we are going to prove (7.9): $L(x,y)\,P(x) = P(x)\,L(y,x) = P(P(x)y, x)$

1-19-17

continued 1-20-17 on p. 7

$$\boxed{L(x,y) \stackrel{?}{=} 2\,[L(x), L(y)] + 2\,L(xy)}$$

$$(7.6) \qquad P(x,y) = 2\big(L(x)L(y) + L(y)L(x) - L(xy)\big)$$

NOTE: $P(x,y) = P(y,x)$

def $\quad L(x,y)z = \{xyz\} = P(x,z)(y)$

$$L(x,y)z = P(x,z)y = 2\big(L(x)L(z) + L(z)L(x) - L(xz)\big)(y)$$
$$= 2\big(x(zy) + z(xy) - (xz)y\big)$$

$$2\,[L(x), L(y)](z) + 2\,L(xy)(z) =$$
$$2\,L(x)L(y)z - 2\,L(y)L(x)z + 2\,(xy)z$$
$$= 2x(yz) - 2y(xz) + 2(xy)z$$

agree, proving

$$L(x)P(x) = P(x)L(x) \qquad (\text{known})$$

$$(7.7) \qquad L(y)P(x) + \underbrace{P(x)L(y)} = P(xy, x) \qquad (\text{proved already})$$

$$\tfrac{1}{2}P(x)L(y,x) = \tfrac{1}{2}P(x)\big(2\,[L(y), L(x)] + 2\,L(yx)\big)$$

$$= P(x)L(y)L(x) - \overbrace{P(x)L(x)}^{\text{commute}}L(y) + P(x)L(yx)$$

$$= \big(P(xy,x) - L(y)P(x)\big)L(x) - L(x)\big(P(xy,x) - L(y)P(x)\big) + P(x)L(xy)$$

$$= P(xy,x)L(x) - L(y)L(x)P(x) - L(x)P(xy,x) + L(x)L(y)P(x) + P(x)L(xy)$$

$$= [L(x), L(y)]P(x) + \underbrace{[P(xy,x), L(x)]} + P(x)L(xy)$$

↑ use (7.8) on this

Recall 7.8

$$(7.8)\quad [P(x,u), L(x)] = [L(u), P(x)]$$

put $u = xy$ to get

$$[P(x,xy), L(x)] = [L(xy), P(x)]$$

---

continuing from the bottom of p. 1

$$= [L(x), L(y)]\, P(x) + \underbrace{\cancel{P(x,xy)L(x) - L(x)P(x,xy)}}_{[L(xy),\, P(x)]} + P(x)\, L(xy)$$

$$= [L(x), L(y)]\, P(x) + L(xy)P(x) - P(x)\cancel{L(xy)} + P(x)\cancel{L(xy)}$$

$$\boxed{\text{so}\quad \tfrac{1}{2} P(x)\, L(y,x) = [L(x), L(y)]\, P(x) + L(xy)\, P(x)}$$

Look at the top of p. 1

$$L(x,y) = 2\,[L(x), L(y)] + 2\,L(xy)$$

$$\Rightarrow\ \tfrac{1}{2} L(x,y)\, P(x) = [L(x), L(y)]\, P(x) + L(xy)\, P(x)$$

proving $\boxed{P(x)\, L(y,x) = L(x,y)\, P(x)}$ which is part of (7.9) (see top of p. 1)

apply to element u

Next $P(x)\{yxu\} = \{xy\, P(x)u\} = \{xu\, P(x)y\} = P(x, P(x)y)\, u$

symmetric, so

i.e. $L(x,y)\, P(x) = P(P(x)y, x)$ completing the proof of (7.9) $\square$

An attempt to prove (7.10)

Next, let's linearize (7.7): $L(y)P(x) + P(x)L(y) = P(xy, x)$

replace x by u+w and act on v

$$L(y)P(u+w)v + P(u+w)(yz) = P((u+w)y, u+w)v$$

$$L(y)\left(2(L(u+w))^2 - L((u+w)^2)\right)v =$$
$$+ \left(2(L(u+w))^2 - L((u+w)^2)\right)(yz)$$

start over

(7.7) $\boxed{L(y)\left(2L(x)^2 - L(x^2)\right) + \left(2L(x)^2 - L(x^2)\right)L(y)}$

$$= P(xy+x) - P(xy) - P(x)$$

$$= 2L((xy+x))^2 - L((xy+x)^2)$$
$$-2L(xy)^2 + L((xy)^2)$$
$$-2L(x)^2 + L(x^2)$$

$$= 2\,\overset{1}{L(xy)}^2 + 2L(xy)L(x) + 2\cancel{L(x)}^2 + 2L(x)L(xy) - L(\cancel{(xy)}^2 + 2x(xy) + \cancel{x^2})$$

$$- 2\cancel{L(xy)}^2 + L(\cancel{(xy)^2}) - 2\cancel{L(x)}^2 + L(\cancel{x^2})$$

$$= \boxed{2L(x)L(xy) + 2L(xy)L(x) - L(2x(xy))}$$

these are equal by (7.7)

