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On Description of Leibniz Algebras Corresponding to sl_2

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Abstract In this paper we describe finite-dimensional complex Leibniz algebras whose quotient algebra with respect to the ideal I generated by squares is isomorphic to the simple Lie algebra sl_2. It is shown that the number of isomorphism classes such of Leibniz algebras coincides with the number of partitions of $dim I$.

Keywords Leibniz algebra · Lie algebra · Irreducible module · Simple Leibniz algebra

Mathematics Subject Classifications (2010) 17A32 · 17A60 · 17B10 · 17B20

1 Introduction

Leibniz algebra is a generalization of Lie algebra. Leibniz algebras have been first introduced by Cuvier and Loday in [5, 11] as a non-antisymmetric version of Lie algebras.

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There a marked difference is noted between the structural theory of semisimple and solvable or nilpotent Lie algebras.

The problem of description of finite dimensional Lie algebras can be represented as three separate tasks:

- Description of nilpotent Lie algebras;
- Description of solvable algebras with given nilradical;
- Description of Lie algebras with given radical.

The latter two problems are the most studied part of the series and brought to fruition for the complex algebras in the middle of the last century. The third problem is reduced to a description of semisimple subalgebras of derivations of solvable algebras [14]. The problem how to construct by a given solvable algebra R and a semisimple algebra S, all the algebras L with the radical R and the quotient algebra L/R being isomorphic to S has been also solved. It turned out that such algebras L are finite in number, and they correspond in a one-to-one way to semisimple subalgebras of $Der\ R$. For semisimple algebras over the complex numbers one has the Killing form, Dynkin diagrams, root space decompositions, the Serre presentation, the theory of highest weight representations, the Weyl character formula for finite-dimensional representations and much more. The second problem is reduced to the description of the orbits of some nilpotent linear algebraic groups [15]. Thus the problem in Lie algebras case has been reduced to the study of nilpotent algebras.

The present note concerns Leibniz algebras. Many results of the theory of Lie algebras have been extended to Leibniz algebras since Loday's introduction of Leibniz algebras in 1993. However, the majority of the results have been devoted to (co)homological problems [6, 10, 12, 13], e.c.t., the classification problems of nilpotent part and its subclasses [1, 2, 4, 16–21], e.c.t.[1] Less attention has been paid to the semisimple part of the Leibniz algebras. Hardly original consideration belongs to Dzhumadil'daev and Abdykassymova [3, 7] who suggested a notion of simple Leibniz algebra and studied its properties in characteristic p. In this paper we describe the class of Leibniz algebras whose quotient algebra with respect to the ideal generated by squares is a Lie algebra isomorphic to the simple Lie algebra sl_2.

The outline of the paper is as follows. Section 2 is a brief introduction that includes a few facts needed in Section 3 which contains the main results of the paper. Here one considers the ideal I, generated by squares as an irreducible sl_2-module (due to the simplicity of the Leibniz algebra) and distinguishes three cases when the dimension of I is even, is equal to three and it is an odd natural number greater than three. We give results for each of these three cases as Propositions 3.1., 3.2. and 3.3., respectively. The main results of the paper are summarized in two theorems (Theorems 3.4. and 3.5.).

2 Preliminaries

In this section we give necessary definitions and preliminary results.

[1] Evidently, this is not at all complete list of references.

Definition 2.1 An algebra $(L, [\cdot, \cdot])$ over a field F is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]] \tag{1}$$

holds.

Every Lie algebra is a Leibniz algebra, but the bracket in a Leibniz algebra need not be skew-symmetric.

Let L be a Leibniz algebra and $I = \langle [x, x] \,|\, x \in L \rangle$ be the ideal of L generated by all squares. Then I is the minimal ideal with respect to the property that $G := L/I$ is a Lie algebra. The quotient mapping $\pi : L \longrightarrow G$ is a homomorphism of Leibniz algebras.

