(left!) Meyberg chpt 1 p. 6–7 ①

10-13-16

<u>powers</u> $a^1 = a$ $a^{n+1} = a^n a$ (note $a^n a \neq a a^n$)

$a^2 = aa$ $a^3 = a^2 a = (aa)a$ $a^4 = a^3 a = ((aa)a)a$

$a^{n+1} = a^n \cdot a = (a^{n-1} a) a = ((a^{n-2} a) a) a$

$= (((a^{n-3} a) a) a) a$

~~I~~ $\cancel{= (\cdots(((aa)a)a)a) \cdots a)}$

$\cdots(((((aa)a)a)a)a)a) \cdots$

<u>power associative</u> $a^n a^m = a^{n+m}$ $(aa)a = a(aa)$

$(a^{n-1} a)(a^{m-1} a) = (a^{n+m-1}) a$ $a^2 a \stackrel{?}{=} a a^2$

$a^2 a^3 = a^5$ $a^3 a^2 = [((aa)a)](aa)$

$(aa)((aa)a) = ([(aa)a]a)a$

<u>nilpotent</u> $a^n = 0$ (a = 0 is considered nilpotent)

nil ideal (all elements are nilpotent)

Lemma 3   Assume $(a^n)^m = a^{nm}$ $\forall\, a,\ n, m \geq 1$

B, C nil ideals $\Rightarrow$ $B+C$ is nil ideal

proof   let $b+c \in B+C$ For $n = 1, 2, \ldots$

$$(b+c)^n = (b+c)^{n-1}(b+c)$$

assume $(b+c)^{n-1} = b^{n-1} + d$ where $d \in C$

then $(b+c)^n = (b^{n-1}+d)(b+c) = b^n + \underbrace{db + b^{n-1}c + dc}_{\in C}$

true for $n-1 = 1$ so induction proceeds

So $(b+c)^n = b^n + d$ with $d \in C$

For some n $b^n = 0$ so

for such n $(b+c)^n = d$ $(\in C)$

$\exists\, m$ $d^m = 0$ so $((b+c)^n)^m = d^m = 0$

but $((b+c)^n)^m = (b+c)^{nm}$ so $b+c$ is nilpotent

and $B+C$ is a nil ideal.

DEFN The nil radical of A, A a power associative algebra, is the unique maximal nil ideal of A.

<u>Zorn's Lemma</u> Let $\mathcal{S}$ be a partially ordered set. Suppose each <u>linearly ordered</u> (totally ordered) subset of $\mathcal{S}$ has an upper bound. Then $\mathcal{S}$ has a maximal element.

<u>application</u> Let $\mathcal{S}$ be the set of nil ideals in the algebra A, ordered by inclusion.

Let $\mathcal{C}$ be a linearly ordered (chain) subset of $\mathcal{S}$

$\mathcal{C} = (B_\alpha)_{\alpha \in \Lambda}$ B_α nil ideal & $\forall\, \alpha, \beta$

either $B_\alpha \subseteq B_\beta$ or $B_\beta \subseteq B_\alpha$

Let $B = \bigcup_{\alpha \in \Lambda} B_\alpha$ Then B is a nil ideal

and B is an upper bound ~~of~~ for $\mathcal{C}$

By Zorn Lemma $\mathcal{S}$ has a maximal element

i.e. a nil ideal $N(A)$ such that if

$$N(A) \subset C \subsetneq A \qquad C \text{ a nil ideal}$$

then $N(A) = C$.

(Note: if A itself is a nil ideal, there is nothing to prove)

Why is $N(A)$ unique? Suppose $N_1(A)$ and $N_2(A)$ are each maximal nil ideals.

Since $N_1(A) \subset \underbrace{N_1(A) + N_2(A)}_{\text{nil ideal by Lemma 3}} \subset A$

we have $N_1(A) = N_1(A) + N_2(A)$

so $N_2(A) \subseteq N_1(A)$ and therefore $N_1(A) = N_2(A)$.