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ON ISOMORPHISMS AND INVARIANTS OF FINITE DIMENSIONAL COMPLEX FILIFORM LEIBNIZ ALGEBRAS

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In this article, we propose an approach classifying a class of filiform Leibniz algebras. The approach is based on algebraic invariants. The method allows to classify all filiform Leibniz algebras (including filiform Lie algebras) in a given fixed dimensional case.

Key Words: Filiform Leibniz algebra; Isomorphism criterion; Lie algebra; Natural gradation.

2000 Mathematics Subject Classification: Primary 17A32, 17A60, 17B30, 17B70; Secondary 13A50.

1. INTRODUCTION

It is well known that the natural gradation of nilpotent Lie and Leibniz algebras is very helpful in investigation of their structural properties. This technique is more effective when the length of the natural gradation is sufficiently large. In the case when it is maximal, the algebra is called *filiform*. For applications of this technique, for instance, see Vergne [18], Goze et al. [10] (for Lie algebras) and Ayupov et al. [2] (for Leibniz algebras) case.

The present article deals with a class of nonassociative algebras that generalizes the class of Lie algebras. These algebras satisfy certain identities that were suggested by Loday [12] and Cuvier [5]. When one uses the tensor product instead of external product in the definition of the nth cochain, in order to prove the differential property, that is defined on cochains, it suffices to replace the anticommutativity and Jacobi identity by the Leibniz identity. This is one of the motivations to appear for this class of algebras. It turned out later that they appeared to be related in a natural way to several topics such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, quantum physics etc., as a generalization of the corresponding applications of Lie algebras to these topics.

The (co)homology theory, representations, and related problems of Leibniz algebras were studied by Cuvier [4], Loday et al. [14], Liu et al. [15], and others. A good survey about these all and related problems is Loday et al. [13].

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The problems related to the group theoretical realizations of Leibniz algebras are studied by Kinyon et al. [11] and others.

Deformation theory of Leibniz algebras and related physical applications of it are initiated by Fialowski et al. [7].

Problems concerning Cartan subalgebras, solvability, and weight spaces were studied by Albeverio et al. [1] and Omirov [16].

The notion of simple Leibniz algebra was suggested by Dzhumadil'daev et al. [6], who obtained some results concerning special cases of simple Leibniz algebras.

The article is organized as follows. Section 2 collects basic definitions, notations, and conventions used in the article. Section 3 is devoted to the adapted basis and the adapted transformations. Here we mention an isomorphism criterion from Gómez et al. [8] for filiform Leibniz algebras whose naturally gradation is non-Lie filiform Leibniz algebra. Then we rewrite it adjusting to our purpose. The main results of the article are in Sections 4 and 5. In Section 4 we give an algorithm for algebraic classification of finite dimensional complex filiform Leibniz algebras derived from the naturally graded non-Lie filiform Leibniz algebra in terms of invariant functions (Sections 4.1 and 4.2). Section 4.3 deals with the class of filiform Leibniz algebras whose natural gradation is a filiform Lie algebra. Here we simplify the table of multiplication and keep track of the behavior of the structure constants under the adapted base change. Section 5 contains implementations of the results in some low dimensional cases.

2. PRELIMINARIES

Let V be a vector space of dimension n over an algebraically closed field K (char $K=0$). Bilinear maps $V \times V \to V$ form a vector space $Hom(V \otimes V, V)$ of dimensional n^3, which can be considered together with its natural structure of an affine algebraic variety over K and denoted by $Alg_n(K) \cong K^{n^3}$. An n-dimensional algebra L over K can be considered as an element $\lambda(L)$ of $Alg_n(K)$ via the bilinear mapping $\lambda : L \otimes L \to L$ defining a binary algebraic operation on L: let $\{e_1, e_2, \ldots, e_n\}$ be a basis of the algebra L. Then the table of multiplication of L is represented by point (γ_{ij}^k) of this affine space as follows:

$$\lambda(e_i, e_j) = \sum_{k=1}^{n} \gamma_{ij}^k e_k.$$

Here γ_{ij}^k are called *structure constants* of L. The linear reductive group $GL_n(K)$ acts on $Alg_n(K)$ by $(g * \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y)))$ ("transport of structure"). Two algebras λ_1 and λ_2 are isomorphic if and only if they belong to the same orbit under this action.

Recall that an algebra L over a field F is called a *Leibniz algebra* if it satisfies the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \tag{1}$$

where $[\cdot, \cdot]$ denotes the multiplication in L.

A skew-symmetric Leibniz algebra is a Lie algebra. In this case, (1) is just the Jacobi identity.

Let $LB_n(K)$ be a subvariety of $Alg_n(K)$ consisting of all n-dimensional Leibniz algebras over K. It is stable under the above mentioned action of $GL_n(K)$. As a subset of $Alg_n(K)$ the set $LB_n(K)$ is specified by the system of equations with respect to structure constants γ_{ij}^k

$$\sum_{l=1}^{n}(\gamma_{jk}^l\gamma_{il}^m - \gamma_{ij}^l\gamma_{lk}^m + \gamma_{ik}^l\gamma_{lj}^m) = 0, \qquad \text{where } i, j, k, m = 1, 2, \ldots, n.$$

The first naive way to describe $LB_n(K)$ is to solve this quadratic system with respect to γ_{ij}^k, which is somewhat cumbersome. It has been done for low-dimensional Leibniz algebras ($n \leq 3$). The complexity of the computations increases much with increasing of dimension. Therefore, usually one has to create some appropriate methods of investigation. However, to classify whole $LB_n(K)$ for any fixed n is a hopeless task. Hence one considers some subclasses of $LB_n(K)$ to be classified.

Let L be a Leibniz algebra. We define the lower central series

$$L^1 = L, \qquad L^{k+1} = [L^k, L], \qquad k \geq 1.$$

Definition 2.1. A Leibniz algebra L is called nilpotent if there exists $s \in \mathbb{N}$ such that $L^s = 0$.

Definition 2.2. A Leibniz algebra L is said to be filiform if $dimL^i(L) = n - i$, where $n = dim\, L$ and $2 \leq i \leq n$.

Let Lb_n denote the class of all n-dimensional filiform Leibniz algebras. Clearly, it is a subclass of nilpotent Leibniz algebras. Let L be a nilpotent Leibniz algebra with nilindex s. Consider $L_i = L^i/L^{i+1}$, $1 \leq i \leq s-1$,, and $grL = L_1 \oplus L_2 \oplus \cdots L_{n-1}$. Then $[L_i, L_j] \subseteq L_{i+j}$, and we obtain the graded algebra grL.

Definition 2.3. If a Leibniz algebra L' is isomorphic to a filiform naturally graded algebra grL, then L' is said to be naturally graded filiform Leibniz algebra.

Later on all algebras are supposed to be over the field of complex numbers $\mathbb{C}$ and omitted products of basis vectors are supposed to be zero. The following theorem summarizes the results of Ayupov et al. [2] and Vergne [18].

Theorem 2.1. *Any complex $(n+1)$-dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non-isomorphic algebras*:

$$NGF_1 = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \end{cases}$$

$$NGF_2 = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \leq i \leq n-1, \end{cases}$$

$$NGF_3 = \begin{cases} [e_i, e_0] = -[e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = \alpha(-1)^{i+1}e_n, & 1 \leq i \leq n-1, \\ \alpha \in \{0, 1\} \text{ for odd } n \text{ and } \alpha = 0 \text{ for even } n. \end{cases}$$

It is clear that NGF_3 is a Lie algebra, but neither NGF_1 nor NGF_2 is the case.

Based on this theorem, the class of fixed dimensional filiform Leibniz algebras can be split into three disjoint classes as follows (see Gómez et al. [8]).

Theorem 2.2. *Any $(n+1)$-dimensional complex filiform Leibniz algebra admits a basis $\{e_0, e_1, \ldots, e_n\}$ called adapted, such that the table of multiplication of the algebra has one of the following forms*:

$$FLb_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1, \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + \cdots + \alpha_{n-1} e_{n-1} + \theta e_n, & \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + \cdots + \alpha_{n+1-j} e_n, & 1 \le j \le n-2, \\ \alpha_3, \alpha_4, \ldots, \alpha_n, \theta \in \mathbb{C}. & \end{cases}$$

$$SLb_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \le i \le n-1, \\ [e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + \cdots + \beta_n e_n, & \\ [e_1, e_1] = \gamma e_n, & \\ [e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + \cdots + \beta_{n+1-j} e_n, & 2 \le j \le n-2, \\ \beta_3, \beta_4, \ldots, \beta_n, \gamma \in \mathbb{C}. & \end{cases}$$

$$TLb_{n+1} = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le n-1, \\ [e_0, e_0] = b_{0,0} e_n, & \\ [e_0, e_1] = -e_2 + b_{0,1} e_n, & \\ [e_1, e_1] = b_{1,1} e_n, & \\ [e_i, e_j] = -[e_j, e_i] \in span_{\mathbb{C}}\{e_{i+j+1}, e_{i+j+2}, \ldots, e_n \mid 1 \le i \le n-3, & \\ \qquad\qquad\qquad\qquad\qquad\qquad 2 \le j \le n-1-i\}, & \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^i b_{i,n-i} e_n, & \\ \text{where } a^k_{i,j}, b_{i,j} \in \mathbb{C} \text{ and } b_{i,n-i} = b \text{ whenever } 1 \le i \le n-1, & \\ \quad \text{and } b = 0 \text{ for even } n. & \end{cases}$$

The above theorem means that the natural gradation of a filiform Leibniz algebra may be an algebra from one of NGF_i for $i = 1, 2, 3$.