Conversely, let G be a Lie algebra, L be a G-module, and $\pi : L \longrightarrow G$ be a G-module morphism. Assume without loss of generality that π is an epimorphism, and define the Leibniz algebra structure on L by $[x, y] := \pi(x)y$. Then π becomes an epimorphism of Leibniz algebras. To view Leibniz algebras in this way is to regard them as being Lie algebra objects in the infinitesimal tensor category of linear maps [13]. The Lie algebra G is called the corresponding Lie algebra of the Leibniz algebra L.

Definition 2.2 A Leibniz algebra L ($[L, L] \neq I$) is said to be simple if the only ideals of L are $\{0\}, \ I, \ L$.

Obviously, if L is Lie then the ideal I is trivial. Therefore, this definition agrees with the definition of simple Lie algebra.

Here is an example of simple Leibniz algebra from [3].

Example 2.3 Let G be a simple Lie algebra and M be an irreducible skew-symmetric $G-$module (*i.e.*$[x, m] = 0$ for all $x \in G, \ m \in M$). Then the vector space $Q = G + M$ equipped with the multiplication

$$[x + m, y + n] = [x, y] + [m, y],$$

is a simple Leibniz algebra, where $m, n \in M, \ x, y \in G$.

Let L be a finite-dimensional simple Leibniz algebra. Then the quotient algebra L/I is a simple Lie algebra. It should be noted, that the ideal I may be viewed as a right L/I-module via the action:

$$m * (a + I) = [m, a], \text{ where } m \in I. \tag{2}$$

For later use we mention a couple of well-known results from the classical theory of Lie algebras concerning representations of split three-dimensional simple Lie algebras. According to [8] a three-dimensional simple Lie algebra L is said to be split if L contains an element h such that $ad\ h$ has a non-zero characteristic root ρ

belonging to the base field. It is well-known that any such algebra has a basis $\{e, f, h\}$ with the multiplication table

$$[e, h] = 2e, \qquad [f, h] = -2f, \qquad [e, f] = h,$$
$$[h, e] = -2e, \qquad [h, f] = 2f, \qquad [f, e] = -h.$$

This simple 3-dimensional Lie algebra is denoted by sl_2 and the basis $\{e, f, h\}$ is called canonical basis. Note that any three-dimensional simple Lie algebra is isomorphic to sl_2. The representation of sl_2 is determined by the images E, F, H of the basis elements e, f, h and we have

$$[E, H] = 2E, \ [F, H] = -2F, \ [E, F] = H,$$
$$[H, E] = -2E, \ [H, F] = 2F, \ [F, E] = -H.$$

Conversely, any three linear transformations E, F and H satisfying these relations determine a representation of sl_2 and hence an sl_2-module.

Definition 2.4 A nonzero module M whose only submodules are the module itself and the zero module is called irreducible module. A nonzero module M which is a direct sum of irreducible modules is said to be completely reducible.

We make use of the following two theorems from [8].

Theorem 2.5 *For each integer $m = 0, 1, 2, \dots$ there exists up to isomorphism only one irreducible sl_2-module M of dimension $m+1$. The module M has a basis $\{x_0, x_1, ..., x_m\}$ such that the representing transformations E, F and H corresponding to the canonical basis $\{e, f, h\}$ are given by:*

$$H(x_k) = (m - 2k)x_k, \ k = 0, \dots, m,$$
$$F(x_m) = 0, \ F(x_k) = x_{k+1}, \ k = 0, \dots, m-1,$$
$$E(x_0) = 0, \ E(x_k) = -k(m+1-k)x_{k-1}, \ k = 1, \dots, m.$$

Theorem 2.6 *If L is finite-dimensional semi-simple Lie algebra over a field of characteristic 0, then every finite-dimensional module over L is completely reducible.*

3 Main Result

This section is devoted to the description of finite dimensional complex Leibniz algebras whose corresponding Lie algebra is isomorphic to sl_2.

