Meyberg Exercises 1-3 chapter 1 10/6/16 ①

Meyberg (Reduced2) ch1 marked.pdf (desktop)

Exercise 1 $[a,a] = aa - aa = a^2 - a^2 = 0$

$$[[a,b],c] = (ab-ba)c - c(ab-ba) = \overset{(1)}{\cancel{abc}} - \overset{(2)}{\cancel{bac}} - \overset{(3)}{\cancel{cab}} + \overset{(4)}{\cancel{cba}}$$

$$[[b,c],a] = (bc-cb)a - a(bc-cb) = \overset{(5)}{\cancel{bca}} - \overset{(4)}{\cancel{cba}} - \overset{(1)}{\cancel{abc}} + \overset{(6)}{acb}$$

$$[[c,a],b] = (ca-ac)b - b(ca-ac) = \overset{(3)}{\cancel{cab}} - \overset{(6)}{\cancel{acb}} - \overset{(5)}{\cancel{bca}} + \overset{(2)}{\cancel{bac}}$$

(I forgot to include this)

Exercise 1b $a \circ b = ab + ba = ba + ab = b \circ a$

$$a \circ ((a \circ a) \circ b) = a \circ (a^2 b + ba^2) = a(a^2 b + ba^2) + (a^2 b + ba^2)a$$

$$= \overset{(1)}{a^3 b} + \overset{(2)}{aba^2} + \overset{(3)}{a^2 ba} + \overset{(4)}{ba^3}$$

$$(a \circ a) \circ (a \circ b) = a^2 \circ (ab + ba) = a^2(ab+ba) + (ab+ba)a^2$$

$$= \overset{(1)}{a^3 b} + \overset{(3)}{a^2 ba} + \overset{(2)}{aba^2} + \overset{(4)}{ba^3}$$

□ End of Exercise 1

Exercise 2 (Proof of Th 1 on p. 3)

(i) Let $B \subset A$ be an ideal. Then $\overline{A} := A/B$ is an algebra and the map $A \ni a \to a + B$ is a homomorphism with kernel B.

Conversely if $\alpha : A \to C$ is a homomorphism with kernel B then B is an ideal since

for $a \in A, b \in B$ $\alpha(ab) = \alpha(a)\alpha(b) = \alpha(a)\cdot 0 = 0$

so $AB \subseteq B$ and similarly $BA \subseteq B$

Of course if $b_1, b_2 \in B$ then $\alpha(b_1 + b_2) = \alpha(b_1) + \alpha(b_2) = 0$

so B is a linear space. QED for (i)

(FIRST ISOMORPHISM THEOREM)

(ii) Consider the map $\alpha : f(A) \to A/_{\ker f}$

defined by $\alpha(f(a)) = a + \ker f$. It is easy to verify that
- α is well defined
- α is one-to-one & onto (bijective) (injective) (surjective)
- α is a homomorphism (hence isomorphism)

For $a, b \in A$,

$f(a) = f(b) \Rightarrow$ ~~$a + \ker f = b + \ker f \Rightarrow a - b \in \ker f$~~

$f(a-b) = 0 \Rightarrow a - b \in \ker f \Rightarrow a + \ker f = b + \ker f$ $\therefore$ well defined

if $\alpha(f(a)) = \alpha(f(b))$ then $a + \ker f = b + \ker f \Rightarrow a - b \in \ker f$

$\Rightarrow f(a-b) = 0 \Rightarrow f(a) - f(b) = 0$ $\therefore \alpha$ injective

if $a + \ker f \in A/_{\ker f}$ then $a + \ker f = \alpha(f(a))$ $\therefore \alpha$ surjective

$\alpha(f(a) + f(b)) = \alpha(f(a+b)) = (a+b) + \ker f = (a + \ker f) + (b + \ker f)$

$= \alpha(f(a)) + \alpha(f(b))$

$\therefore \alpha$ is linear

$\alpha(f(a)f(b)) = \alpha(f(ab)) = ab + \ker f$

$= (a + \ker f)(b + \ker f) = \alpha(f(a))\,\alpha(f(b))$ $\therefore \alpha$ is homomorphism

