STRUCTURE OF BERNSTEIN ALGEBRAS

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1. GENERALITIES.

Recall that A, a commutative, non-associative algebra of finite dimension over $\mathbb{R}$, is called a Bernstein algebra (see [1]) if

$$(x^2)^2 = (\omega(x))^2 x^2, \text{ for all } x \in A$$

where $\omega : A \to \mathbb{R}$ is a surjective $\mathbb{R}$-algebra morphism. Polarisation of the above identity yields

$$(xy)(zw)+(xz)(yw)+(xw)(zy) = \frac{1}{2}(\omega(x)w(y)zw+\omega(x)\omega(z)yw+\omega(x)\omega(w)yz+$$
$$+\omega(z)\omega(w)xy+\omega(y)\omega(w)xz+\omega(y)\omega(z)xw, \quad \text{for all } x,y,z,w \in A \qquad (1)$$

We note that the surjectivity of ω yields a vector space direct sum decomposition $A = e\,\mathbb{R} \oplus K$, where $K = \mathrm{Ker}(\omega)$ is an ideal in A and e is an idempotent. We have $\omega(e) = 1$. Thus to exploit the above identity we can in view of multilinearity, substitute e's and k's for the x,y,z,w without loss of information.

Immediate deductions from (1).

(i) Take $x = k \in K$, $y = z = w = e$. We find that the $\mathbb{R}$-linear map $\pi : K \to K$, such that $\pi(x) = 2ex$, is idempotent, that is $\pi^2 = \pi$. Hence K is the vector space direct sum $U \oplus V$, where $U = \pi(K)$, $V = \mathrm{Ker}(\pi)$.

The multilinearity of (1) now shows that we need only substitute u's and v's for k's without loss of information.

If we take $x = y = e$, $z = k_1$, $w = k_2$, we obtain $\pi(k_1k_2)+\pi(k_1)\pi(k_2) = k_1k_2$, which is equivalent to

$$U^2 \subseteq V,\ UV \subseteq U,\ V^2 \subseteq U. \qquad (1,ii)$$

Next take $x = e$, $y = u_1$, $z = u_2$, $w = u_3$. We have Jacobi's identity

$$u_1(u_2u_3)+u_2(u_3u_1)+u_3(u_1u_2) = 0. \qquad (1,iii,a)$$

Take $x = e$, $y = u_1$, $z = u_2$, $w = v_3$. We have

$$u_1(u_2v_3)+u_2(u_1v_3) = 0 \; ; \qquad (1,iii,b)$$

Take $x = e$, $y = u_1$, $z = v_2$, $w = v_3$. We have

$$u_1(v_2v_3) = 0 \; ; \qquad (1,iii,c)$$

Finally,

$$(xy)(zw)+(xz)(yw)+(xw)(yz) = 0 \quad \text{when } x,y,z,w \in K. \qquad (1,iv)$$

2. IDEALS IN NUCLEAR BERNSTEIN ALGEBRAS.

Recall that a commutative, non-associative algebra is said to be nuclear if $A = A^2$.

Proposition 2.1. (i) If A is Bernstein, A^2 is nuclear Bernstein.
(ii) A Bernstein algebra A is nuclear if and only if $V = U^2$, for every choice of e.

Proof. (i) $B = A^2$ is a subalgebra, in fact an ideal, of A. All its elements satisfy equation (1). Hence it is Bernstein. Moreover $B = (e\mathbb{R}\oplus U\oplus V)^2 = e\mathbb{R}\oplus U\oplus U^2$, since $U^2 \subseteq U$, $UV \subseteq U$, $V^2 \subseteq U$. Then $B^2 = e\mathbb{R}+U+U^2U+(U^2)^2+U^2 = e\mathbb{R}\oplus U\oplus U^2$ since $U^2U \subseteq UV \subseteq U$, $(U^2)^2 \subseteq V^2 \subseteq U$, whence $B^2 = B$.
(ii) As in (1), $A^2 = e\mathbb{R}\oplus U\oplus U^2 = A$, which implies $V = U^2$. Note that A^2 is always an ideal in A, and A/A^2 is a zero-algebra, that is one in which any product $xy = 0$. We now concentrate on nuclear Bernstein algebras.