Let L be a simple Leibniz algebra such that $L/I \cong sl_2$. Then L is isomorphic to $sl_2 \oplus I$ as a vector space. Due to the simplicity of L, the ideal I can be regarded as an irreducible module over sl_2. Let $\dim I = m + 1$.

Proposition 3.1 *Let m be an odd positive integer. Then there exists a basis $\{e, h, f, x_0, x_1, \ldots, x_m\}$ of L such that the multiplication table on this basis has the following form:*

$$\begin{array}{lll} [e, h] = 2e, & [h, f] = 2f, & [e, f] = h, \\ [h, e] = -2e & [f, h] = -2f, & [f, e] = -h, \\ [x_k, h] = (m-2k)x_k & k = 0, \ldots, m, & \\ [x_k, f] = x_{k+1}, & k = 0, \ldots, m-1, & \\ [x_k, e] = -k(m+1-k)x_{k-1}, & k = 1, \ldots, m, & \end{array}$$

where the omitted products are equal to zero.

Proof Let $\{x_0, x_1, \ldots, x_m\}$ be a basis of I. We put

$$[e, h] = 2e + \sum_{k=0}^{m} \alpha_k x_k, \ [f, h] = -2f + \sum_{k=0}^{m} \beta_k x_k, \text{ and } [e, f] = h + \sum_{k=0}^{m} \gamma_k x_k.$$

Without loss of generality, we may assume that $[e, f] = h$ (by taking $h' = h + \sum_{k=0}^{m} \gamma_k x_k$).

Due to Theorem 2.5 along with Eq. 2 we may suppose that the algebra $L = Span\{e, f, h, x_0, x_1, \ldots, x_m\}$ has the following multiplication table:

$$[x_k, h] = (m-2k)x_k, \ k = 0, \ldots, m,$$
$$[x_k, f] = x_{k+1}, \ k = 0, \ldots, m-1, \ [x_m, f] = 0,$$
$$[x_k, e] = -k(m+1-k)x_{k-1}, \ k = 1, \ldots, m, \ [x_0, e] = 0.$$

Applying the base change

$$e' = e - \sum_{k=0}^{m} \frac{\alpha_k}{m-2k-2} x_k,$$
$$f' = f - \sum_{k=0}^{m} \frac{\beta_k}{m-2k+2} x_k,$$
$$h' = h - \sum_{k=0}^{m-1} \frac{\alpha_k}{m-2k-2} x_{k+1} \tag{3}$$

(note that m being an odd integer the denominators of the fractions involved are non zero) we obtain

$$[e', h'] = 2e', \quad [f', h'] = -2f', \quad [e', f'] = h'.$$

Writing the basis elements without primes we have

$$\begin{array}{lll} [e,e]=a, & [e,h]=2e, & [h,e]=-2e+p,\\ [f,f]=b, & [f,h]=-2f, & [h,f]=2f+q,\\ [h,h]=c, & [e,f]=h, & [f,e]=-h+r,\\ [x_k,h]=(m-2k)x_k & k=0,\dots,m, & \\ [x_k,f]=x_{k+1}, & k=0,\dots,m-1, & [x_m,f]=0,\\ [x_k,e]=-k(m+1-k)x_{k-1}, & k=1,\dots,m, & [x_0,e]=0 \end{array}$$

for some $a, b, c, p, q, r \in I$.

By using the Leibniz identity we derive

$$[a,h]=[[e,e],h]=[[e,h],e]+[e,[e,h]]=4a.$$

Let $a=\sum_{k=0}^{m}\lambda_k x_k$. Then from the table of multiplication we get

$$4\sum_{k=0}^{m}\lambda_k x_k=4a=[a,h]=\sum_{k=0}^{m}\lambda_k(m-2k)x_k,$$

and since m is an odd integer, this implies that $\lambda_k=0$ for $0\le k\le m$, i.e., $a=0$.