QED for (ii)

(iii) "Define" $\alpha : \frac{U+V}{V} \longrightarrow U/U\cap V$ by $\alpha(u+v+V) = u+(U\cap V)$

<u>well-defined</u> $u+v+V = u'+v'+V \overset{?}{\Longrightarrow} u+(U\cap V) = u'+(U\cap V)$

$u-u'+v-v' \in V \Rightarrow u-u' \in V - v+v' \subseteq V-V+V \subseteq V$

$\Rightarrow u-u' \in V\cap U \Rightarrow u+(U\cap V) = u'+(U\cap V)$

<u>surjective</u> given $u+(U+V)$ consider $\alpha(u+0+V) = u+(U\cap V)$

$\therefore$ onto (can replace $u+0+V$ with $u+v+V$ any $v\in V$)

<u>injective</u> $u+(U\cap V) = u'+(U\cap V) \overset{?}{\Longrightarrow} u+v+V = u'+v'+V$

$u-u' \in U\cap V$ so $u+v-u'-v' = u-u'+v-v'$

$\in V+V-V \subseteq V$

so $u+v+V = u'+v'+V$

<u>homomorphism</u> $\alpha(u+v+V+(u'+v')+V) = \alpha(u+u'+v+v'+V)$

$= u+u'+U\cap V = u+U\cap V + u'+U\cap V$

$= \alpha(u+v+V) + \alpha(u'+v'+V)$ $\therefore$ α is linear

$\alpha((u+v+V)(u'+v'+V)) = \alpha(uu'+\underbrace{vu'+uv'+vv'}_{\in V}+V)$

$= uu' + U\cap V = (u+U\cap V)(u'+U\cap V) =$

$\alpha(u+v+V)\alpha(u'+v'+V)$.

Exercise 3

(a)(i) $L(xy)(z) = (xy)z$

$L(x)L(y)z = L(x)(yz) = x(yz)$

(ii) $R(yz)x = x(yz)$

$R(z)R(y)x = R(z)(xy) = (xy)z$

(iii) $L(x)R(z)y = L(x)(yz) = x(yz)$

$R(z)L(x)y = R(z)(xy) = (xy)z$

let $D = L(x) - R(x)$ Then $D(ab) = \overset{①}{x(ab)} - \overset{②}{ab(x)}$

$aD(b) = a(xb - bx) = \overset{③}{a(xb)} - \overset{②}{a(bx)}$

$(Da)b = (xa - ax)b = \overset{①}{(xa)b} - \overset{③}{(ax)b}$

$D(ab) = aD(b) + (D(a))b$ so D is a derivation

(b)(L.1') $xy = -yx$ (for all y) is equivalent to $L(x) = -R(x)$ (obvious)

(L.2') $L(xy) = [L(x), L(y)]$ is equivalent to Jacobi identity

i.e. $(xy)z = x(yz) - y(xz)$

$L(xy)(z) = (xy)z$

$[L(x), L(y)](z) = (L(x)L(y) - L(y)L(x))(z)$

$= L(x)(yz) - L(y)(xz) = x(yz) - y(xz)$

(c)(J.1'): $L(x) = R(x)$ is equivalent to $xy = yx$ for all y (obvious)

(J.2') $L(x)L(xx) = L(xx)L(x)$ is equivalent to $x \cdot ((xx)y) = (xx)(xy)$

$L(x)L(xx)y = L(x)((xx)y) = x((xx)y)$

$L(xx)L(x)y = L(xx)(xy) = (xx)(xy)$ ☑

(d) A linear map D is a derivation $\Longleftrightarrow$ $L(Dy) = [D, L(y)]$

$$L(Dy)(z) = (Dy)z$$

$$[D, L(y)](z) = (DL(y) - L(y)D)(z) = D(yz) - y(Dz)$$

(L, 2') In a Lie algebra $L(x)$ is a derivation.

$$L(x)(yz) = x(yz)$$

$$y \cdot L(x)z + (L(x)y)z = y(xz) + (xy)z$$

this is the Jacobi identity