Proposition 2.2. In a nuclear Bernstein algebra A, the ideals are A and certain vector subspaces of K with the direct sum decomposition $I = U'\oplus V'$, $U' \subseteq U$, $V' \subseteq V$.

Proof. Let $I \neq 0$ be an ideal in A and let $\lambda e+u_0+v_0 \in I$. Then $\lambda e+\frac{1}{2}u_0 = e(\lambda e+u_0+v_0) \in eI \subseteq I$, and also $\lambda e+(1/4)u_0 = e(\lambda e+\frac{1}{2}u_0) \in eI \subseteq I$. Since I is a linear space we may subtract to obtain $(1/4)u_0 \in I$. Further, $\lambda e = (\lambda e+\frac{1}{2}u_0) - \frac{1}{2}u_0 \in I$, and $v_0 = (\lambda e+u_0+v_0) - \lambda e - u_0 \in I$. Then either $\lambda = 0$ for all members of I, or else we have $e \in I$. Then $eU = U \subseteq UI \subseteq I$, which implies that $V = U^2 = UU \subseteq UI \subseteq I$. Thus I contains $e\mathbb{R}$, U and V, and so $I = A$. If λ is always zero, then $I \subseteq K$ and $u_0+v_0 \in I$, which implies that u_0, $v_0 \in I$. Hence $I = U'\oplus V'$ where $U' = U\cap I$, $V' = V\cap I$.

For $S \subseteq A$, let the annihilator of S be defined by $\operatorname{ann}(S) = \{a \mid A\in A, sa = 0, \text{ all } s\in S\}$. For any S, $\operatorname{ann}(S)$ is a vector subspace of A. In contrast with the associative case however, ann S is not in general an ideal. However we do have :

Proposition 2.3. The annihilator of the kernel K of ω, ann(K), is an ideal in A, with $\text{ann}(K) \subseteq K$. In fact $\text{ann}(K) = U' \oplus V'$ with $U' = U \cap \text{ann}(U)$, $V' = \text{ann}(A)$.

Proof. (i) Let $\lambda e + u_0 + v_0 \in \text{ann } K$. Then for all $k \in K$, $k(\lambda e + u_0 + v_0) = 0 \in \text{ann}(K)$. Moreover $e(\lambda e + u_0 + v_0) = \lambda e + \frac{1}{2}u_0$, so we need to show $\lambda e + \frac{1}{2}u_0 \in \text{ann}(K)$. We first show that $u_0 \in \text{ann}(K)$. Now $(\lambda e + u_0 + v_0)U = 0$, hence $u_0 U = 0$, $\frac{1}{2}\lambda u + uv_0 = 0$ for all $u \in U$. Since $u_0 u = 0$ for all $u \in U$ we have $0 = u_0 u = 2u(u_0 u) = -u_0 u^2$ by (1,iii,a). Thus for any $u_1, u_2 \in U$, we have $u_0(u_1+u_2)^2 = u_0(u_1-u_2)^2 = 0$. Subtracting, we have $u_0(u_1 u_2) = 0$. This implies $u_0 V = 0$ as $V = U^2$, and hence $u_0 \in \text{ann}(K)$. Since we have $\lambda e + u_0 + v_0 \in \text{ann}(K)$, and $u_0 \in \text{ann}(K)$, we see that $\lambda e + v_0 \in \text{ann}(K)$. In particular, $0 = u(\lambda e + v_0) = = \frac{1}{2}\lambda u + uv_0$ for all $u \in U$. We now choose $u_1, u_2 \in U$. Then (a) $\frac{1}{2}\lambda u_1 + u_1 v_0 = 0$, (b) $\frac{1}{2}\lambda u_2 + u_2 v_0 = 0$. Multiply (a) by u_2 and (b) by u_1 and add. We obtain $\lambda u_1 u_2 = -u_1(u_2 v_0) - u_2(u_1 v_0) = 0$, by (1,III,b). This last result may be written $\lambda v = 0$. If $\lambda \neq 0$ then $v = 0$, and thus $\lambda e = \lambda e + v_0 \in \text{ann}(K)$. That is, $e \in \text{ann}(K)$. Then $U = eU \subseteq U\,\text{ann}(K) = 0$, which implies $K = 0$ and $\text{ann}(K) = A \cong \mathbb{R}$. Otherwise $\lambda = 0$ and then $e(\lambda e + u_0 + v_0) = e(u_0 + v_0) = \frac{1}{2}u_0 \in \text{ann}(K)$ by the above, which means that ann(K) is an ideal of A, $\text{ann}(K) \subseteq K$. It is obvious from the above that $u_0 \in \text{ann}(K)$ is equivalent to $u_0 \in U \cap \text{ann}(U)$, and it is clear that $V \cap \text{ann}(K) = \text{ann}(A)$.