Exhausting the Leibniz identity as follows

$$0=[a,f]=[[e,e],f]=[[e,f],e]+[e,[e,f]]=[h,e]+[e,h]=p$$

we get $p=0$.

Similarly, setting $b=\sum_{k=0}^{m}\mu_k x_k$, we get

$$[b,h]=[[f,f],h]=[[f,h],f]+[f,[f,h]]=-4[f,f]=-4b$$

which gives $\mu_k(m-2k+4)=0$. Since m is odd, one gets $\mu_k=0$ for $0\le k\le m$ and $b=0$.

Applying the Leibniz identity as follows

$$\begin{aligned}0=[[f,f],e]=[[f,e],f]+[f,[f,e]]&=-[h,f]+[r,f]-[f,h]\\ &=-2f-q+[r,f]+2f=-q+[r,f]\end{aligned}$$

we obtain

$$q=[r,f]. \qquad (*)$$

Putting $c=\sum_{k=0}^{m}\nu_k x_k$ and making use of the following relation

$$[c,e]=[[h,h],e]=[[h,e],h]+[h,[h,e]]=-2[e,h]-2[h,e]=0,$$

we find $\nu_1=\dots=\nu_m=0$ and $c=\nu_0 x_0$.

Now we consider

$$\begin{aligned}\nu_0 x_1 &= [c, f] = [[h, h], f] = [[h, f], h] + [h, [h, f]] \\ &= 2[f, h] + [q, h] + 2[h, f] = -4f + [q, h] + 4f + 2q = 2q + [q, h].\end{aligned}$$

The substitution $q = \sum_{k=0}^{m} \theta_k x_k$ gives $\theta_1 = \frac{\nu_0}{m}$ and $\theta_k = 0, k \neq 1$, i.e., $q = \frac{\nu_0}{m} x_1$.

From the relation (*) we derive $[r, f] = \frac{\nu_0}{m} x_1$. Expanding r via basis vectors $\{x_0, \ldots, x_m\}$ along with the last equality one derives $r = \eta x_m + \frac{\nu_0}{m} x_0$.

Finally, the chain of equalities

$$\begin{aligned}-2h &= -2[e, f] = [[h, e], f] = [[h, f], e] + [h, [e, f]] \\ &= 2[f, e] + [q, e] + [h, h] = -2h + 2r + [q, e] + c,\end{aligned}$$

implies that $0 = 2r + [q, e] + c$, consequently $\eta = \nu_0 = 0$. Hence, $c = q = r = 0$ and we conclude that $a = b = c = p = q = r = 0$ which completes the proof. □

Let us now consider the case $m = 2$.

Proposition 3.2 *Let $m = 2$. Then there exists a basis $\{e, h, f, x_0, x_1, x_2\}$ of L such that the multiplication table on this basis has the form:*

$$\begin{array}{lll} [e, h] = 2e, & [h, f] = 2f, & [e, f] = h, \\ [h, e] = -2e, & [f, h] = -2f, & [f, e] = -h, \\ [x_0, h] = 2x_0, & [x_0, f] = x_1, & \\ [x_1, e] = -2x_0, & [x_1, f] = x_2, & \\ [x_2, h] = -2x_2, & [x_2, e] = -2x_1, & \end{array}$$

where the omitted products are zero.

Proof Thanks to Theorem 2.5 we have the following multiplication table:

$$\begin{array}{lll} [x_0, h] = 2x_0, & [x_0, e] = 0, & [x_0, f] = x_1, \\ [x_1, h] = 0, & [x_1, e] = -2x_0, & [x_1, f] = x_2, \\ [x_2, h] = -2x_2, & [x_2, e] = -2x_1, & [x_2, f] = 0. \end{array}$$

Let $[e, h] = 2e + \alpha_0^{eh} x_0 + \alpha_1^{eh} x_1 + \alpha_2^{eh} x_2$. Making the base change as follows

$$e' = e + \frac{\alpha_1^{eh}}{2} x_1 + \frac{\alpha_2^{eh}}{4} x_2 \tag{4}$$

we obtain $[e', h] = 2e' + \alpha_0^{eh} x_0$.