Now replace x + u+w ~~and act~~

$$L(y)\left(\cancel{2}\left(L(u)+L(w)\right)^2-\tfrac{1}{2}L(u^2+2uw+w^2)\right)$$

$$\downarrow$$

$$+\left(\cancel{2}\left(L(u)+L(w)\right)^2-\tfrac{1}{2}L(u^2+2uw+w^2)\right)L(y)$$

$$=\cancel{2}\left(L(u)+L(w)\right)\left(L(uy)+L(wy)\right)+\cancel{2}\left(L(uy)+L(wy)\right)\left(L(u)+L(w)\right)$$

$$-L\left(\cancel{2}(u+w)((u+w)y)\right)$$

cancel the 2's

$$L(y)\,L(u)^2+L(y)L(u)L(w)+L(y)L(w)L(u)+L(y)\,L(w)^2$$

$$-\tfrac{1}{2}L(y)\,L(u^2)-\tfrac{1}{2}L(y)\,L(2uw)-\tfrac{1}{2}L(y)\,L(w^2)$$

$$+\;L(u)^2L(y)+L(u)L(w)L(y)+L(w)L(u)L(y)+L(w)^2L(y)$$

$$-\tfrac{1}{2}L(u^2)L(y)-\tfrac{1}{2}L(2uw)L(y)-\tfrac{1}{2}L(w^2)L(y)$$

Start over

~~$L(y)P(x) + P(x)L(y) = P(xy, x)$~~

~~replace x by $u+w$~~

~~$L(y)$~~

(7.10) says $\qquad LHS = RHS_1 + RHS_2 + RHS_3$

$$L(y)P(u,w) = P(L(y)u, w) - P(u,w)L(y) + P(u, L(y)w)$$

let's check this ~~on a ve~~ $LHS =$

$$L(y)P(u,w) = L(y)\overset{③}{P}(u+w) - L(y)\overset{①}{P}(u) - L(y)\overset{②}{P}(w)$$

$$RHS_1 = P(yu+w) - P(yu) - P(w)$$

$$RHS_2 = -\overset{③}{P}(u+w)L(y) + \overset{①}{P}(u)L(y) + \overset{②}{P}(w)L(y)$$

$$RHS_3 = P(u+yw) - P(u) - P(yw)$$

③ $L(y)P(u+w) + P(u+w)L(y) \overset{7.7}{=} P(y(u+w), u+w)$

① $L(y)P(u) + P(u)L(y) \overset{7.7}{=} P(yu, u)$

② $L(y)P(w) + P(w)L(y) \overset{7.7}{=} P(yw, w)$

(7.10) same as $\overset{③}{P}(y(u+w), u+w) = \overset{①}{P}(yu,u) + \overset{②}{P}(yw, w)$

$$+ P(yu+w) - P(yu) - P(w) + P(u+yw) - P(u) - P(yw)$$

Let's prove (7.10) for "special Jordan algebras"

$$(7.10) \qquad [redacted] \quad y \circ \{uvw\} \overset{?}{=} \{(y\circ u)vw\} - \{u(y\circ v)w\} + \{uv(y\circ w)\}$$

$$y\,(\overset{1}{uvw} + \overset{8}{wvu}) + (\underset{7}{uvw} + \overset{4}{wvu})\,y \overset{?}{=} \cancel{yuvw + wvyu - uyvw - wyvu}$$

$$= (\overset{1}{yu} + \overset{2}{uy})vw + \overset{3}{wv}(yu + \overset{4}{uy})$$

$$- u(\overset{2}{yv} + \overset{5}{vy})w - w(\overset{6}{yv} + \overset{3}{vy})u$$

$$+ uv(\underset{5}{yw} + \underset{7}{wy}) + (\underset{8}{yw} + \overset{6}{wy})vu$$

Derivation: $DL(a) - L(a)D = L(Da)$

$$D(ab) = aDb + Dab$$

(1) $\{uvw\} \overset{?}{=} 2(uv)w + 2(wv)u - 2(uw)v$ assume this

(2) $D\{uvw\} = D(uv)w + (uv)D(w) + D(wv)u + (wv)D(u) - D(uw)v - (uw)(Dv)$

Proof of (2) using (1):

$\frac{1}{2}D\{uvw\} = (uDv + Du\,v)w + (uv)D(w) + ((Dw)v)u + (w\,D(v))u + (wv)Du$
$- (u(Dw))v - (Du\,w)v - (uw)Dv$

~~Du v w~~ $= (uDv)w + (wDv)u - (uw)Dv$
$+ (Du\,v)w + (wv)Du - (wDu)v$
$+ (uv)Dw + ((Dw)v)u - (uDw)v$