Corollary 2.4. In a nuclear Bernstein algebra, (i) $V^2 \subseteq \text{ann}(K)$, and (ii) $(u_1 v_2)v_3 + (u_1 v_3)v_2 \in \text{ann}(K)$ for all $u_1 \in U$, $v_i \in V$.

Proof. (i) By (1,ii), $V^2 \subseteq U$ and by (1,iii,c), $V^2 \subseteq \text{ann}(U)$. Thus $V^2 \subseteq U \cap \text{Ann}(U) \subseteq \text{ann}(K)$ by Proposition 2.3.

(ii) We note that $(u_1 v_2)v_3 + (u_1 v_3)v_2 \in U$, so it is enough to show that it annihilates U. Now $u((u_1 v_2)v_3) + u((u_1 v_3)v_2) = u(u'v_3) + u(u''v_2)$ where $u' = u_1 v_2$, $u'' = u_1 v_3$. Thus by (1,iv,b) we have $u((u_1 v_2)v_3 + (u_1 v_3)v_2) = = -u'(uv_3) - u''(uv_2) = -(u_1 v_2)(uv_3) - (u_1 v_3)(uv_2) = (u_1 u)(v_2 v_3)$, by (1,iii) $= 0$, since $V^2 \subseteq \text{ann}(K)$.

Our strategy now will be to quotient out ann(K). However we first need to check that quotienting does no damage to nuclear Bernstein algebras.

Proposition 2.5. Let A be nuclear Bernstein, let B be any $\mathbb{R}$-algebra whatever, and let $\varphi : A \to B$ be a non zero $\mathbb{R}$-algebra morphism. Then $\varphi(A)$ is nuclear Bernstein with weight function $\omega^*(\varphi(a)) = \omega(a)$ and weight kernel $\bar{K} = \varphi(K)$. The only idempotents of $\varphi(A)$ are the various $\varphi(e)$, e idempotent in A, and for any choice of e, $\bar{K} = \bar{U} \oplus \bar{V}$ with $\bar{U} = \varphi(U)$, $\bar{V} = \varphi(V)$. Moreover we have $\bar{V}^2 = 0$.

Proof. Since $Ker(\varphi)$ is an ideal in A, and $\varphi \neq 0$, we have $Ker(\varphi) \subseteq K$. Consequently, if $\varphi(a') = \varphi(a)$, then $a'-a \in K$ and $\omega(a') = \omega(a)$. It follows that ω^* is well defined. It is trivial that $\omega^* : \varphi(A) \to \mathbb{R}$ is a surjective $\mathbb{R}$-algebra morphism. Moreover $\omega^*(\varphi(a)) = 0$, $\omega(a) = 0$ and $a \in K$ are all equivalent, and hence are equivalent to $\varphi(a) \in \varphi(K)$. To deal with the idempotents we first prove a simple lemma.

Lemma. If C is any Bernstein algebra, the idempotents are just the elements $e_* = e+u+u^2$, where e is a fixed idempotent, $u \in U$.

Proof. If $e_* = \lambda e+u+v$ then $e_*^2 = \lambda^2 e+\lambda u+u^2+2uv+v^2 = \lambda e+u+v$, which implies $(\lambda^2-\lambda)e= 0$, $v = u^2$, $\lambda u+2uv+v^2 = u$. Since $v = u^2$, we have $2uv = v^2 = 0$ by (1,iii,a) and (iv). Thus for $u \neq 0$, $\lambda = 1$. Hence $e_* = e+u+u^2$. Conversely, we see that any $e+u+u^2$ is idempotent.