For the sake of simplicity we will write e instead of e'. Consider

$$[[e, e], h] = [[e, h], e] + [e, [e, h]] = 4[e, e].$$

Then due to the table of multiplication above, we derive $\alpha_0^{ee} = \alpha_1^{ee} = \alpha_2^{ee} = 0$, i.e. $[e, e] = 0$.

Similarly, letting $[f, h] = -2f + \alpha_0^{fh}x_0 + \alpha_1^{fh}x_1 + \alpha_2^{fh}x_2$ and making the substitution

$$f' = f - \frac{\alpha_0^{fh}}{2}x_0 - \frac{\alpha_1^{fh}}{4}x_1 \tag{5}$$

we get $[f', h] = -2f' + \alpha_2^{eh}x_2$. Further, we use f instead of f'.

Note that the relation $[[f, f], h] = [[f, h], f] + [f, [f, h]] = -4[f, f]$ implies that $[f, f] = 0$.

Let $[h, e] = -2e + \alpha_0^{he}x_0 + \alpha_1^{he}x_1 + \alpha_2^{he}x_2$. Making the substitution

$$h' = h + \frac{\alpha_0^{he}}{2}x_1 - \frac{\alpha_1^{he}}{2}x_2 \tag{6}$$

we obtain $[h', e] = -2e + \alpha_2^{he}x_2$.

Assume that $[h', h']$ is written as $\alpha_0^{hh}x_0 + \alpha_1^{hh}x_1 + \alpha_2^{hh}x_2$. Then putting

$$h'' = h' - \frac{\alpha_0^{hh}}{2}x_0 \tag{7}$$

we obtain

$$[h'', h''] = \alpha_1^{hh}x_1 + \alpha_2^{hh}x_2.$$

From the Leibniz identity

$$[[h'', h''], e] = [[h'', e], h''] + [h'', [h'', e]]$$

we get $\alpha_1^{hh} = \alpha_0^{eh}$, $\alpha_2^{hh} = 0$, $\alpha_2^{he} = 0$ and $[h'', h''] = \alpha_0^{eh}x_1$. Thus, $[h'', e] = -2e$.

We denote h'' by h. Let $[h, f] = 2f + \alpha_0^{hf}x_0 + \alpha_1^{hf}x_1 + \alpha_2^{hf}x_2$.

Applying the Leibniz identity for the triple $\{h, h, f\}$ we obtain

$$[f, h] = -2f + \frac{\alpha_0^{eh}}{2}x_2, \quad [h, f] = 2f + \alpha_2^{hf}x_2.$$

If $[e, f]$ is written as $h + \alpha_0^{ef}x_0 + \alpha_1^{ef}x_1 + \alpha_2^{ef}x_2$, then applying the Leibniz identity to $\{e, f, h\}$ as

$$[[e, f], h] = [[e, h], f] + [e, [f, h]]$$

we derive $[e, f] = h + \alpha_1^{ef}x_1$.

Similarly, from

$$[[e, f], e] = [[e, e], f] + [e, [f, e]] = [e, [f, e]]$$

we get $[e, f] = h + \frac{\alpha_0^{eh}}{2}x_1$.

Let us consider

$$[[h, f], e] = [[h, e], f] + [h, [f, e]].$$

Then the relation $2[f, e] - 2\alpha_2^{hf}x_1 = -2[ef] - [h, h]$ yields $[f, e] = -h + (\alpha_2^{hf} - \alpha_0^{eh})x_1$.

Denote $\alpha = \alpha_0^{eh}$ and $\beta = \alpha_2^{hf}$.

