

(left!)

powers $a^1 = a$ $a^{n+1} = a^n a$ (note $a^n a \neq a a^n$) 10-13-16

$a^2 = aa$ $a^3 = a^2 a = (aa)a$ $a^4 = a^3 a = ((aa)a)a$

$a^{n+1} = a^n \cdot a = \underbrace{(a^{n-1} a)}_{aa} a = ((a^{n-2} a) a) a$

$= (((a^{n-3} a) a) a) a$

~~$= (((((aa)a)a)a) \dots a)$~~

~~$\dots ((aa)a)a) a) \dots$~~

power associative

$a^n a^m = a^{n+m}$

$(aa)a = a(aa)$

$(a^{n-1} a)(a^{m-1} a) = (a^{n+m-1}) a$

~~$a^2 a = a a^2$~~

$a^2 a^3 = a^5$

$a^3 a^2 = [((aa)a)](aa)$

$(aa)((aa)a) = (((aa)a)a)a$

nilpotent $a^n = 0$ ($a=0$ is considered nilpotent)

nil ideal (all elements are nilpotent)

Lemma 3 Assume $(a^n)^m = a^{nm} \quad \forall a, n, m \geq 1$

B, C nil ideals $\Rightarrow B+C$ is nil ideal

proof let $b+c \in B+C$ For $n=1, 2, \dots$

$$(b+c)^n = (b+c)^{n-1} (b+c)$$

assume $(b+c)^{n-1} = b^{\overline{n-1}} + d$ where $d \in C$

$$\text{then } (b+c)^n = (b^{\overline{n-1}} + d)(b+c) = b^n + \underbrace{db + b^{\overline{n-1}}c + dc}_{\in C}$$

true for $n-1=1$ so induction proceeds $\in C$

So $(b+c)^n = b^n + d$ with $d \in C$

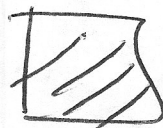
For some n $b^n = 0$ so

$$\text{for some } n \quad (b+c)^n = d \quad (\in C)$$

$$\exists m \quad d^m = 0 \quad \text{so } ((b+c)^n)^m = d^m = 0$$

but $((b+c)^n)^m = (b+c)^{nm}$ so $b+c$ is nilpotent

and $B+C$ is a nil ideal.



DEFN The nilradical of A , A a power associative algebra, is the unique maximal nil ideal of A .

Zorn's Lemma let S be a partially ordered set. Suppose each linearly ordered (totally ordered) subset of S has an upper bound. Then S has a maximal element.

application let S be the set of nil ideals in the algebra A , ordered by inclusion. (chain)

let C be a linearly ordered subset of S

$$C = (B_\alpha)_{\alpha \in \Lambda} \quad B_\alpha \text{ nil ideal} \quad \& \quad \forall \alpha, \beta$$

$$\text{either } B_\alpha \subseteq B_\beta \text{ or } B_\beta \subseteq B_\alpha$$

let $B = \bigcup_{\alpha \in \Lambda} B_\alpha$ Then B is a nil ideal

and B is an upper bound of C

By Zorn lemma S has a maximal element

i.e. a nil ideal $N(A)$ such that if

$$N(A) \subset C \subsetneq A \quad C \text{ a nil ideal}$$

$$\text{then } N(A) = C.$$

(Note: if A itself is a nil ideal, there is nothing to prove)

Why is $N(A)$ unique? Suppose $N_1(A)$ and $N_2(A)$ are each maximal nil ideals.

Since $N_1(A) \subset \underbrace{N_1(A) + N_2(A)}_{\text{nil ideal by Lemma 3}} \subset A$

we have $N_1(A) = N_1(A) + N_2(A)$

so $N_2(A) \subseteq N_1(A)$ and therefore $N_1(A) = N_2(A)$,