Now suppose $\varphi(a)$ is idempotent and non zero. Then $\varphi(a^2-a) = 0$, while $\omega(a) = 1$. Let $a = e+u_0+v_0$. Then $a^2 = e+u_0+2u_0v_0+u_0^2+v_0^2$. Thus $a^2-a =$ $= (2u_0v_0+v_0^2)+(u_0^2-v_0) \in Ker(\varphi)$, where the first bracketed term belongs to U and the second to V. But $Ker(\varphi) \subseteq K$, hence $Ker(\varphi) = Ker(\varphi) \cap U \oplus (Ker(\varphi) \cap V)$. Hence $2u_0v_0+v_0^2 \in Ker(\varphi)$ and $u_0^2-v_0 \in Ker(\varphi)$. Hence $\varphi(v_0) = \varphi(u_0^2)$, and $\varphi(a) = \varphi(e+u_0+u_0^2) = \varphi(e_1)$ where e_1 is idempotent in A, by the lemma. Finally, $\varphi(K) = \varphi(U+V)$. If $\varphi(u) \in \varphi(V)$ then $\varphi(u) = \varphi(v)$, $u-v \in Ker(\varphi)$. Then $u \in Ker(\varphi) \cap U$, $v \in Ker(\varphi) \cap V$. This implies that $\varphi(u) = \varphi(v) = 0$, so that $\varphi(K) = \varphi(U) \oplus \varphi(V)$ as a vector space direct sum. It is trivial that $V^2 \subseteq ann(K)$, so that $\bar{V}^2 = 0$.

3. QUOTIENTS WITH RESPECT TO $ann(K)$.

Theorem 3.1. Let A be nuclear Bernstein and different from $\mathbb{R}$. Then $A/ann(K)$ is a nuclear Bernstein algebra in which the identity $x^3 = \omega(x)x^2$ holds. It is also a Jordan algebra, that is a commutative algebra in which $x^2(xy) =$ $= x(x^2y)$ for all $x,y \in A$.

Proof. $B = A/ann(K) = \varphi(A)$ where φ is the natural map, so B is certainly nuclear Bernstein by Proposition 2.5. and to prove $x^3 = \omega(x)x^2$ in B we need only show that $\varphi(a^3) = \omega(a)\varphi(a^2)$ for all $a \in A$, that is that $a^3-\omega(a)a^2 \in ann(K)$ for all $a \in A$ since $Ker(\varphi) = ann(K)$. Now if $a=\lambda e+u_0+v_0$

we have $\omega(a) = \lambda$ and $a^2 = \lambda^2 e+\lambda u_o+u_o^2+2u_o v+v_o^2$. Hence

$$\omega(a)a^2 = \lambda^3 e + \lambda^2 u_o + \lambda u_o^2 + 2\lambda u_o v_o + \lambda v_o^2 \quad \text{and} \tag{2}$$

$$a^3 = \lambda^3 e+\tfrac{1}{2}\lambda^2 u_o+\lambda u_o v_o+\tfrac{1}{2}\lambda v_o^2+\tfrac{1}{2}\lambda^2 u_o+\lambda u_o^2+u_o(u_o^2)+u_o(2u_o v)+u_o v_o^2+\lambda^2 u_o v_o+\lambda u_o^2 v_o+$$
$$+2\lambda v_o(u_o v_o)+\lambda v_o(v_o^2). \tag{3}$$