We have managed to simplify the initial multiplication table as follows

$$[e,h]=2e+\alpha x_0,\ [h,e]=-2e,\ [e,e]=0,\ [e,\ f]=h+\frac{\alpha}{2}x_1,\ [f,h]=-2f+\frac{\alpha}{2}x_2,$$

$$[h,\ f]=2f+\beta x_2,\ [f,\ f]=0,\ [f,e]=-h+(\beta-\alpha)x_1,\ [h,h]=\alpha x_1.$$

It can be easily seen that the Leibniz identity $[[f,e]f]=[[f,\ f],e]+[f,[e,\ f]]$ implies that $\alpha=0$. Finally, after making the substitution

$$f'=f+\frac{\beta}{2}x_2 \tag{8}$$

we get $[f',e]=-h$ and $[f',h]=-2f'$, which completes the proof. □

In the next proposition we establish the similar result for the case m is even integer greater than 2.

Proposition 3.3 *Let $m=2n,\ n\geq 2$. Then there exists a basis $\{e,h,\ f,x_0,x_1,\dots,x_m\}$ such that the non zero products in L are as follows:*

$$\begin{array}{lll}
[e,h]=2e, & [h,\ f]=2f, & [e,\ f]=h,\\
[h,e]=-2e & [f,h]=-2f, & [f,e]=-h,\\
[x_k,h]=(m-2k)x_k & k=0,\dots,m, & \\
[x_k,\ f]=x_{k+1}, & k=0,\dots,m-1, & \\
[x_k,e]=-k(m+1-k)x_{k-1}, & k=1,\dots,m. &
\end{array}$$

Proof Let $[e,h]=2e+\sum\limits_{k=0}^{m}\alpha_k^{eh}x_k$. After making the substitution

$$e'=e-\sum_{k\neq n-1}\frac{\alpha_k^{eh}}{m-2k-2}x_k \tag{9}$$

one obtains $[e',h]=2e'+\alpha_{n-1}^{eh}x_{n-1}$.

Assume that $[e',e']=\sum\limits_{k=0}^{m}\alpha_k^{ee}x_k$. Then applying the multiplication rules we get

$$[[e',e'],h]=\sum_{k=0}^{m}\alpha_k^{ee}(m-2k)x_k.$$

On the other hand, the Leibniz identity yields

$$[[e',e'],h]=[[e',h],e']+[e',[e',h]]=4\sum_{k=0}^{m}\alpha_k^{ee}x_k-(n-1)(n+2)\alpha_{n-1}^{eh}x_{n-2}.$$

Comparing the coefficients of the basis vectors we derive

$$\alpha_k^{ee}=\alpha_{n-1}^{eh}=0,\ k\neq n-2.$$

Therefore,

$$[e',h]=2e' \text{ and } [e',e']=\alpha_{n-2}^{ee}x_{n-2}.$$

Similarly, assuming that $[f, h] = -2f + \sum_{k=0}^{m} \alpha_k^{fh} x_k$ and making the substitution

$$f' = f - \sum_{k \neq n+1} \frac{\alpha_k^{fh}}{m - 2k + 2} x_k \tag{10}$$

one gets

$$[f', h] = -2f' + \alpha_{n+1}^{fh} x_{n+1}.$$

Writing $[f', f']$ as a linear combination of x_i as $[f', f'] = \sum_{k=0}^{m} \alpha_k^{ff} x_k$ we have

$$[[f', f'], h] = \sum_{k=0}^{m} \alpha_k^{ff} (m - 2k) x_k.$$

On the other hand, the Leibniz identity gives

$$[[f', f'], h] = [[f', h], f'] + [f', [f', h]] = -4 \sum_{k=0}^{m} \alpha_k^{ff} x_k + \alpha_{n+1}^{fh} x_{n+2},$$

which implies that $\alpha_k^{ff} = \alpha_{n+1}^{fh} = 0, \ k \neq n+2$. Hence,

$$[f', h] = -2f', \quad [f', f'] = \alpha_{n+2}^{ff} x_{n+2}.$$

Let now $[h, e] = -2e + \sum_{k=0}^{m} \alpha_k^{he} x_k$ and $[h, h] = \sum_{k=0}^{m} \alpha_k^{hh} x_k$.