3. ADAPTED BASE CHANGE AND ISOMORPHISM CRITERIA

In this section we simplify the isomorphic action of GL_n ("transport of structure") on the class of algebras coming out from the naturally graded non-Lie filiform Leibniz algebras. The details of the proofs can be found in Gómez et al. [8].

Let L be a filiform Leibniz algebra defined on a vector space V and $\{e_0, e_1, \ldots, e_n\}$ be an adapted basis of L.

The closed subgroup of $GL(V)$ consisting of all linear transformation sending one adapted basis to another is said to be *adapted* for the structure of L. This subgroup is denoted by G_{ad}. In G_{ad}, we consider the following isomorphisms, called elementary:

$$\text{first type } -\tau(a,b,k) = \begin{cases} f(e_0) = e_0 + ae_k, \\ f(e_1) = e_1 + be_k, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n-1, 2 \leq k \leq n, \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$$

$$\text{second type } -\vartheta(a,b) = \begin{cases} f(e_0) = ae_0 + be_1, \\ f(e_1) = (a+b)e_1 + b(\theta - \alpha_n)e_{n-1}, & a(a+b) \neq 0, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \leq i \leq n-1, \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$$

$$\text{third type } -\sigma(b,n) = \begin{cases} f(e_0) = e_0, \\ f(e_1) = e_1 + be_n, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 2 \leq i \leq n-1, \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$$

$$\text{fourth type } -\eta(a,k) = \begin{cases} f(e_0) = e_0 + ae_k, \\ f(e_1) = e_1, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 2 \leq i \leq n-1, 2 \leq k \leq n, \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$$

$$\text{fifth type } -\delta(a,b,d) = \begin{cases} f(e_0) = ae_0 + be_1, \\ f(e_1) = de_1 - \frac{bd\gamma}{a}e_{n-1}, & ad \neq 0, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 2 \leq i \leq n-1, \\ f(e_2) = [f(e_0), f(e_0)]. \end{cases}$$

where $a, b, d \in \mathbb{C}$.

Proposition 3.1.

(a) *Let f be an adapted transformation of FLb_{n+1}; then*

$$f = \tau(a_n, b_n, n) \circ \tau(a_{n-1}, a_{n-1}, n-1) \circ \cdots \circ \tau(a_2, a_2, 2) \circ \vartheta(a_0, a_1).$$

(b) *Let f be an adapted transformation of SLb_{n+1}; then*

$$f = \sigma(b_n, n) \circ \eta(a_n, n) \circ \eta(a_{n-1}, n-1) \circ \cdots \circ \eta(a_2, 2) \circ \delta(a_0, a_1, b_1).$$

Proposition 3.2.

(a) *The transformation*

$$\tau(a_n, b_n, n) \circ \tau(a_{n-1}, a_{n-1}, n-1) \circ \cdots \circ \tau(a_2, a_2, 2)$$

preserves the structure constants of algebras from FLb_{n+1}.

(b) *The transformation*

$$\sigma(b_n, n) \circ \eta(a_n, n) \circ \eta(a_{n-1}, n-1) \circ \cdots \circ \eta(a_2, 2)$$

preserves the structure constants of algebras from SLb_{n+1}.

Thus the action of $GL_{ad}(V)$ on FLb_{n+1} and SLb_{n+1} can be reduced to the action of the elementary transformations of the second and the fifth types, respectively.

The next two theorems are reformulations of the corresponding results of Gómez et al. [8] on isomorphism criteria for filiform Leibniz algebras appearing from the naturally graded non Lie filiform Leibniz algebras.

Introduce the following series of functions:

$$\begin{aligned}
&\varphi_t(y; z)\\
&= \varphi_t(y; z_3, z_4, \ldots, z_n, z_{n+1})\\
&= (1+y)z_t - \sum_{k=3}^{t-1}\left(\binom{k-1}{k-2} y z_{t+2-k} + \binom{k-1}{k-3} y^2 \sum_{i_1=k+2}^{t} z_{t+3-i_1}\, z_{i_1+1-k}\right.\\
&\quad + \binom{k-1}{k-4} y^3 \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{i_2} z_{t+3-i_2}\, z_{i_2+3-i_1}\, z_{i_1-k} + \ldots\\
&= + \binom{k-1}{1} y^{k-2} \sum_{i_{k-3}=2k-2}^{t} \sum_{i_{k-4}=2k-2}^{i_{k-3}} \cdots \sum_{i_1=2k-2}^{i_2} z_{t+3-i_{k-3}} z_{i_{k-3}+3-i_{k-4}} \cdots z_{i_2+3-i_1} z_{i_1+5-2k}\\
&\quad \left. + y^{k-1} \sum_{i_{k-2}=2k-1}^{t} \sum_{i_{k-3}=2k-1}^{i_{k-2}} \cdots \sum_{i_1=2k-1}^{i_2} z_{t+3-i_{k-2}} z_{i_{k-2}+3-i_{k-3}} \cdots z_{i_2+3-i_1} z_{i_1+4-2k}\right)\varphi_k(y; z),
\end{aligned}$$

for $3 \le t \le n$.

Theorem 3.1. *Two algebras* $L(\alpha)$ *and* $L(\alpha')$ *from* FLb_{n+1}, *where* $\alpha = (\alpha_3, \alpha_4, \ldots, \alpha_n, \theta)$, *and* $\alpha' = (\alpha'_3, \alpha'_4, \ldots, \alpha'_n, \theta')$ *are isomorphic if and only if there exist complex numbers A and B such that* $A(A + B) \neq 0$ *and the following conditions hold*:

$$\alpha'_t = \frac{1}{A^{t-2}} \varphi_t\left(\frac{B}{A}; \alpha\right), \qquad 3 \le t \le n,$$

$$\theta' = \frac{1}{A^{n-2}} \varphi_{n+1}\left(\frac{B}{A}; \alpha\right).$$

Let

$$\psi_t(y;z)$$
$$=\psi_t(y;z_3,z_4,\ldots,z_n,z_{n+1})$$
$$=z_t-\sum_{k=3}^{t-1}\left(\binom{k-1}{k-2}yz_{t+2-k}+\binom{k-1}{k-3}y^2\sum_{i_1=k+2}^{t}z_{t+3-i_1}z_{i_1+1-k}\right.$$
$$+C_{k-1}^{k-4}y^3\sum_{i_2=k+3}^{t}\sum_{i_1=k+3}^{i_2}z_{t+3-i_2}z_{i_2+3-i_1}z_{i_1-k}+\ldots$$
$$+C_{k-1}^{1}y^{k-2}\sum_{i_{k-3}=2k-2}^{t}\sum_{i_{k-4}=2k-2}^{i_{k-3}}\cdots\sum_{i_1=2k-2}^{i_2}z_{t+3-i_{k-3}}z_{i_{k-3}+3-i_{k-4}}\ldots z_{i_2+3-i_1}\;z_{i_1+5-2k}$$
$$\left.+y^{k-1}\sum_{i_{k-2}=2k-1}^{t}\sum_{i_{k-3}=2k-1}^{i_{k-2}}\cdots\sum_{i_1=2k-1}^{i_2}z_{t+3-i_{k-2}}\;z_{i_{k-2}+3-i_{k-3}}\ldots z_{i_2+3-i_1}\;z_{i_1+4-2k}\right)\psi_k(y;z),$$

where $3\le t\le n$,

and

$$\psi_{n+1}(y;z)=z_{n+1}.$$

Theorem 3.2. *Two algebras $L(\beta)$ and $L(\beta')$ from SLb_{n+1}, where $\beta=(\beta_3,\beta_4,\ldots,\beta_n,\gamma)$, and $\beta'=(\beta'_3,\beta'_4,\ldots,\beta'_n,\gamma')$, are isomorphic if and only if there exist complex numbers A, B, and D such that $AD\neq 0$ and the following conditions hold:*

$$\beta'_t=\frac{1}{A^{t-2}}\frac{D}{A}\psi_t\left(\frac{B}{A};\beta\right),\qquad 3\le t\le n-1,$$
$$\beta'_n=\frac{1}{A^{n-2}}\frac{D}{A}\frac{B}{A}\gamma+\psi_n\left(\frac{B}{A};\beta\right),$$

and

$$\gamma'=\frac{1}{A^{n-2}}\left(\frac{D}{A}\right)^2\psi_{n+1}\left(\frac{B}{A};\beta\right).$$

Now we commence to create the classification procedure for Lb_{n+1}.