On the right of (3), the term $\tfrac{1}{2}\lambda v_o^2 \in V^2 \subseteq \text{ann}(K)$, and $u_o u_o^2 = 0$ by (1,iii,a), while $2u_o(u_o v) = 0$ by (1,iii,b). Further, $uv_o^2 = 0$ by the Corollary to Proposition 2.3, $\lambda u_o^2 v_o \in V^2 \subseteq \text{ann}(K)$, $v_o(u_o v_o) \in \text{ann}(K)$ by Proposition 2.3, and $v_o^2 v_o = 0$ since $V^2 \subseteq \text{ann}(K)$. On subtraction we see that $a^3-\omega(a)a^2 \in \text{ann}(K)$, so that

$$x^3 = \omega(x)x^2 \quad \text{for all} \quad x \in B. \tag{4}$$

To establish the Jordan formula, we polarise (4) with respect to x :

$$x^2 y+2x(xy) = \omega(y)x^2+2\omega(x)xy \tag{5}$$

Then $x(x^2 y)+2x(x(xy)) = \omega(y)x^3+2\omega(x)x(xy) = \omega(y)\omega(x)x^2+2\omega(x)x(xy)$. Further, substitution of xy for y in (5) leads to $x^2(xy)+2x(x(xy)) = \omega(xy)x^2 + 2\omega(x)x(xy)$ which gives the result immediately.

Proposition 3.2. Let B be any commutative non-associative finite dimensional $\mathbb{R}$-algebra, equipped with a surjective $\mathbb{R}$-algebra morphism $\omega : B\to\mathbb{R}$, such that $x^3 = \omega(x)x^2$. Then B is (as above) a Jordan algebra, in which, for every choice of idempotent, $V^2 = 0$ and $(uv_1)v_2+(uv_2)v_2 = 0$ for all $u \in U$, $v_i \in V$.

Proof. The Jordan property comes, as before, from polarisation. To derive the Bernstein identity we polarise once to obtain

$$2x(xy)+x^2 y = 2\omega(x)xy+\omega(y)x^2 \tag{6}$$

and then substitute $y = x^2$. Then $2x^3x+(x^2)^2 = 2\omega(x)x^3+\omega^2(x)x^2$ which implies

$$2\omega(x)x^3+(x^2)^2 = 2\omega^2(x)x^2+\omega^2(x)x^2.$$

This implies that $(x^2)^2 = \omega^2(x)x^2$, that is that B is Bernstein. On polarising the relation (5) again, we have

$$2z(xy)+2x(yz)+2(xz)y = 2\omega(z)xy+2\omega(x)zy+2\omega(y)xz \tag{7}$$

Put $x = y = v$, $z = e$. Then since $eV = 0$, $v^2 = 2v^2$, which implies $V^2 = 0$. Next put $x = u$, $y = z = v$. Then we have $2v(uv)+2uv^2+2(uv)v = 0$. The right hand side is $4v(uv)$, hence $(uv)v = 0$. This polarises to $(uv_1)v_2+(uv_2)v_1 = 0$. We will call a Bernstein algebra satisfying the identity $x^3 = \omega(x)x^2$ a reduced Bernstein algebra. Relation (7) above shows that the Jacobi identity

$x(yz)+z(xy)+y(zx) = 0$, all $x,y,z \in K$ holds for any reduced Bernstein algebra.

The fact that a reduced Bernstein algebra is Jordan is very useful since there is a vast literature on Jordan algebras, and quite a good structure theory for them. Results for reduced nuclear Bernstein algebras A' can sometimes be lifted back to the corresponding general nuclear Bernstein A from $A/\mathrm{ann}(K)$, as in the next section.

4. THE K_n SEQUENCE OF A NUCLEAR BERNSTEIN ALGEBRA.

Let A be nuclear Bernstein. We prove :

Proposition 4.1. The subspaces K_n, defined inductively by $K_0 = A$, $K_1 = KA = K, \ldots, K_{n+1} = KK_n$ are all ideals in A, with $K_{n+1} \subseteq K_n$ for $n \geq 0$.