The transformation

$$h' = h + \sum_{k=0}^{m-1} \frac{\alpha_k^{he}}{(k+1)(m-k)} x_{k+1} - \frac{\alpha_0^{hh}}{m} x_0 \tag{11}$$

yields $[h', e] = -2e + \alpha_m^{he} x_m$ and $[h', h'] = \sum_{k=1}^{m} \alpha_k^{hh} x_k$.

Analogously, from the following identity

$$-\sum_{k=1}^{m} \alpha_k^{hh} k(m+1-k) x_{k-1} = [[h', h'], e]$$
$$= [[h', e], h'] + [h', [h', e]] = -\alpha_m^{he} (m+2) x_m$$

we get

$$[h', e] = -2e \ \text{ and } \ [h', h'] = 0.$$

Again, we use h instead of h' and the Leibniz identity written as follows

$$[[e, h], f] = [[e, f], h] + [e, [h, f]]$$

gives $[[e, f], h] = 0$ and $[e, f] = h + \alpha_n^{ef} x_n$.

Similarly, from the chain of equalities

$$-2[f, e] = [[f, h], e] = [[f, e], h] + [f, [h, e]] = [[f, e], h] - 2[f, e]$$

we derive $[[f, e], h] = 0$ and hence $[f, e] = -h + \alpha_n^{fe} x_n$.

In addition, by using the Leibniz identity for the triple $\{f, e, f\}$ it is easy to get

$$[h, f] = 2f + (n+2)(n-1)\alpha_{n+2}^{ff} x_{n+1} + \alpha_n^{fe} x_{n+1}.$$

The Leibniz identity applied to $\{h, e, f\}$ gives

$$\alpha_{n+2}^{ff} = \frac{1}{(n+2)(n-1)} \left(\frac{2\alpha_n^{fe} + 2\alpha_n^{ef}}{n(n+1)} - \alpha_n^{fe} \right).$$

As a result we get $[h, f] = 2f + \frac{2}{n(n+1)}(\alpha_n^{fe} + \alpha_n^{ef}) x_{n+1}$.

Next, due to the Leibniz identity

$$[[e, e], f] = [[e, f], e] + [e, [e, f]]$$

we obtain $\alpha_{n-2}^{ee} x_{n-1} = -\alpha_n^{ef} n(n+1) x_{n-1}$ and if one denotes $\alpha = \alpha_n^{ef}$, $\beta = \alpha_n^{fe}$, then we get

$$\alpha_{n-2}^{ee} = -n(n+1)\alpha, \ \alpha_{n+2}^{ff} = \frac{1}{(n+2)(n-1)} \left(\frac{2\beta + 2\alpha}{n(n+1)} - \beta \right).$$

Finally, making the basis transformation as follows

$$e' = e - \frac{n(n+1)\alpha}{(n-1)(n+2)} x_{n-1}, \ h' = h - \frac{2\alpha}{(n-1)(n+2)} x_n \tag{12}$$

$$f' = f + \left(\frac{\beta}{n(n+1)} - \frac{2\alpha}{n(n-1)(n+1)(n+2)} \right) x_{n+1} \tag{13}$$

we obtain the required result. □

Summarizing all the observations above the final result can be written as follows.

Theorem 3.4 *Let L be a complex $n-$dimensional ($n \geq 4$) simple Leibniz algebra and let I be the ideal generated by squares in L. Assume that the quotient L/I is isomorphic to the simple Lie algebra sl_2. Then there exist a basis $\{e, f, h, x_0, x_1, \dots, x_{n-4}\}$ of L such that the non zero products of the basis vectors in L are represented as follows:*

$$\begin{aligned}
&[x_k, h] = (n-4-2k)x_k, \ (0 \leq k \leq n-4) \\
&[x_k, f] = x_{k+1}, \ (0 \leq k \leq n-5) \\
&[x_k, e] = k(k+3-n)x_{k-1}, \ (1 \leq k \leq n-4) \\
&[e, h] = 2e, [h, e] = -2e, [f, h] = -2f, \\
&[h, f] = 2f, [e, f] = h, [f, e] = -h.
\end{aligned}$$

The next theorem presents a generalization of Theorem 3.4 for arbitrary finite-dimensional Leibniz algebras with the corresponding algebra sl_2.