To simplify notation let us agree that in the above case for transition from $L(\alpha)$ to $L(\alpha')$, and from $L(\beta)$ to $L(\beta')$ we write $\alpha'=\rho\left(\frac{1}{A},\frac{B}{A};\alpha\right)$ and $\beta'=\nu\left(\frac{1}{A},\frac{B}{A},\frac{D}{A};\beta\right)$, respectively, where

$$\rho\left(\frac{1}{A},\frac{B}{A};\alpha\right)=\left(\rho_1\left(\frac{1}{A},\frac{B}{A};\alpha\right),\rho_2\left(\frac{1}{A},\frac{B}{A};\alpha\right),\ldots,\rho_{n-1}\left(\frac{1}{A},\frac{B}{A};\alpha\right)\right),$$

with

$$\rho_t(x,y;z)=x^t\varphi_{t+2}(y;z)\qquad\text{for } 1\le t\le n-2,$$
$$\rho_{n-1}(x,y;z)=x^{n-2}\varphi_{n+1}(y,z),$$

and

$$v\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right) = \left(v_1\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right), v_2\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right), \ldots, v_{n-1}\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right)\right),$$

with

$$\begin{aligned} v_t(x, y, v; z) &= x^t v\psi_{t+2}(y; z) \qquad \text{for } 1 \leq t \leq n-3, \\ v_{n-2}(x, y, v; z) &= x^{n-2} v(yz_{n+1} + \psi_n(y; z)), \\ v_{n-1}(x, y, v; z) &= x^{n-2}\psi_{n+1}(y, z), \qquad \text{respectively.} \end{aligned}$$

Here are the main properties of the operators ρ and v used in this article:

1^0. $\rho(1, 0; \cdot)$ is the identity operator;
2^0. $\rho\left(\frac{1}{A_2}, \frac{B_2}{A_2}; \rho\left(\frac{1}{A_1}, \frac{B_1}{A_1}; \alpha\right)\right) = \rho\left(\frac{1}{A_1 A_2}, \frac{A_1 B_2 + A_2 B_1 + B_1 B_2}{A_1 A_2}; \alpha\right)$;
3^0. If $\alpha' = \rho\left(\frac{1}{A}, \frac{B}{A}; \alpha\right)$, then $\alpha = \rho\left(A, -\frac{B}{A+B}; \alpha'\right)$.

1^0. $v(1, 0, 1; \cdot)$ is the identity operator;
2^0. $v\left(\frac{1}{A_2}, \frac{B_2}{A_2}, \frac{D_2}{A_2}; v\left(\frac{1}{A_1}, \frac{B_1}{A_1}, \frac{D_1}{A_1}; \beta\right)\right) = v\left(\frac{1}{A_1 A_2}, \frac{B_1 A_2 + B_2 D_1}{A_1 A_2}, \frac{D_1 D_2}{A_1 A_2}; \beta\right)$;
3^0. If $\beta' = v\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta, \right)$, then $\beta = v\left(A, -\frac{B}{D}, \frac{A}{D}; \beta'\right)$.

From here on, we assume that $n \geq 4$ since there are complete classifications of complex nilpotent Leibniz algebras of dimension at most four in Albeverio et al. [3].

4. CLASSIFICATION PROCEDURE

Definition 4.1. An action of algebraic group G on a variety X is a morphism $\sigma : G \times X \longrightarrow X$ with:

(i) $\sigma(e, x) = x$, where e is an identity element of G and $x \in X$;
(ii) $\sigma(g, \sigma(h, x)) = \sigma(gh, x)$, for all $g, h \in G$ and $x \in X$.

One writes gx for $\sigma(g, x)$ and call X a G-variety. $O(x) = \{y \in X \mid \exists\, g \in G,\ y = gx\}$ is the orbit of x. A function $f : X \longrightarrow K$ is said to be invariant if $f(gx) = f(x)$ for all $g \in G$ and $x \in X$.

We consider the case when $G = G_{ad}$ and $X = Lb_{n+1}$. Then the orbits with respect to the action of $G = G_{ad}$ on $X = Lb_{n+1}$ consist of all isomorphic to each other algebras.

4.1. Classification Algorithm and Invariants for FLb_{n+1}

Consider the following representation of FLb_{n+1}: $FLb_{n+1} = U \cup F$, where

$$U = \{L(\alpha) \in FLb_{n+1} : \alpha_3 \neq 0\} \qquad \text{and} \qquad F = \{L(\alpha) \in FLb_{n+1} : \alpha_3 = 0\}.$$

Then U can be represented as a disjoint union of the subsets

$$U_1 = \{L(\alpha) \in U : \alpha_4 \neq -2\alpha_3^2\} \qquad \text{and} \qquad F_1 = \{L(\alpha) \in U : \alpha_4 = -2\alpha_3^2\}.$$

Theorem 4.1.

(i) *Two algebras* $L(\alpha)$ *and* $L(\alpha')$ *from* U_1 *are isomorphic if and only if*

$$\rho_i\left(\frac{2\alpha_3}{\alpha_4+2\alpha_3^2}, \frac{\alpha_4}{2\alpha_3^2}; \alpha\right) = \rho_i\left(\frac{2\alpha'_3}{\alpha'_4+2\alpha'^2_3}, \frac{\alpha'_4}{2\alpha'^2_3}; \alpha'\right),$$

whenever $i = 3, 4, \ldots, n-1$.

(ii) *For any* $(a_3, a_4, \ldots, a_{n-1}) \in \mathbb{C}^{n-3}$, *there is an algebra* $L(\alpha)$ *from* U_1 *such that*

$$\rho_i\left(\frac{2\alpha_3}{\alpha_4+2\alpha_3^2}, \frac{\alpha_4}{2\alpha_3^2}; \alpha\right) = a_i, \qquad i = 3, 4, \ldots, n-1.$$

Proof. (i). "If" part. Let two algebras $L(\alpha)$ and $L(\alpha')$ be isomorphic. Then there exist $A, B \in \mathbb{C}$ such that $A(A+B) \neq 0$ and $\alpha' = \rho\big(\frac{1}{A}, \frac{B}{A}; \alpha\big)$. Hence, $\alpha = \rho\big(A, \frac{-B}{A+B}; \alpha'\big)$. Consider the algebra $L(\alpha^0)$, where $\alpha^0 = \rho\big(\frac{1}{A_0}, \frac{B_0}{A_0}; \alpha\big)$ and $A_0 = \frac{\alpha_4+2\alpha_3^2}{2\alpha_3}$, $B_0 = \frac{\alpha_4(\alpha_4+2\alpha_3^2)}{4\alpha_3^3}$. Then $\alpha^0 = \rho\big(\frac{2\alpha_3}{\alpha_4+2\alpha_3^2}, \frac{\alpha_4}{2\alpha_3^2}; \alpha\big) = \rho\big(\frac{1}{A_0}, \frac{B_0}{A_0}; \rho\big(A, \frac{-B}{A+B}; \alpha'\big)\big) = \rho\big(\frac{A}{A_0}, \frac{B_0A-A_0B}{A(A+B)}; \alpha'\big)$. It is easy to check that $\frac{A}{A_0} = \frac{2\alpha'_3}{\alpha'_4+2\alpha'^2_3}$ and $\frac{B_0A-A_0B}{A(A+B)} = \frac{\alpha'_4}{2\alpha'^2_3}$.

Therefore,

$$\rho\left(\frac{2\alpha_3}{\alpha_4+2\alpha_3^2}, \frac{\alpha_4}{2\alpha_3^2}; \alpha\right) = \rho\left(\frac{2\alpha'_3}{\alpha'_4+2\alpha'^2_3}, \frac{\alpha'_4}{2\alpha'^2_3}; \alpha'\right)$$

and, hence,

$$\rho_i\left(\frac{2\alpha_3}{\alpha_4+2\alpha_3^2}, \frac{\alpha_4}{2\alpha_3^2}; \alpha\right) = \rho_i\left(\frac{2\alpha'_3}{\alpha'_4+2\alpha'^2_3}, \frac{\alpha'_4}{2\alpha'^2_3}; \alpha'\right),$$

for all $i = 3, 4, \ldots, n-1$.

This procedure can be shown schematically as follows:

$$\begin{array}{ccccc} \alpha & & \overset{(A_0,B_0)}{\longrightarrow} & & \alpha^0 \\ & (A,B)\searrow & & \nearrow\left(A_0A^{-1}, \frac{B_0A-A_0B}{A_0(A+B)}\right). & \\ & & \alpha' & & \end{array}$$

"Only if" part. Let the equalities

$$\rho_i\left(\frac{2\alpha_3}{\alpha_4+2\alpha_3^2}, \frac{\alpha_4}{2\alpha_3^2}; \alpha\right) = \rho_i\left(\frac{2\alpha'_3}{\alpha'_4+2\alpha'^2_3}, \frac{\alpha'_4}{2\alpha'^2_3}; \alpha'\right), \qquad i = 3, 4, \ldots, n-1$$

hold. Then it is easy to see that

$$\rho_i\left(\frac{2\alpha_3}{\alpha_4+2\alpha_3^2}, \frac{\alpha_4}{2\alpha_3^2}; \alpha\right) = \rho_i\left(\frac{2\alpha'_3}{\alpha'_4+2\alpha'^2_3}, \frac{\alpha'_4}{2\alpha'^2_3}; \alpha'\right) \qquad \text{for } i = 1, 2$$

as well and, therefore, $\rho\big(\frac{2\alpha_3}{\alpha_4+2\alpha_3^2}, \frac{\alpha_4}{2\alpha_3^2}; \alpha\big) = \rho\big(\frac{2\alpha'_3}{\alpha'_4+2\alpha'^2_3}, \frac{\alpha'_4}{2\alpha'^2_3}; \alpha'\big)$, which means that the algebras $L(\alpha)$ and $L(\alpha')$ are isomorphic.