Proof. Firstly, $AK = (eR+U+U^2)(U+U^2) = K$ as asserted ; K_1 is certainly an ideal. We work by induction on n. Assume $K_1,\ldots,K_n$ are ideals. Then clearly $K = K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n$ and, in particular, $K_n = U_n \oplus V_n$ by Proposition 2.2, with $U_n = U \cap K_n$, $B_n = V \cap K_n$. Since K_n is an ideal, $KK_n \subseteq K_n$, and so $UU_n \oplus (UV_n + VU_n + VV_n) \subseteq U_n \oplus V_n$. That is,

$$U_{n+1} = UV_n + VU_n + VV_n \subseteq U_n \tag{8}$$

and

$$V_{n+1} = UU_n \subseteq V_n . \tag{9}$$

We can now show that KK_n $(= K_{n+1})$ is an ideal. First, $e(K_{n+1}) = U_{n+1}$ K_{n+1}. Secondly, by (8), $uU_{n+1} \subseteq uU_n \subseteq UU_n = V_{n+1} \subseteq K_{n+1}$, while on using (9) as well we have $uV_{n+1} = uUU_n \subseteq uV_n \subseteq UV_n \subseteq U_{n+1} \subseteq K_{n+1}$. Finally, $vU_{n+1} \subseteq vU_n \subseteq VU_n \subseteq U_{n+1} \subseteq K_{n+1}$ and $vV_{n+1} \subseteq vV_n \subseteq VV_n \subseteq U_{n+1} \subseteq K_{n+1}$, which completes the proof.

Remark 4.2. Let $u', u'' \in U$, $u_n \in U_n$. Then by (1,iii,a), $(u'u'')u_n = -u'(u_n u'') - u''(u_n u') \in U(U_n U) + U(U_n U) = U(U_n U) = UV_{n+1} \subseteq UV_n$. Thus we have $VU_n \subseteq UV_{n+1} \subseteq UV_n$, and so in fact we have the stronger recurrence relations

$$U_{n+1} = UV_n + VV_n \ , \quad V_{n+1} = UU_n \tag{10}$$

Let A be nuclear Bernstein. The ideals K_n form a decreasing chain of vector subspaces of A, so that $\dim(K_{n+1}) \leq \dim(K_n)$ for all $n \geq 1$. Then the sequence $d_n = \dim(K_n)$ is a non increasing sequence of positive integers since $\dim(A)$ is finite. Thus d_n is constant for $n > n_0$, and this implies $K_n = K_{n+1} = \ldots$ for $n > n_0$. Let us call this ideal $K(\infty)$. It is of course $\bigcap_{n=1}^{\infty} K_n$.

Theorem 4.3. In any nuclear Bernstein algebra A, $K(\infty) = 0$. Further, if $W \subseteq A$ is any subspace of A satisfying $KW = W$, then $W = 0$.

Proof. Suppose first that the theorem is true for reduced nuclear Bernstein algebras, and let A be any nuclear Bernstein algebra. Then by section 3, $A/\text{ann}(K)$ is reduced. The sequence $\varphi(K)$, $(\varphi(K))^2, \ldots, \varphi(K)_{n+1} = \varphi(K)(\varphi(K))_n$, ..., where $\varphi : A \to A/\text{ann}(K)$ is the natural map, converges finitely to 0. That is $\varphi(K)_N = 0$ for some N. Now since φ is a homomorphism, it follows by induction that $\varphi(K)_n = \varphi(K_n)$ for all n. Hence $\varphi(K_N) = 0$, that is $K_N \subseteq \text{ann}(K)$. But then $K_{N+1} = KK_N \subseteq K(\text{ann}(K)) = 0$. It remains to prove the theorem for reduced algebras. If A is reduced it is a Jordan algebra in which $x^3 = \omega(x)x^2$. Then K is also Jordan, and $x^3 = 0$ in K. A theorem of Albert ([2], pp. 91-97) shows that if in a Jordan algebra J, each element is nilpotent, then there exists a $c > 0$ such that any product of c elements of J, no matter how associated, is 0. Note that the Jordan identity $x^2(xy) = x(x^2y)$ implies that the powers of a given element x generate an associative algebra, so that the concept of nilpotence makes sense. It follows from Albert's theorem that $K_M = 0$ for some $M \geq 1$, and thus we have shown that $K(\infty) = 0$ in all nuclear Bernstein algebras. Finally, if $W = KW$, then $W = KW = K(KW) = K(K(KW)) = \ldots = 0$, since $K_M = 0$ for some M.

Corollary 4.4. If $\dim(K) = \varkappa$, then $K_{1+\varkappa} = 0$.

