K} = \mathcal{M} \oplus \mathcal{U}$ where $\mathcal{M}$ is a subspace of $\mathcal{H}$ and $\mathcal{U}$ is a subspace of $\mathcal{J}$.

Suppose $\mathcal{K}$ is an (Lie algebra) ideal in $\mathcal{L}$. Then $\forall\ K = M \oplus u \in \mathcal{K}$ and $X = H \oplus x \in \mathcal{L}$ we have $[X, K] \in \mathcal{K}$

$$[X, K] = [H \oplus x, M \oplus u] = \big(\underbrace{[H, M] + L(x,u)}_{\in \mathcal{M}}\big) \oplus \big(\underbrace{Hu - Mx}_{\in \mathcal{U}}\big)$$

This is equivalent to

$$\left.\begin{array}{ll} \text{(i)} & [H, M],\ L(x,u) \in \mathcal{M} \\ \text{(ii)} & Hu,\ Mx \in \mathcal{U} \end{array}\right\} \ \forall\ H \in \mathcal{H},\ x \in \mathcal{J},\ M \in \mathcal{M},\ u \in \mathcal{U}$$

def
$$i(\mathcal{U}) = L(\mathcal{J}, \mathcal{U}) = \{ L(x,u) : x \in \mathcal{J},\ u \in \mathcal{U} (\subseteq \mathcal{J}) \}$$
$$j(\mathcal{U}) = \{ A \in \mathcal{H} : A(\mathcal{J}) \subseteq \mathcal{U} (\subseteq \mathcal{J}) \}$$

Lemma 4 on p. 48 A ~~θ-invariant~~ subspace $\mathcal{K} \subset \mathcal{L}$ is ~~an ideal~~ a θ-invariant (Lie algebra) ideal of $\mathcal{J}$ $\Longleftrightarrow$

$\mathcal{K} = \mathcal{M} \oplus \mathcal{U}$ where • $\mathcal{U}$ is an (Lie triple system) ideal of $\mathcal{J}$

• $\mathcal{M}$ is an (Lie algebra) ideal of $\mathcal{H}$ • $i(\mathcal{U}) \subseteq \mathcal{M} \subseteq j(\mathcal{U})$.

$L(\mathcal{J},\mathcal{U}) \subseteq \mathcal{M} \subseteq \left\{ \sum_{i=1}^{n} L(x_i,y_i) : x_i, y_i \in \mathcal{J}, \ni \sum_{i=1}^{n} [x_i y_i u] \in \mathcal{U} \right\}$

Proof Assume $\mathcal{K}$ is a θ-invariant ideal of $\mathcal{L}$

By (i) on the previous page <u>$\mathcal{M}$ is an ideal of $\mathcal{H}$</u>. What about $\mathcal{U}$ being an ideal in $\mathcal{J}$. We need

① $[\mathcal{U}\mathcal{J}\mathcal{J}] \subseteq \mathcal{U}$, ② $[\mathcal{J}\mathcal{J}\mathcal{U}] \subseteq \mathcal{U}$, ③ $[\mathcal{J}\mathcal{U}\mathcal{J}] \subseteq \mathcal{U}$.

Let $u \in \mathcal{U}$, $x, y \in \mathcal{J}$ Recall (6.2) on p. 44 $[xyz] = -[yxz]$

① $[uxy] = -[xuy] = -\underbrace{L(x,u)}_{\in \mathcal{M}} y \in \mathcal{U}$ by (ii) on the previous page $\mathcal{M}x \in \mathcal{U}$

Thus $[\mathcal{U}\mathcal{J}\mathcal{J}] \subseteq \mathcal{U}$

② $[xyu] = \underbrace{L(x,y)}_{\in \mathcal{H}} u \in \mathcal{U}$ by (ii) on the previous page $\mathcal{H}u \in \mathcal{U}$

Thus $[\mathcal{J}\mathcal{J}\mathcal{U}] \subseteq \mathcal{U}$

③ $[xuy] = L(x,u)y \in \mathcal{U}$ as shown in ① above.

Thus $[\mathcal{J}\mathcal{U}\mathcal{J}] \subseteq \mathcal{U}$ & <u>$\mathcal{U}$ is a (Lie triple) ideal of $\mathcal{J}$</u>.

$i(\mathcal{U}) = \{ L(x,u) : x \in \mathcal{J},\ u \in \mathcal{U} \} \subseteq \mathcal{M}$ by (i) on the previous page

what about $\mathcal{M} \subseteq j(\mathcal{U}) = \{ A \in \mathcal{H} : A(\mathcal{J}) \subset \mathcal{U} \}$

Let $M \in \mathcal{M}$, $x \in \mathcal{J}$. Then $Mx \in \mathcal{U}$ by (ii) on the previous page.

This proves that a θ-invariant ideal $\mathcal{K}$ in $\mathcal{L}$ is of the form $\mathcal{K} = \mathcal{M} \oplus \mathcal{U}$ where $\mathcal{M}$ is an ideal in $\mathcal{H}$, $\mathcal{U}$ is an ideal in $\mathcal{J}$, and $i(\mathcal{U}) \subset \mathcal{M} \subset j(\mathcal{U})$.

What about the converse? Assume $\mathcal{K}$ has the above form. It is obviously a linear subspace.

Look at $[H \oplus x,\ M \oplus u]$ for $M \oplus u \in \mathcal{K}$, $H \oplus x \in \mathcal{L}$

$$= \Big(\underbrace{[H, M]}_{\in \mathcal{M}} + \underbrace{L(x,u)}_{\in \mathcal{M}} \Big) \oplus \underbrace{Hu - Mx}_{\in \mathcal{U}} \in \mathcal{K},$$

so $\mathcal{K}$ is an ideal in $\mathcal{L}$

(since $\mathcal{M}$ is an ideal in $\mathcal{H}$) (since $i(\mathcal{U}) \subseteq \mathcal{M}$) (by (ii) on p. 3 of these notes and p. 48 of Meyberg)

Need to also show that $\mathcal{K}$ is θ-invariant, i.e.

$$M \oplus u \in \mathcal{K} \Rightarrow (-M) \oplus u \in \mathcal{K}$$

This is obvious since $-M \in \mathcal{M}$ $\blacksquare$