(ii). The system of equations

$$\rho_i\left(\frac{2\alpha_3}{\alpha_4+2\alpha_3^2}, \frac{\alpha_4}{2\alpha_3^2}; \alpha\right) = a_i, \quad 3 \le i \le n-1; \tag{2}$$

where $(a_3, a_4, \ldots, a_{n-1})$ is given and $\alpha = (\alpha_3, \alpha_4, \ldots, \alpha_{n-1}, \theta)$ is unknown, has a solution as far as for any $3 \le i \le n-1$ in $\rho_i\left(\frac{2\alpha_3}{\alpha_4+2\alpha_3^2}, \frac{\alpha_4}{2\alpha_3^2}; \alpha\right)$ only variables $\alpha_3, \alpha_4, \ldots, \alpha_i$ occur and each of these equations is a linear equation with respect to the last variable occurred in it. Hence, making each of α_i the subject of (2), where $i = 3, \ldots, n-1$, one can find the required algebra $L(\alpha)$. □

Let us now consider the isomorphism criterion for F_1. This set in its turn can be written as a disjoint union of the subsets

$$V_1 = \{L(\alpha) \in F_1 : \alpha_5 \neq 5\alpha_3^3\} \quad \text{and} \quad G_1 = \{L(\alpha) \in F_1 : \alpha_5 = 5\alpha_3^3\},$$

and V_1 can be represented as a disjoint union of the subsets

$$U_2 = \{L(\alpha) \in V_1 : \alpha_6 + 6\alpha_3\alpha_5 - 16\alpha_3^4 \neq 0\} \quad \text{and}$$
$$G_2 = \{L(\alpha) \in V_1 : \alpha_6 + 6\alpha_3\alpha_5 - 16\alpha_3^4 = 0\}.$$

Then the isomorphism criterion for U_2 can be spelled out as follows.

Theorem 4.2.

(i) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_2 are isomorphic if and only if*

$$\rho_i\left(\frac{5\alpha_3^3-\alpha_5}{\alpha_6+6\alpha_3\alpha_5-16\alpha_3^4}, \frac{\alpha_6+7\alpha_3\alpha_5-21\alpha_3^4}{\alpha_3\left(5\alpha_3^3-\alpha_5\right)}; \alpha\right) = \rho_i\left(\frac{5\alpha_3'^3-\alpha_5'}{\alpha_6'+6\alpha_3'\alpha_5'-16\alpha_3'^4}, \frac{\alpha_6'+7\alpha_3'\alpha_5'-21\alpha_3'^4}{\alpha_3'\left(5\alpha_3'^3-\alpha_5'\right)}; \alpha'\right)$$

for $i = 4, \ldots, n-1$.

(ii) *For any $(a_4, \ldots, a_{n-1}) \in \mathbb{C}^{n-4}$, there is an algebra $L(\alpha)$ from U_2 such that*

$$\rho_i\left(\frac{5\alpha_3^3-\alpha_5}{\alpha_6+6\alpha_3\alpha_5-16\alpha_3^4}, \frac{\alpha_6+7\alpha_3\alpha_5-21\alpha_3^4}{\alpha_3\left(5\alpha_3^3-\alpha_5\right)}; \alpha\right) = a_i, \quad i = 4, 5, \ldots, n-1.$$

Proof can be carried out with minor changing in the proof of the Theorem 4.1. As for the subsets F, G_1, and G_2, the isomorphism criteria for them can be treated likewise.

4.2. Classification Algorithm and Invariants for SLb_{n+1}

In this section we consider SLb_{n+1}. The classification algorithm in this case works effectively as well. However, in this case, instead of the representation ρ we have to use the representation v (see Section 3). To illustrate it, we represent SLb_{n+1} as a union of two stable subsets:

$$SLb_{n+1} = U \cup F, \tag{3}$$

where

$$U = \{L(\beta) : (4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3)(4\beta_3\beta_5 - 5\beta_4^2) \neq 0\}$$

and

$$F = \{L(\beta) : (4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3)(4\beta_3\beta_5 - 5\beta_4^2) = 0\}.$$

Theorem 4.3.

(i) *Two algebras $L(\beta)$ and $L(\beta')$ from U are isomorphic if and only if*

$$v_i\left(\frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_5 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_4^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta\right) = v_i\left(\frac{4\beta_3'^2\beta_6' - 12\beta_3'\beta_4'\beta_5' + \beta_4'^3}{4\beta_3'\beta_5' - 5\beta_4'^2}, \frac{\beta_4'}{2\beta_3'^2}, \frac{1}{\beta_3'}; \beta'\right),$$

whenever $i = 3, 4, \ldots, n-1$.

(ii) *For any $(b_3, b_4, \ldots, b_{n-1}) \in \mathbb{C}^{n-3}$, there is an algebra $L(\beta)$ from U such that*

$$v_i\left(\frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_6 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_4^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta\right) = b_i, \qquad i = 3, 4, \ldots, n-1.$$

Proof. i). Let $L(\beta)$ and $L(\beta')$ be isomorphic. Then there exist $A, B, D \in \mathbb{C}$ such that $AD \neq 0$ and $\beta' = v\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right)$. Consider the algebra $L(\beta^0)$, where $\beta^0 = v\left(\frac{1}{A_0}, \frac{B_0}{A_0}, \frac{D_0}{A_0}; \beta\right)$, and

$$A_0 = \frac{4\beta_3\beta_5 - 5\beta_4^2}{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_5 + \beta_4^3}, \qquad B_0 = \frac{\beta_4(4\beta_3\beta_5 - 5\beta_4^2)}{2\beta_3^2(4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_5 + \beta_4^3)},$$
$$D_0 = \frac{4\beta_3\beta_5 - 5\beta_4^2}{\beta_3(4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_5 + \beta_4^3)}.$$

Since $\beta = v\left(A, \frac{-B}{D}, \frac{A}{D}; \beta'\right)$, then

$$\beta^0 = v\left(\frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_5 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_4^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta\right) = v\left(\frac{1}{A_0}, \frac{B_0}{A_0}, \frac{D_0}{A_0}; v\left(A, \frac{-B}{D}, \frac{A}{D}; \beta'\right)\right)$$
$$= v\left(\frac{A}{A_0}, \frac{B_0A - A_0B}{A_0D}, \frac{D_0A}{A_0D}; \alpha'\right).$$

One can easily check that

$$\frac{A}{A_0} = \frac{4{\beta'_3}^2\beta'_6 - 12\beta'_3\beta'_4\beta'_5 + \beta'^3_4}{4\beta'_3\beta'_5 - 5{\beta'_4}^2}, \qquad \frac{B_0A - A_0B}{A_0D} = \frac{\beta'_4}{2{\beta'_3}^2}, \qquad \text{and} \qquad \frac{D_0A}{A_0D} = \frac{1}{\beta'_3}.$$

Therefore,

$$v\left(\frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_5 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_4^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta\right) = v\left(\frac{4\beta'^2_3\beta'_6 - 12\beta'_3\beta'_4\beta'_5 + \beta'^3_4}{4\beta'_3\beta'_5 - 5\beta'^2_4}, \frac{\beta'_4}{2\beta'^2_3}, \frac{1}{\beta'_3}; \beta'\right)$$

and

$$v_i\left(\frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_5 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_4^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta\right) = v_i\left(\frac{4\beta'^2_3\beta'_6 - 12\beta'_3\beta'_4\beta'_5 + \beta'^3_4}{4\beta'_3\beta'_5 - 5\beta'^2_4}, \frac{\beta'_4}{2\beta'^2_3}, \frac{1}{\beta'_3}; \beta'\right), \tag{4}$$

for all $i = 3, 4, \ldots, n-1$.

This procedure can be shown schematically by the following picture:

$$\begin{array}{ccc} \beta & \overset{\left(\frac{1}{A_0},\frac{B_0}{A_0},\frac{D_0}{A_0}\right)}{\longrightarrow} & \beta^0 \\ \left(\frac{1}{A},\frac{B}{A},\frac{D}{A}\right)\searrow & & \nearrow\left(\frac{A}{A_0},\frac{AB_0-BA_0}{A_0D},\frac{D_0A}{A_0D}\right). \\ & \beta' & \end{array}$$

Conversely, let the equalities

$$v_i\left(\frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_5 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_4^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta\right) = v_i\left(\frac{4\beta'^2_3\beta'_6 - 12\beta'_3\beta'_4\beta'_5 + \beta'^3_4}{4\beta'_3\beta'_5 - 5\beta'^2_4}, \frac{\beta'_4}{2\beta'^2_3}, \frac{1}{\beta'_3}; \beta'\right)$$

hold for $i = 3, 4, \ldots, n-1$. Then it is easy to see that

$$v_i\left(\frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_5 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_4^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta\right) = v_i\left(\frac{4\beta'^2_3\beta'_6 - 12\beta'_3\beta'_4\beta'_5 + \beta'^3_4}{4\beta'_3\beta'_5 - 5\beta'^2_4}, \frac{\beta'_4}{2\beta'^2_3}, \frac{1}{\beta'_3}; \beta'\right)$$

for $i = 1, 2$ as well and, therefore,

$$v\left(\frac{4\beta_3^2\beta_6 - 12\beta_3\beta_4\beta_5 + \beta_4^3}{4\beta_3\beta_5 - 5\beta_4^2}, \frac{\beta_4}{2\beta_3^2}, \frac{1}{\beta_3}; \beta\right) = v\left(\frac{4\beta'^2_3\beta'_6 - 12\beta'_3\beta'_4\beta'_5 + \beta'^3_4}{4\beta'_3\beta'_5 - 5\beta'^2_4}, \frac{\beta'_4}{2\beta'^2_3}, \frac{1}{\beta'_3}; \beta'\right)$$

that means the algebras $L(\beta)$ and $L(\beta')$ are isomorphic.

Part ii) is similar to that of the Theorem 4.1. □

In regard to the set F, it can be split into subsets and the algorithm can be applied with v, instead of ρ, by using the properties of v.

4.3. Simplifications in TLb_{n+1}

In this section we treat filiform Leibniz algebras whose natural gradation is an algebra from NGF_3. This class has been denoted as TLb_{n+1}. Here we clarify the table of multiplications of algebras from TLb_{n+1} and investigate the behavior of structure constants under the base change. We recall that $(n+1)$-dimensional filiform Lie algebras are in TLb_{n+1}. A study of TLb_{n+1} was initiated in Omirov et al. [17].

