Meyberg p. 5

$$A = A^{(0)} \supset A^{(1)} \supset A^{(2)} \supset \cdots \supset A^{(k)}$$

$$A^{(k+1)} = A^{(k)} A^{(k)} \quad \text{in part } A^{(1)} = AA$$

$A^{(0)}$ is an algebra ! ~~obv~~ (obvious: $A^{(0)} = A$)

$A^{(1)}$ is an algebra

First: $A^{(1)}$ is a linear space. Let $x = \sum_i a_i b_i$, $y = \sum_j c_j d_j \in A^{(1)}$

$a, b, c, d \in$

Need $xy \in A^{(1)}$

$$xy = \sum_i \sum_j \underbrace{(a_i b_i)}_{\in A^{(0)}} \underbrace{(c_j d_j)}_{\in A^{(0)}} \in A^{(1)}$$

$\therefore A^{(1)}$ is an algebra

Next: $A^{(1)}$ is an algebra

~~Exercise 4 Let L be a Lie algebra~~

Now let $x \in A^{(1)}$ $y \in A$ $x = \sum a_i b_i$

$$xy = \sum_i x(a_i b_i) \in A^{(1)}$$

$$yx = \sum (a_i b_i) x \in A^{(1)} \quad \therefore A^{(1)} \text{ is an ideal}$$

Exercise 4

Let L be a Lie algebra i.e.

$$\left\{ \begin{array}{l} a^2 = 0 \\ a(bc) = (ab)c + b(ac) \end{array} \right\}$$

L and $L^{(1)}$ are ideals. Is $L^{(2)}$ an ideal?

$L^{(2)} = L^{(1)} L^{(1)}$ let $x \in L^{(2)}$ $y \in L$

$x = \sum_i a_i b_i$ $a_i, b_i \in L^{(1)}$

$$xy = \sum_i (a_i b_i) y \overset{\text{(Jacobi)}}{=} \sum_i \Big[\underset{L^{(1)}}{a_i} \underset{L^{(1)}}{(b_i y)} - \underset{L^{(1)}}{b_i} \underset{L^{(1)}}{(a_i y)} \Big] \in L^{(2)}$$

similar for yx & the general case follows by induction (details omitted)

Exercise 5

Let $B \subset A$ be a subalg & $A^{(n)} = (0)$ (for some $n \geq 1$)

By induction $B^{(k)} \subset A^{(k)}$ (details omitted) so $B^{(n)} = (0)$ ▨

Let $\alpha : A \to B$ be an <u>onto</u> homomorphism.

check that $\alpha(A^{(1)}) = B^{(1)}$ and by induction

$\alpha(A^{(k)}) = B^{(k)}$ (details omitted) So $A^{(n)} = 0 \Rightarrow B^{(n)} = 0$. ▨

Exercise 6 (Meyberg p. 8) $\Phi = \mathbb{R}$ or $\mathbb{C}$

A is an algebra, $\hat{A} := \{(\alpha, a) : \alpha \in \Phi, a \in A\} = \Phi \times A$

If A is associative, then so is $\hat{A}$

Write $(\alpha 1 + a)(\beta 1 + b) = (\alpha\beta) 1 + \alpha b + \beta a + ab$

OR $(\alpha, a)(\beta, b) = (\alpha\beta, \alpha b + \beta a + ab)$

So $\big((\alpha 1 + a)(\beta 1 + b)\big)(\gamma 1 + c) = (\alpha\beta 1 + \underline{\beta a + \alpha b + ab})(\gamma 1 + c)$

$$= \alpha\beta\gamma 1 + \gamma(\beta a + \alpha b + ab) + \alpha\beta c + (\beta a + \alpha b + ab)c$$

$$= \underset{1}{\alpha\beta\gamma 1} + \underset{2}{\gamma\beta a} + \underset{3}{\gamma\alpha b} + \underset{4}{\gamma ab} + \underset{5}{\alpha\beta c} + \underset{6}{\beta ac} + \underset{7}{\alpha bc} + \underset{8}{(ab)c}$$

On the other hand

$$(\alpha 1 + a)\big((\beta 1 + b)(\gamma 1 + c)\big) = (\alpha 1 + a)(\beta\gamma 1 + \underline{\gamma b + \beta c + bc})$$

$$= \alpha\beta\gamma 1 + \beta\gamma a + \alpha(\gamma b + \beta c + bc) + \gamma ab + \beta ac + a(bc)$$

$$= \underset{1}{\alpha\beta\gamma 1} + \underset{2}{\beta\gamma a} + \underset{3}{\alpha\gamma b} + \underset{5}{\alpha\beta c} + \underset{7}{\alpha bc} + \underset{4}{\gamma ab} + \underset{6}{\beta ac} + \underset{8}{a(bc)}$$