$= \{u, Dv, w\} + \{Du, v, w\} + \{u, v, Dw\}$

Thus proves (1) $\Rightarrow$ (2) Proof of (1) follows:

$\{uvw\} = P(u,w)v = P(u+w)v - P(u)v - P(w)v$

$= 2L(u+w)^2 v - L((u+w)^2)v$
$- 2L(u)^2 v + L(u^2)v$
$- 2L(w)^2 v + L(w^2)v$

$= (2\cancel{L(u)^2} + 2L(u)L(w) + 2L(w)L(u) + \cancel{2L(w)^2})v$
$- (\cancel{L(u^2)} + L(2uw) + \cancel{L(w^2)})v - \cancel{2L(u)^2 v} + \cancel{L(u^2)v}$
$- \cancel{2L(w)^2 v} + \cancel{L(w^2)v}$

~~$P(u)v + P(w)v$~~

$$= 2u(wv) + 2w(uv) - 2(uw)v$$

$$= 2(wv)u + 2(uv)w - 2(uw)v$$

except for the 2 it works.

(1) is true & (2) follows.

---

but this is not how Meyberg does it.

Exercise 3 — prove 7.10 for all Jordan algebras

Exercise 4 — prove $D\{uvw\} = \{Du\,vw\} + \{u\,Dv\,w\} + \{uv\,Dw\}$ for derivations on all Jordan algebras

apply this to $D = [L(x), L(y)]$

$$[L(x), L(y)]\{uvw\} = \{[L(x),L(y)]u, v, w\} + \{u, [L(x),L(y)]v, w\} + \{u, v, [L(x),L(y)]w\}$$

From p. 68 or the top of p. 1 we have

$$L(x,y) = 2\overbrace{[L(x),L(y)]}^{D} + 2\overbrace{L(xy)}^{E}$$

$$\text{so}\quad L(x,y)\{uvw\} = 2\overbrace{[L(x),L(y)]}^{\text{call this } D}\{uvw\} + 2\underbrace{\overbrace{L(xy)}^{\text{call this } E}\{uvw\}}_{\text{use (7.10) here}}$$

(7.10) says $y\{uvw\} = \{(yu)vw\} - \{u(yv)w\} + \{uv(yw)\}$

OR $\quad L(y)\{uvw\} = \{L(y)u, v, w\} - \{u, L(y)v, w\} + \{u, v, L(y)w\}$

OR $(y \to xy)\quad L(xy)\{uvw\} = \{L(xy)u, v, w\} - \{u, L(xy)v, w\} + \{u, v, L(xy)w\}$

$$L(x,y) = 2D + 2E$$

$$\therefore \quad L(x,y)\{uvw\} = 2D\{uvw\} + 2E\{uvw\}$$

$$= 2\{Du,v,w\} + 2\{u\,Dv\,w\} + 2\{uv\,Dw\}$$
$$+ 2\{Eu,v,w\} - 2\{u,Ev,w\} + 2\{u,v,Ew\}$$

$$= \{L(x,y)u,v,w\} + \; ? \; + \{u,v,L(x,y)w\}$$

Look at $L(x,y) = 2[L(x),L(y)] + 2L(xy)$

interchange x,y

$$\Longrightarrow \quad L(y,x) = 2[L(y),L(x)] + 2L(\cancel{yx}\,xy) \qquad (xy = yx)$$
$$= -2[L(x),L(y)] + 2L(xy)$$
$$= -2D + 2E$$
$$-L(y,x) = 2D - 2E$$

$$? = -\{u, L(y,x)v, w\}$$

This proves (7.11)

$$(7.11) \quad \{xy\{uvw\}\} = \{\{xyu\}vw\} - \{u,\{yxv\},w\} + \{uv\{xyw\}\}$$

OR in operator form

$$(7.11') \quad [L(x,y), L(u,v)] = L(\{xyu\},v) - L(u,\{yxv\})$$

Replace u by x, v by y

i.e. $\quad 0 = L(P(x)y,y) - L(x,P(y)x)$

$$(7.12) \quad L(P(x)y,y) = L(x,P(y)x)$$

Finished with p. 69

In (7.11′) interchange the pairs (x,y), (u,v)
(i.e. ~~replace~~ interchange x ~~by~~ and u and interchange y and v)

$$\begin{aligned} [L(u,v), L(x,y)] &= L(\{uvx\},y) - L(x,\{vuy\}) \\ \| \quad & \\ -[L(x,y), L(u,v)] &= -L(\{xyu\},v) + L(u,\{yxv\}) \end{aligned}$$

This proves (7.13)

(7.13) $\{\{xyu\}vw\} - \{u\{yxv\}w\} = \{x\{vuy\}w\} - \{\{uvx\},y,w\}$

one more formula to go. (7.14) on p. 70

Exercise 5 — Prove (7.14) on p. 70 by filling in the details of the outline on p. 70.