Proposition 4.1.

$$TLb_{n+1} = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0, e_0] = b_{0,0}e_n, & \\ [e_0, e_1] = -e_2 + b_{0,1}e_n, & \\ [e_1, e_1] = b_{1,1}e_n, & \\ [e_i, e_j] = a^1_{i,j}e_{i+j+1} + \cdots + a^{n-(i+j+1)}_{i,j}e_{n-1} + b_{i,j}e_n, & 1 \leq i < j \leq n-2, \\ [e_i, e_j] = -[e_j, e_i], & 1 \leq i < j \leq n-1, \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^i b_{i,n-i}e_n, & 1 \leq i \leq n-1, \end{cases} \tag{5}$$

where $a^k_{i,j}, b_{i,j} \in \mathbb{C}$ and $b_{i,n-i} = b$, whenever $1 \leq i \leq n-1$, $b \in \{0, 1\}$ for odd n and $b = 0$ for even n.

Proof. Let $L \in TLb_{n+1}$ and $e_0, e_1, \ldots, e_n$ be a basis of L. Then due to Theorem 2.2 $[e_i, e_j] \in span\{e_{i+j+1}, \ldots, e_n\}$ for any $i, j \neq 0$.

Then

$$[e_i, e_0] = e_{i+1} + (*)e_{i+2} + \cdots + (*)e_n, \qquad 1 \leq i \leq n-1.$$

Putting $e'_1 = e_1, e'_0 = e_0, e'_{i+1} := [e'_i, e'_0]$, we may assume that $[e_i, e_0] = e_{i+1}, 1 \leq i \leq n-1$.

Now consider

$$[e_0, e_i] = -e_{i+1} + \alpha^{i+2}_{0,i}e_{i+2} + \alpha^{i+3}_{0,i}e_{i+3} + \cdots + \alpha^n_{0,i}e_n, \qquad 1 \leq i \leq n-1.$$

Then we get

$$[e_i, e_0] + [e_0, e_i] = \alpha^{i+2}_{0,i}e_{i+2} + \alpha^{i+3}_{0,i}e_{i+3} + \cdots + \alpha^n_{0,i}e_n, \qquad 1 \leq i \leq n-1. \tag{6}$$

Note that the Leibniz identity implies that $[x, y] + [y, x] \in \Re(L)$, for any $x, y \in L$, where $\Re(L)$ is the right annihilator of L. Therefore, if we multiply the both sides of (6) from the right-hand side $(n-i-2)$ times by e_0, we obtain $\alpha^{i+2}_{0,i} = 0$. Substituting and repeating it, we get

$$\alpha^{i+k}_{0,i} = 0, \qquad 2 \leq k \leq n-1-i.$$

Applying the above to $[e_i, e_i]$, $0 \leq i \leq [\frac{n}{2}]$, we get $[e_i, e_i] = \alpha^n_{i,i}e_n$.

The chain of equalities

$$\begin{aligned}[e_0, e_i] &= [e_0, [e_{i-1}, e_0]] = [[e_0, e_{i-1}], e_0] - [[e_0, e_0], e_{i-1}] \\ &= [-e_i + \alpha^n_{0,i-1}e_n, e_0] = -[e_i, e_0] = -e_{i+1}\end{aligned}$$

leads to $[e_i, e_0] = -[e_0, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$, i.e. $[e_0, x] = -[x, e_0]$ for any $x \in L^2$.

We claim that

$$[e_i, e_j] = -[e_j, e_i], \qquad 1 \leq i < j \leq n-1. \tag{7}$$

The induction by i for any j and the following chain of equalities:

$$\begin{aligned}
[e_i, e_{j+1}] &= [e_i, [e_j, e_0]] = [[e_i, e_j], e_0] - [[e_i, e_0], e_j] = (\text{since } [e_i, e_j] \in L^2) \\
&= -[e_0, [e_i, e_j]] + [[e_0, e_i] - \alpha_{0,i}^n e_n, e_j] = -[e_0, [e_i, e_j]] + [[e_0, e_i], e_j] \\
&= -[[e_0, e_i], e_j] + [[e_0, e_j], e_i] + [[e_0, e_i], e_j] = -[e_{j+1}, e_i], \qquad 1 \leq j \leq n-1
\end{aligned}$$

show (7).

The above observations lead to the required table of multiplication of $L \in TLb_{n+1}$. □

Let $L \in TLb_{n+1}$. The subspace spanned by $\{e_n\}$ is an ideal of L and the quotient algebra $L/\langle e_n \rangle$ is the n-dimensional filiform Lie algebra μ_n with the composition law

$$\begin{aligned}
&[e_i, e_0] = e_{i+1}, && i = 1, 2, \ldots, n-1, \\
&[e_i, e_j] = a_{i,j}^1 e_{i+j+1} + \cdots + a_{i,j}^{n-(i+j+1)} e_{n-1}, && 1 \leq i < j \leq n-1.
\end{aligned}$$

Moreover, e_n is in the center of L. Therefore, L can be considered as a Leibniz central extension of μ_n.

Lemma 4.1. *Let $L \in TLb_{n+1}$. Then*

$$\sum_{s=1}^{n-(i+j+k+1)} a_{j,k}^s b_{i,j+k+s} = \sum_{s=1}^{n-(i+j+k+1)} (a_{i,j}^s b_{i+j+s,k} - a_{i,k}^s b_{i+k+s,j}). \tag{8}$$

Proof. The Leibniz identity for e_i, e_j, and e_k gives the required relations between the structure constants. Indeed,

$$\begin{aligned}
[e_i, [e_j, e_k]] &= \left[e_i, \sum_{s=1}^{n-(j+k+1)} a_{j,k}^s e_{j+k+s} + b_{j,k} e_n \right] \\
&= \sum_{s=1}^{n-(i+j+k+1)} a_{j,k}^s \left(\sum_{t=1}^{n-(i+j+k+s+1)} a_{i,j+k+s}^t e_{i+j+k+s+t} + b_{i,j+k+s} e_n \right), \\
[[e_i, e_j], e_k] &= \left[\sum_{s=1}^{n-(i+j+1)} a_{i,j}^s e_{i+j+s} + b_{i,j} e_n, e_k \right] \\
&= \sum_{s=1}^{n-(i+j+k+1)} a_{i,j}^s \left(\sum_{t=1}^{n-(i+j+k+s+1)} a_{i+j+s,k}^t e_{i+j+k+s+t} + b_{i+j+s,k} e_n \right),
\end{aligned}$$

$$[[e_i, e_k], e_j] = \left[\sum_{s=1}^{n-(i+k+1)} a_{i,k}^s e_{i+k+s} + b_{i,k}e_n, e_j\right]$$
$$= \sum_{s=1}^{n-(i+j+k+1)} a_{i,k}^s \left(\sum_{t=1}^{n-(i+j+k+s+1)} a_{i+k+s,j}^t e_{i+j+k+s+t} + b_{i+k+s,j}e_n\right),$$

and then it implies that

$$\sum_{s=1}^{n-(i+j+k+1)} a_{j,k}^s b_{i,j+k+s} = \sum_{s=1}^{n-(i+j+k+1)} (a_{i,j}^s b_{i+j+s,k} - a_{i,k}^s b_{i+k+s,j}).$$ □

Here are several useful remarks regarding (8) that permit much simplify the multiplication table of TLb_{n+1}:

1. It is symmetric with respect to i, j, k (since $a_{s,t}^k = -a_{t,s}^k$ and $b_{s,t} = -b_{t,s}$ for any s and t, except for $(s, t) = (0, 0), (1, 1), (0, 1), (1, 0)$).
2. In the case when $(i, j, k) = (0, j, k)$, we get

$$\sum_{s=1}^{n-(j+k+1)} a_{j,k}^s b_{0,j+k+s} = \sum_{s=1}^{n-(j+k+1)} (a_{0,j}^s b_{j+s,k} - a_{0,k}^s b_{k+s,j}),$$

   where $j \neq 0, k \neq 0$.
3. Since $a_{0,t}^s = 0$ as $s \neq 1$ and $a_{0,t}^1 = -1$, we get

$$a_{j,k}^1 b_{0,j+k+1} + a_{j,k}^2 b_{0,j+k+2} + \cdots + a_{j,k}^{n-(j+k+1)} b_{0,n-1} = -b_{j+1,k} + b_{k+1,j}.$$

4. Since $b_{0,t} = 0$ as $t = 2, \ldots, n-2$ and $b_{0,n-1} = -1$, it implies that

$$a_{j,k}^{n-(j+k+1)} = b_{j+1,k} - b_{k+1,j},$$

   for $k = j+1, j+2, \ldots, n-j-2$ and $j = 1, 2, \ldots, [\frac{n-3}{2}]$.

Lemma 4.2. *Let $L \in TLb_{n+1}$. Then*

$$[e_i, e_{j+k}] = \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} [e_{i+k-s}, e_j] R_{e_0}^s, \tag{9}$$

where $1 \leq i, j, k \leq n$ and $yR_x = [y, x]$ is the right multiplication operator on L.

