**Complex Analysis Math 147—Winter 2008**
**Solutions to First Midterm—February 4, 2008**

1. (**10 points**) Describe (or sketch the graph of) the set of points in the complex plane that satisfies each of the following

   (a) $|z| > 6$

   **Solution: exterior of the circle centered at 0 of radius 6**

   (b) $\operatorname{Re} z \geq 4$

   **Solution: all points to the right of the line $x = 4$ including the line**

   (c) $|z-1| + |z+1| = 7$

   **Solution: an ellipse with foci at 1 and -1**

2. (a) (**5 points**) Prove that if $|z| = \operatorname{Re} z$ then $z$ is a non-negative real number.

   **Proof:** $x = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = x^2 \Rightarrow y^2 = 0 \Rightarrow y = 0 \Rightarrow x = \mathbf{Re}\, z = |z| \geq 0$

   (b) (**10 points**) Show that if the nonzero complex numbers $z_1$ and $z_2$ satisfy $|z_1+z_2| = |z_1| + |z_2|$ then they have the same argument. (Hint: square both sides and use the polar form and (a))

   **Proof:** $|z_1+z_2|^2 = (|z_1|+|z_2|)^2 \Rightarrow \mathbf{Re}\, z_1\bar{z}_2 = |z_1\overline{z_2}| \Rightarrow z_1\bar{z}_2$ **is a non-negative real number** $\Rightarrow \mathbf{arg}\, z_1 - \mathbf{arg}\, z_2 = 0 + k\pi i$

3. (a) (**5 points**) Write $\exp z$ where $z = 4\exp(i\pi/3)$ in the form $a+bi$ with $a, b \in \mathbf{R}$.

   **Solution:** $z = 4\cos(\pi/3) + i4\sin(\pi/3) = 2 + i2\sqrt{3}$ **so** $\exp z = e^2\cos(2\sqrt{3}) + ie^2\sin(2\sqrt{3})$

   (b) (**5 points**) Write $(1+i)^6$ in rectangular form $a+bi$ with $a, b \in \mathbf{R}$ (Hint: use the polar form $r\cos\theta + ir\sin\theta$ first).

   **Solution:** $(1+i)^6 = (\sqrt{2}e^{i\pi/4})^6 = 8e^{3i\pi/2} = -8i$

4. (**10 points**) True or false.

   (a) $\exp z \neq 0$ for every $z$ **True**

   (b) $z \mapsto \exp z$ is one-to-one **False**

   (c) $\exp z$ is defined for all complex $z$ **True**

   (d) $\exp(-z) = 1/\exp z$ **True**

   (e) $\exp z$ is a bounded function . **False**

5. (**10 points**) Write each of the following functions in the form $w = u(x,y) + iv(x,y)$.

   (a) $f(z) = 1/\bar{z}$

   **Solution:** $f(z) = \frac{1}{x-iy} = \frac{x+iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{y}{x^2+y^2}$

   (b) $F(z) = \exp z + \exp(-z)$

   **Solution:** $F(z) = e^x(\cos y + i\sin y) + e^{-x}(\cos(-y) + i\sin(-y)) = e^x(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y) = \cos y(e^x + e^{-x} + i\sin y(e^x - e^{-x})$

6. (**5 points**) Prove that the function $f(z) = \overline{z}$ is continuous for every $z \in \mathbf{C}$.

   **Proof: The real and imaginary parts of $f$ are $u(x, y) = x$ and $v(x, y) = -y$, which are continuous. Hence $f$ is continuous.**

   **Another proof:** $|f(z) - f(z_0)|^2 = (x - x_0)^2 + (-y + y_0)^2 = |z - z_0|^2$**, so you can take** $\delta = \epsilon$

7. (**10 points**) (a) Where is the following function differentiable? $x^2 + y^2 + 2xyi$

   **Solution:** $u = x^2 + y^2, v = 2xy \Rightarrow u_x = 2x, u_y = 2y, v_x = 2y, v_y = 2x$ **so the Cauchy-Riemann equations hold if and only if** $y = 0$**. Since the partial derivatives of $u$ and $v$ are continuous, $f$ is differentiable if and only if** $y = 0$**, that is, on the $x$-axis.**

   (b) Where is it analytic?

   **Solution: Nowhere; $f$ is not differentiable on any open disk centered on the real axis.**

8. (**10 points**) Show that if $f$ is analytic on an open disk and is purely imaginary, that is, $\operatorname{Re} f(z) = 0$ for all $z$, then $f$ is a constant function.

   **Solution: Since** $u = 0$**, by the Cauchy-Riemann equations,** $v_x = -u_y = 0$ **and** $v_y = u_x = 0$ **so that $v$ is a (real) constant. Hence** $f = iv$ **is a (complex) constant.**

9. (**10 points**) Find all values of the following.

   (a) $\log(1 - i)$

   **Solution:** $\log(1 - i) = \log|1 - i| + i\,\mathbf{arg}\,(1 - i) = \log\sqrt{2} + i(-\pi/4 + 2k\pi).\ k \in \mathbf{Z}$

   (b) $\operatorname{Log}(-i)$

   **Solution:** $\mathbf{Log}\,(-i) = \log|-i| + i\,\mathbf{Arg}\,(-i) = 0 + i(-\pi/2)$

10. (**10 points**) Find all values of the following.

    (a) $i^i$

    **Solution:** $i^i = \exp(i \log i) = \exp(i[\log|i| + i\,\mathbf{arg}\,i]) = \exp(-(\pi/2 + 2k\pi)),\ k \in \mathbf{Z}$

    (b) $(-1)^{2/3}$

    **Solution:** $(-1)^{2/3} = \exp((2/3)\log(-1)) = \exp((2/3)(i\,\mathbf{arg}\,(-1)) = \exp(i(2/3)(2k + 1)\pi)$ **(which by the way equals** $\{1, -1/2 + \sqrt{3}i/2, -1/2 - \sqrt{3}i/2\}$**)**