Theorem 3.5 *Let L be a finite-dimensional complex Leibniz algebra such that $L/I \cong sl_2$. Then there exists a basis $\{e, f, h, x_1^1, \dots, x_{t_1}^1, x_1^2, \dots, x_{t_2}^2, \dots, x_1^p, \dots, x_{t_p}^p\}$ of L such that the only non-zero products in the table of multiplication of L with respect to this basis are given as follows*

$$[e, h] = 2e, \quad [f, h] = -2f, \quad [e, f] = h,$$

$$[h, e] = -2e, \quad [h, f] = 2f, \quad [f, e] = -h,$$

$$\begin{aligned} \left[x_k^j, h\right] &= (t_j - 2k)x_k^j, & k &= 0, \dots, t_j \\ \left[x_k^j, f\right] &= x_{k+1}^j, & k &= 0, \dots, t_j - 1 \\ \left[x_k^j, e\right] &= -k(t_j + 1 - k)x_{k-1}^j, & k &= 1, \dots, t_j, \end{aligned}$$

$1 \le j \le p$.

Proof In view of Theorem 2.6 the ideal I as a sl_2-module is a direct sum of irreducible sl_2-modules $I_1, \dots, I_p$. Therefore, L decomposes into a direct sum of vector spaces $sl_2 \oplus I_1 \oplus I_2 \oplus \cdots \oplus I_p$. Denote $t_j = \dim I_j$ and let $\{x_1^j, \dots, x_{t_j}^j\}$ be the basis of I_j for $1 \le j \le p$ which satisfies the assertion of Theorem 2.5.

Note that I is in the right annihilator of the Leibniz algebra L. Therefore, the only products which should be determined are $[sl_2, sl_2]$ and $[I, sl_2]$.

For any fixed j from $\{1, 2, \dots, p\}$ similar to the proof of Propositions 3.1–3.3, depending on the parity of $t_j = \dim I_j$ one can arrange transformations for the basis vectors e, f and h via $\{x_1^j, \dots, x_{t_j}^j\}$ similar to those of Eqs. 4–13 in such a way that the basis expansions of the products of new basis vectors e, f and h from sl_2 do not contain elements of $\{x_1^j, \dots, x_{t_j}^j\}$. We apply this argument to sl_2-modules $I_1, \dots, I_p$, sequentially. Then, eventually, we get the required basis of L. □

Let L be an n-dimensional complex Leibniz algebra with the corresponding algebra sl_2. Then according to Theorem 3.5 L corresponds to a partition of $n-3$. Conversely, to a partition of the number $n-3$ we can assign a unique up to isomorphism precise n-dimensional Leibniz algebra L with the corresponding algebra sl_2. Evidently, this assignment is one to one correspondence between the isomorphism classes of n-dimensional complex Leibniz algebras with corresponding algebra sl_2 and the set of all partitions of $n-3$. If we denote the number of partitions of a number s by $f(s)$ then it is well known that $\lim_{s\to\infty} \frac{f(s)}{\frac{1}{4s\sqrt{3}}e^{A\sqrt{s}}} = 1$ [9], where $A = \pi\sqrt{\frac{2}{3}}$. Therefore, the number of non isomorphic n-dimensional complex Leibniz algebras with corresponding algebra sl_2 can be estimated approximately by $\frac{1}{4(n-3)\sqrt{3}}e^{\pi\sqrt{\frac{2(n-3)}{3}}}$.

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