Proof. The proof will be proceed by the induction with respect to k. Let $k = 1$. Then $[e_i, e_{j+1}] = [e_i, [e_j, e_0]] = -[e_{i+1}, e_j] + [[e_i, e_j], e_0]$, i.e., (9) holds at $k = 1$. This is a base of the induction. Then the following chain of equalities lead to the claim:

$$[e_i, e_{j+k+1}] = [e_i, [e_{j+k}, e_0]] = [[e_i, e_{j+k}], e_0] - [[e_i, e_0], e_{j+k}]$$
$$= \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} [e_{i+k-s}, e_j] R_{e_0}^{s+1} - \sum_{s=0}^{k} (-1)^{k-s} \binom{k}{s} [e_{i+k+1-s}, e_j] R_{e_0}^s$$

$$= -\sum_{s=1}^{k+1}(-1)^{k-s}\binom{k}{s-1}[e_{i+k+1-s}, e_j]R^s_{e_0} - \sum_{s=0}^{k}(-1)^{k-s}\binom{k}{s}[e_{i+k+1-s}, e_j]R^s_{e_0}$$

$$= \sum_{s=1}^{k}(-1)^{k+1-s}\left(\binom{k}{s-1}+\binom{k}{s}\right)[e_{i+k+1-s}, e_j]R^s_{e_0}$$

$$+ [e_{i+k+1}, e_j]R^{k+1}_{e_0} - (-1)^k[e_{i+1+k}, e_k]$$

$$= \sum_{s=0}^{k+1}(-1)^{k+1-s}\binom{k+1}{s}[e_{i+k+1-s}, e_j]R^s_{e_0}.$$

□

The following proposition specifies elements of G_{ad} corresponding to the structure of $L \in TLb_{n+1}$.

Proposition 4.2. *Let $L \in TLb_{n+1}$ and f be an adapted transformation of L. Then f can be represented as follows*:

$$f(e_0) = e'_0 = \sum_{i=0}^{n} A_i e_i,$$

$$f(e_1) = e'_1 = \sum_{i=1}^{n} B_i e_i,$$

$$f(e_i) = e'_i = [f(e_{i-1}), f(e_0)], \qquad 2 \le i \le n,$$

$A_0, A_i, B_j,\ (i, j = 1, \ldots, n)$ are complex numbers, and $A_0 B_1(A_0 + A_1 b) \neq 0$.

Proof. Since a filiform Leibniz algebra is 2-generated, it is sufficient to consider the adapted action of f on the generators e_0, e_1:

$$f(e_0) = e'_0 = \sum_{i=0}^{n} A_i e_i, \qquad \text{and} \qquad f(e_1) = e'_1 = \sum_{i=0}^{n} B_i e_i.$$

Then $f(e_i) = [f(e_{i-1}), f(e_0)] = A_0^{i-2}(A_1B_0 - A_0B_1)e_i + \sum_{j=i+1}^{n}(*)e_j,\ 2 \le i \le n$. Note that $A_0 \neq 0,\ (A_1B_0 - A_0B_1) \neq 0$; otherwise, $f(e_n) = 0$. The condition $A_0 B_1(A_0 + A_1b) \neq 0$ appears naturally since f is not singular.

Let now consider $[f(e_1), f(e_2)] = B_0\,(A_1\,B_0 - A_0\,B_1)\,e_3 + \sum_{j=4}^{n}(*)e_j$. Then for the basis $\{f(e_0), f(e_1), \ldots, f(e_n)\}$ to be adapted, $B_0(A_1B_0 - A_0B_1)$ must not be zero. But according to the above observation, $(A_1B_0 - A_0B_1) \neq 0$. Therefore, $B_0 = 0$. □

The following elements of G_{ad} are said to be elementary with respect to the structure of $L \in TLb_{n+1}$:

$$\sigma(b,k) = \begin{cases} f(e_0) = e_0, & \\ f(e_1) = e_1 + b\,e_k, & b \in \mathbb{C}, \quad 2 \le k \le n, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \le i \le n-1, \end{cases}$$

$$\tau(a,k)=\begin{cases} f(e_0)=e_0+a\,e_k, & a\in\mathbb{C},\quad 1\le k\le n,\\ f(e_1)=e_1, & \\ f(e_{i+1})=[f(e_i),f(e_0)], & 1\le i\le n-1,\end{cases}$$

$$\upsilon(a,b)=\begin{cases} f(e_0)=a\,e_0, & \\ f(e_1)=b\,e_1, & a,b\in\mathbb{C}^*,\\ f(e_{i+1})=[f(e_i),f(e_0)], & 1\le i\le n-1.\end{cases}$$

Proposition 4.3. *Let f be an adapted transformation of L. Then it can be represented as a composition*:

$$f=\tau(A_n,n)\circ\tau(A_{n-1},n-1)\circ\cdots\circ\tau(A_2,2)\circ\sigma(B_n,n)\circ\sigma(B_{n-1},n-1)$$
$$\circ\cdots\circ\sigma(B_2,2)\circ\tau(A_1,1)\circ\upsilon(A_0,B_1).$$

Proof. The proof is straightforward. □

Proposition 4.4. *The transformations*

$$g=\tau(A_n,n)\circ\tau(A_{n-1},n-1)\circ\tau(A_{n-2},n-2)\circ\tau(A_{n-3},n-3)\circ\tau(A_{n-4},n-4)$$
$$\circ\sigma(B_n,n)\circ\sigma(B_{n-1},n-1)\circ\sigma(B_{n-2},n-2)\circ\sigma(B_{n-3},n-3),\qquad \textit{if } n \textit{ even},$$

and

$$g=\tau(A_n,n)\circ\tau(A_{n-1},n-1)\circ\tau(A_{n-2},n-2)\circ\tau(A_{n-3},n-3)$$
$$\circ\sigma(B_n,n)\circ\sigma(B_{n-1},n-1)\circ\sigma(B_{n-2},n-2),\qquad \textit{for odd } n$$

do not change the structure constants of algebras from TLb_{n+1}.

Remark. From the propositions above, one easily can see that the class TLb_{n+1} is less yieldable to simplification of adapted transformations than the first two classes. Nevertheless, the following lemma tracks out the behavior of some structure constants.

Lemma 4.3. *Let L and L' be filiform Leibniz algebras from TLb_{n+1} with the parameters $(b_{0,0},b_{0,1},b_{1,1},b_{1,2},\ldots,b_{i,j})$ and $(b'_{0,0},b'_{0,1},b'_{1,1},b'_{1,2},\ldots,b'_{i,j})$, respectively, where $1\le j\le n-1$. Suppose that L' is obtained from L by the adapted base change. Then*

$$b'_{0,0}=\frac{A_0^2b_{0,0}+A_0A_1b_{0,1}+A_1^2b_{1,1}}{A_0^{n-2}B_1(A_0+A_1b)}$$
$$b'_{0,1}=\frac{A_0b_{0,1}+2A_1b_{1,1}}{A_0^{n-2}(A_0+A_1b)},$$
$$b'_{1,1}=\frac{B_1b_{1,1}}{A_0^{n-2}(A_0+A_1b)}.$$

Proof. Consider the product $[f(e_0), f(e_0)] = b'_{0,0}f(e_n)$. Equating the coefficients of e_n in it, we get

$$A_0^2 b_{0,0} + A_0A_1b_{0,1} + A_1^2 b_{1,1} = b'_{0,0}A_0^{n-2}B_1(A_0 + A_1b).$$

Then $b'_{0,0} = \frac{A_0^2 b_{0,0}+A_0A_1b_{0,1}+A_1^2b_{1,1}}{A_0^{n-2}B_1(A_0+A_1b)}$.

The product $[f(e_1), f(e_1)] = b'_{1,1}f(e_n)$ yields

$$b'_{1,1} = \frac{B_1b_{1,1}}{A_0^{n-2}(A_0 + A_1b)}.$$

Consider the equality

$$b'_{0,1}f(e_n) = [f(e_1), f(e_0)] + [f(e_0), f(e_1)].$$

Then $b'_{0,1}A_0^{n-2}B_1(A_0 + A_1b) = A_0B_1b_{0,1} + 2A_1B_1b_{1,1}$, and it implies that

$$b'_{0,1} = \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0^{n-2}(A_0 + A_1b)}.$$

□

The complete implementation of the procedure for some low dimensional cases will be given in the next section.

5. APPLICATIONS

The objective of this section is to provide the isomorphism classes of complex filiform Leibniz algebras Lb_n for $n = 5, 6$.

For the computational purpose, we establish the following notations and conventions:

$\Delta_4 = \alpha_4 + 2\alpha_3^2$, $\Delta_5 = \alpha_5 - 5\alpha_3^3$, $\Theta_i = \theta - \alpha_i$, $i = 4, 5$, and the letters Δ_4, Δ_5, Θ_4, and Θ_5 with $'$ (prime) denote the same expression depending on parameters $\alpha'_3, \alpha'_4, \alpha'_5, \theta'$. Notice that $\Delta_i = \alpha_i$ $(i = 4, 5)$ as $\alpha_3 = 0$.

$\Lambda = 4\beta_3\beta_5 - 5\beta_4^2$ and $\Lambda' = 4\beta'_3\beta'_5 - 5\beta_4'^2$.

$\Delta = 4b_{0,0}b_{1,1} - b_{0,1}^2$ and $\Delta' = 4b'_{0,0}b'_{1,1} - b_{0,1}'^2$.