Proof. If, for some n, $K_{n+1} = K_n$ then $K_n = K(K_n)$, which implies $K_n = 0$ by Theorem 4.3. Thus the non zero K_n are strictly decreasing, $\dim(K_1) \leq \varkappa$, $\dim(K_2) \leq \varkappa-1$, $\dim(K_3) \leq \varkappa-2, \ldots, \dim(K_{1+\varkappa}) = 0$.

5. FURTHER PROPERTIES OF K_n IN THE REDUCED CASE.

In a reduced Bernstein algebra A, the recurrence relations (10) above reduce to $U_{n+1} = UV_n$, $V_{n+1} = UU_n$, $n \geq 1$, since $V^2 = 0$ in these algebras. For small n we have $U_1 = U$, $V_1 = U^2 = V$, $U_2 = UV$, $V_2 = U^2 = V$, $U_3 = UV$, $V_3 = U(UV)$; $U_4 = U(U(UV))$, $V_4 = U(UV)$. In fact we have, in general, by induction :

Proposition 5.1. For $n \geq 1$, $V_{2n-1} = V_{2n} = U(U(U\ldots(UV)\ldots)$, $U_{2n} = U_{2n-1} = U(U(U\ldots(UV)\ldots)$, where there are $2n-2$ U factors in the first product, and $2n-1$ in the second.

Proposition 5.2. Let S be any vector subspace of K, with $K_{n+1} \subseteq S \subseteq K_n$ for some $n \geq 1$. Then S is an ideal in A if and only if $S = (U\cap S)\oplus(V\cap S)$.

Proof. If $S = (U\cap S)\oplus(V\cap S) = U'\oplus V'$, then $eS = U' \subseteq S$, and $KS \subseteq KK_n = K_{n+1} \subseteq S$, hence S is an ideal. If S is an ideal, then $S = (U\cap S)\oplus(V\cap S)$ by proposition 2.2.

Remark 5.3. In the above Proposition, there is no requirement that A should be reduced.

We can use Proposition 5.1. and 5.2 to obtain a convenient basis for A, giving rise to a multiplication table that has a particularly attractive form when A is reduced. We observe that K_n and K_{n+1} either have the same U-part or the same V-part. If the U-parts are equal, $U_{n+1} = U_n$, we choose an ascending chain

$$S_n^{(0)} = K_{n+1} \subset S_n^{(1)} \subset \ldots \subset S_n^{(t)} = K_n$$

of vector subspace $S_n^{(i)}$, increasing in dimension by 1 at each stage and of the form $U_{n+1}\oplus V_n^{(i)}$. If $V_{n+1} = V_n$ then we take $S_n^{(i)} = U_n^{(i)}\oplus V_{n+1}$. If both $U_n = U_{n+1}$ and $U_n = V_{n+1}$ then $K_{n+1} = K_n = K(K_n)$ with implies that $K_n = 0 = K_{n+1}$, by Theorem 4.2. By this process we obtain :

Theorem 5.3. If A is a reduced nuclear Bernstein algebra, then there exists a chain of ideals $I_0 = 0 \subset I_1 \subset \ldots \subset I_m = A$, in which $\dim I_r = r$, $0 \leq r \leq m$, and such that the sequence K_n is embedded in the chain of I_n.

Corollary 5.4. For each a in A, let R_a be the linear map $A \to A$ such that $x \mapsto xa$ for all x in A. Then there exists a basis of A such that the matrices representing the maps R_a with respect to this basis are all simultaneously upper triangular.

Proof. We use the chain $I_0 \subset I_1 \subset \ldots$, of Theorem 5.3. Let i_1 be a basis for I_1, which has dimension 1. Next choose i_2 such that i_1, i_2 is a basis for I_2, choose i_3 such that i_1, i_2, i_3 is a basis for I_3 and so on. Since each I_r is an ideal we have $R_a(I_r) = aI_r \subseteq I_r$ for all a in A, $0 \leq r \leq \dim A$. Thus with respect to the basis $i_1, i_2, \ldots, i_m$ of A, each R_a corresponds to an upper triangular matrix.

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