5.1. The Isomorphism Classes in FLb_5 and FLb_6

5.1.1. Dimension 5. The class FLb_5 can be represented as a disjoint union of the following subsets:

$$FLb_5 = U_1 \bigcup U_2 \bigcup U_3 \bigcup U_4 \bigcup U_5 \bigcup U_6 \bigcup U_7,$$

where

$$U_1 = \{L(\alpha) \in FLb_5 : \alpha_3 \neq 0, \Delta_4 \neq 0\},$$

$$U_2 = \{L(\alpha) \in FLb_5 : \alpha_3 \neq 0, \Delta_4 = 0, \Theta_4 \neq 0\},$$

$$U_3 = \{L(\alpha) \in FLb_5 : \alpha_3 \neq 0, \Delta_4 = 0, \Theta_4 = 0\},$$
$$U_4 = \{L(\alpha) \in FLb_5 : \alpha_3 = 0, \Delta_4 \neq 0, \Theta_4 \neq 0\},$$
$$U_5 = \{L(\alpha) \in FLb_5 : \alpha_3 = 0, \Delta_4 \neq 0, \Theta_4 = 0\},$$
$$U_6 = \{L(\alpha) \in FLb_5 : \alpha_3 = 0, \Delta_4 = 0, \Theta_4 \neq 0\},$$
$$U_7 = \{L(\alpha) \in FLb_5 : \alpha_3 = 0, \Delta_4 = 0, \Theta_4 = 0\}.$$

Now we consider the isomorphism problem for each of these sets separately.

Proposition 5.1. *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_1 are isomorphic if and only if*

$$\left(\frac{\alpha_3}{\Delta_4}\right)^2 \Theta_4 = \left(\frac{\alpha'_3}{\Delta'_4}\right)^2 \Theta'_4$$

The expression

$$\left(\frac{\alpha_3}{\Delta_4}\right)^2 \Theta_4$$

can be taken as a parameter λ, and orbits from the set U_1 can be parameterized as $L(1, 0, \lambda)$, $\lambda \in \mathbb{C}$.

Proposition 5.2. *The subsets U_2, U_3, U_4, U_5, U_6, and U_7 are single orbits with the representatives $L(1, -2, 0)$, $L(1, -2, -2)$, $L(0, 1, 0)$, $L(0, 1, 1)$, $L(0, 0, 1)$, and $L(0, 0, 0)$.*

We summarize these all in the following theorem.

Theorem 5.1. *Let L be a non-Lie complex filiform Leibniz algebra in FLb_5. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

1) $L(0, 0, 0)$:

$$L_5^f = \{e_0e_0 = e_2, e_ie_0 = e_{i+1}, 1 \leq i \leq 3\}.$$

2) $L(0, 0, 1)$:

$$L_5^f, \quad e_0e_1 = e_4.$$

3) $L(0, 1, 1)$:

$$L_5^f, \quad e_0e_1 = e_4, \quad e_1e_1 = e_4.$$

4) $L(0, 1, 0)$:

$$L_5^f, \quad e_1e_1 = e_4.$$

5) $L(1, -2, -2)$:

$$L_5^f, \quad e_0e_1 = e_3 - 2e_4, \quad e_1e_1 = e_3 - 2e_4, \quad e_2e_1 = e_4.$$

6) $L(1, -2, 0)$:

$$L_5^f, \quad e_0e_1 = e_3, \quad e_1e_1 = e_3 - 2e_4, \quad e_2e_1 = e_4.$$

7) $L(1, 0, \lambda)$:

$$L_5^f, \quad e_0e_1 = e_3 + \lambda e_4, \quad e_1e_1 = e_3, \quad e_2e_1 = e_4, \quad \lambda \in \mathbb{C}.$$

5.1.2. Dimension 6. The set FLb_6 can be represented as a disjoint union of its subsets as follows:

$$FLb_6 = U_1 \bigcup U_2 \bigcup U_3 \bigcup U_4 \bigcup U_5 \bigcup U_6 \bigcup U_7 \bigcup U_8 \bigcup U_9 \bigcup U_{10} \bigcup U_{11},$$

where

$$\begin{aligned}
U_1 &= \{L(\alpha) \in FLb_6 : \alpha_3 \neq 0, \Delta_4 \neq 0\},\\
U_2 &= \{L(\alpha) \in FLb_6 : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 \neq 0\},\\
U_3 &= \{L(\alpha) \in FLb_6 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 \neq 0\},\\
U_4 &= \{L(\alpha) \in FLb_6 : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 = 0\},\\
U_5 &= \{L(\alpha) \in FLb_6 : \alpha_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 \neq 0\},\\
U_6 &= \{L(\alpha) \in FLb_6 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Theta_5 \neq 0\},\\
U_7 &= \{L(\alpha) \in FLb_6 : \alpha_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Theta_5 = 0\},\\
U_8 &= \{L(\alpha) \in FLb_6 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 \neq 0\},\\
U_9 &= \{L(\alpha) \in FLb_6 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 = 0\},\\
U_{10} &= \{L(\alpha) \in FLb_6 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 \neq 0\},\\
U_{11} &= \{L(\alpha) \in FLb_6 : \alpha_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 = 0\}.
\end{aligned}$$

Proposition 5.3. *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_1 are isomorphic if and only if*

$$\frac{\alpha_3(\Delta_5 + 5\alpha_3\Delta_4)}{\Delta_4^2} = \frac{\alpha'_3(\Delta'_5 + 5\alpha'_3\Delta'_4)}{\Delta_4'^2}$$

and

$$\frac{\alpha_3^3\Theta_5}{\Delta_4^3} = \frac{\alpha_3'^3\Theta'_5}{\Delta_4'^3}.$$

The following two expressions can be taken as parameters λ_1, λ_2:

$$\frac{\alpha_3(\Delta_5 + 5\alpha_3\Delta_4)}{\Delta_4^2}, \qquad \frac{\alpha_3^3\Theta_5}{\Delta_4^3}$$

and orbits in U_1 are parameterized as

$$L(1, 0, \lambda_1, \lambda_2), \qquad \lambda_1, \lambda_2 \in \mathbb{C}.$$

Proposition 5.4. *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_2 are isomorphic if and only if*

$$\frac{\Delta_5^3}{\alpha_3^3\Theta_5^2} = \frac{\Delta_5'^3}{\alpha_3'^3\Theta_5'^2}.$$

In the set U_2, the expression

$$\frac{\Delta_5^3}{\alpha_3^3\Theta_5^2}$$

can be taken as a parameter and orbits from U_2 are parameterized as

$$L(1, -2, \lambda, 2\lambda - 5), \qquad \lambda \in \mathbb{C}.$$

Proposition 5.5. *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_3 are isomorphic if and only if*

$$\frac{\alpha_4^3\Theta_5}{\alpha_5^3} = \frac{\alpha_4'^3\Theta_5'}{\alpha_5'^3},$$

and the expression

$$\frac{\alpha_4^3\Theta_5}{\alpha_5^3}$$

can be taken as a parameter and orbits from the set U_3 can be represented as a union of orbits with representatives

$$L(0, 1, 1, \lambda), \qquad \lambda \in \mathbb{C}.$$

Proposition 5.6. *Subsets U_4, U_5, U_6, U_7, U_8, U_9, U_{10}, and U_{11} are single orbits with the representatives $L(1, -2, 0, 0)$, $L(1, -2, 5, 0)$, $L(0, 1, 0, 1)$, $L(0, 1, 0, 0)$, $L(0, 0, 1, 0)$, $L(0, 0, 1, 1)$, $L(0, 0, 0, 1)$, and $L(0, 0, 0, 0)$, respectively.*

Theorem 5.2. *Let L be a non-Lie complex filiform Leibniz algebra in FLb_6. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

1) $L(0, 0, 0, 0)$:

$$L_6^f = \{e_0e_0 = e_2, e_ie_0 = e_{i+1}, 1 \le i \le 4\}.$$

2) $L(0, 0, 0, 1)$:

$$L_6^f, \quad e_0e_1 = e_5.$$

3) $L(0, 0, 1, 1)$:

$$L_6^f, \quad e_0e_1 = e_5, \quad e_1e_1 = e_5.$$

4) $L(0, 0, 1, 0)$:

$$L_6^f, \quad e_1e_1 = e_5.$$

5) $L(0, 1, 0, 0)$:

$$L_6^f, \quad e_0e_1 = e_4, \quad e_1e_1 = e_4, \quad e_2e_1 = e_5.$$

6) $L(0, 1, 0, 1)$:

$$L_6^f, \quad e_0e_1 = e_4 + e_5, \quad e_1e_1 = e_4, \quad e_2e_1 = e_5.$$

7) $L(1, -2, 5, 0)$:

$$L_6^f, \quad e_0e_1 = e_3 - 2e_4, \quad e_1e_1 = e_3 - 2e_4 + 5e_5, \quad e_2e_1 = e_4 - 2e_5, \quad e_3e_1 = e_5.$$

8) $L(1, -2, 0, 0)$:

$$L_6^f, \quad e_0e_1 = e_3 - 2e_4, \quad e_1e_1 = e_3 - 2e_4, \quad e_2e_1 = e_4 - 2e_5, \quad e_3e_1 = e_5.$$

9) $L(0, 1, 1, \lambda)$:

$$L_6^f, e_0e_1 = e_4 + \lambda e_5, \quad e_1e_1 = e_4 + e_5, \quad e_2e_1 = e_5, \quad \lambda \in \mathbb{C}.$$

10) $L(1, -2, \lambda, 2\lambda - 5)$:

$$L_6^f, \quad e_0e_1 = e_3 - 2e_4 + (2\lambda - 5)e_5, \quad e_1e_1 = e_3 - 2e_4 + \lambda e_5, \quad e_2e_1 = e_4 - 2e_5, \quad e_3e_1 = e_5, \quad \lambda \in \mathbb{C}.$$

11) $L(1, 0, \lambda_1, \lambda_2)$:

$$L_6^f, e_0e_1 = e_3 + \lambda_1 e_5, \quad e_1e_1 = e_3 + \lambda_2 e_5, \quad e_2e_1 = e_4, e_3e_1 = e_5, \quad \lambda_1, \lambda_2 \in \mathbb{C}.$$

The Propositions 5.1, 5.3–5.5 are variation of the Theorem 4.1. Propositions 5.2 and 5.6 can be proven by precise indication of base change leading to the representatives.

5.2. The Isomorphism Classes in SLb_5 and SLb_6

5.2.1. Dimension 5. It is easy to see that there is the following representation of SLb_5 as a disjoint union of its subsets as follows:

$$SLb_5 = U_1 \bigcup U_2 \bigcup U_3 \bigcup U_4 \bigcup U_5,$$

where

$$\begin{aligned}
U_1 &= \{L(\beta) \in SLb_5 : \beta_3 \neq 0, \gamma - 2\beta_3^2 \neq 0\},\\
U_2 &= \{L(\beta) \in SLb_5 : \beta_3 \neq 0, \gamma - 2\beta_3^2 = 0, \beta_4 \neq 0\},\\
U_3 &= \{L(\beta) \in SLb_5 : \beta_3 = 0, \gamma \neq 0\},\\
U_4 &= \{L(\beta) \in SLb_5 : \beta_3 = 0, \gamma = 0, \beta_4 = 0\},\\
U_5 &= \{L(\beta) \in SLb_5 : \beta_3 = 0, \gamma = 0, \beta_4 \neq 0\}.
\end{aligned}$$

Proposition 5.7. *Two algebras $L(\beta)$ and $L(\beta')$ from U_1 are isomorphic if and only if*

$$\frac{\gamma}{\beta_3^2} = \frac{\gamma'}{\beta_3'^2}.$$

Then orbits from the set U_1 can be parameterized as $L(1, 0, \lambda)$ $\lambda \in \mathbb{C}$.

Proposition 5.8. *Subsets U_2, U_3, U_4, and U_5 are single orbits with the representatives $L(1, 1, 2)$, $L(0, 0, 1)$, $L(0, 1, 0)$, and $L(0, 0, 0)$, respectively.*

Theorem 5.3. *Let L be a non-Lie complex filiform Leibniz algebra in SLb_5. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

1) $L(0, 0, 0)$:

$$L_5^s = \{e_0e_0 = e_2, e_ie_0 = e_{i+1}, 2 \leq i \leq 3\}.$$

2) $L(0, 1, 0)$:

$$L_5^s, \quad e_0e_1 = e_4.$$

3) $L(0, 0, 1)$:

$$L_5^s, \quad e_1e_1 = e_4.$$

4) $L(1, 1, 2)$:

$$L_5^s, \quad e_0e_1 = e_3 + e_4, \quad e_1e_1 = 2e_4, \quad e_2e_1 = e_4.$$

5) $L(1, 0, \lambda)$:

$$L_5^s, \quad e_0e_1 = e_3, \quad e_1e_1 = \lambda e_4, \quad e_2e_1 = e_4, \quad \lambda \in \mathbb{C}.$$

5.2.2. Dimension 6. The set SLb_6 can be represented as a disjoint union of its subsets

$$SLb_6 = U_1 \bigcup U_2 \bigcup U_3 \bigcup U_4 \bigcup U_5 \bigcup U_6 \bigcup U_7 \bigcup U_8 \bigcup U_9,$$

where

$$\begin{aligned}
U_1 &= \{L(\beta) \in SLb_6 : \beta_3 \neq 0, \gamma \neq 0\},\\
U_2 &= \{L(\beta) \in SLb_6 : \beta_3 \neq 0, \gamma = 0, \Lambda \neq 0\},\\
U_3 &= \{L(\beta) \in SLb_6 : \beta_3 \neq 0, \gamma = 0, \Lambda = 0\},\\
U_4 &= \{L(\beta) \in SLb_6 : \beta_3 = 0, \beta_4 \neq 0, \gamma \neq 0\},\\
U_5 &= \{L(\beta) \in SLb_6 : \beta_3 = 0, \beta_4 \neq 0, \gamma = 0, \beta_5 \neq 0\},\\
U_6 &= \{L(\beta) \in SLb_6 : \beta_3 = 0, \beta_4 \neq 0, \gamma = 0, \beta_5 = 0\},\\
U_7 &= \{L(\beta) \in SLb_6 : \beta_3 = 0, \beta_4 = 0, \gamma \neq 0\},\\
U_8 &= \{L(\alpha) \in SLb_6 : \beta_3 = 0, \beta_4 = 0, \gamma = 0, \beta_5 \neq 0\},\\
U_9 &= \{L(\alpha) \in SLb_6 : \beta_3 = 0, \beta_4 = 0, \gamma = 0, \beta_5 = 0\}.
\end{aligned}$$

Proposition 5.9. *Two algebras $L(\beta)$ and $L(\beta')$ from U_1 are isomorphic if and only if*

$$\frac{2\beta_3\beta_4\gamma + \beta_3^2\Lambda}{\gamma^2} = \frac{2\beta'_3\beta'_4\gamma' + \beta_3'^2\Lambda'}{\gamma'^2}.$$

Orbits in U_1 can be parameterized as $L(1, 0, \lambda, 1)$, $\lambda \in \mathbb{C}$.

Proposition 5.10. *Algebras* $L(1, 0, 1, 0)$, $L(1, 0, 0, 0)$, $L(0, 1, 0, 1)$, $L(0, 1, 1, 0)$, $L(0, 1, 0, 0)$, $L(0, 0, 0, 1)$, $L(0, 0, 1, 0)$, *and* $L(0, 0, 0, 0)$ *are representatives of the single orbits* U_2, U_3, U_4, U_5, U_6, U_7, U_8, *and* U_9, *respectively.*

Theorem 5.4. *Let L be a non-Lie complex filiform Leibniz algebra in SLb_6. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

1) $L(0, 0, 0, 0)$:

$$L_6^s = \{e_0e_0 = e_2, e_ie_0 = e_{i+1}, 2 \leq i \leq 4\}.$$

2) $L(0, 0, 1, 0)$:

$$L_6^s, \quad e_0e_1 = e_5.$$

3) $L(0, 0, 0, 1)$:

$$L_6^s, \quad e_1e_1 = e_5.$$

4) $L(0, 1, 0, 0)$:

$$L_6^s, \quad e_0e_1 = e_4, \quad e_2e_1 = e_5.$$

5) $L(0, 1, 1, 0)$:

$$L_6^s, \quad e_0e_1 = e_4 + e_5, \quad e_2e_1 = e_5.$$

6) $L(0, 1, 0, 1)$:

$$L_6^s, \quad e_0e_1 = e_4, \quad e_1e_1 = e_5, \quad e_2e_1 = e_5.$$

7) $L(1, 0, 0, 0)$:

$$L_6^s, \quad e_0e_1 = e_3, \quad e_2e_1 = e_4, \quad e_3e_1 = e_5.$$

8) $L(1, 0, 1, 0)$:

$$L_6^s, \quad e_0e_1 = e_3 + e_5, \quad e_2e_1 = e_4, \quad e_3e_1 = e_5.$$

9) $L(1, 0, \lambda, 1)$:

$$L_6^s, \quad e_0e_1 = e_3 + \lambda e_5, \quad e_1e_1 = e_5, \quad e_2e_1 = e_4, \quad e_3e_1 = e_5, \quad \lambda \in \mathbb{C}.$$

The Propositions 5.7, 5.9 are variation of the Theorem 4.3. Propositions 5.8 and 5.10 can be proven by precise indication of base change leading to the representatives.

5.3. The Description of TLb_n, $n = 5, 6$

5.3.1. 5-Dimensional Case. By virtue of Proposition 4.1, we represent TLb_5 as follows:

$$TLb_5 = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \le i \le 3, \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le 3, \\ [e_0, e_0] = b_{0,0}e_4, & \\ [e_0, e_1] = -e_2 + b_{0,1}e_4, & \\ [e_1, e_1] = b_{1,1}e_4, & \\ [e_1, e_2] = -[e_2, e_1] = b_{1,2}e_4, & \\ b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2} \in \mathbb{C}. & \end{cases}$$

Here elements of TLb_5 will be denoted by $L(\alpha) = (b_{0,1}, b_{0,1}, b_{1,1}, b_{1,2})$.

Theorem 5.5 (Isomorphism Criterion for TLb_5). *Two algebras $L(\alpha)$ and $L(\alpha')$ from TLb_5 are isomorphic if and only if there exist complex numbers A_0, A_1, and B_1: $A_0 B_1 \neq 0$, and the following conditions hold*:

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_1 A_0 b_{0,1} + A_1^2 b_{1,1}}{A_0^3 B_1}, \tag{10}$$

$$b'_{0,1} = \frac{A_0 b_{0,1} + 2\, A_1 b_{1,1}}{A_0^3}, \tag{11}$$

$$b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^3}, \tag{12}$$

$$b'_{1,2} = \frac{B_1 b_{1,2}}{A_0^2}. \tag{13}$$

Proof. Part "If". Let L_1 and L_2 from TLb_5 be isomorphic: $f : L_1 \cong L_2$. We choose the corresponding adapted bases $\{e_0, e_1, e_2, e_3, e_4\}$ and $\{e'_0, e'_1, e'_2, e'_3, e'_4\}$ in L_1 and L_2, respectively. Then in these bases the algebras will be presented as $L(\alpha)$ and $L(\alpha')$.

According to Proposition 4.1 one has

$$\begin{aligned} e'_0 &= f(e_0) = A_0 e_0 + A_1 e_1 + A_2 e_2 + A_3 e_3 + A_4 e_4, \\ e'_1 &= f(e_1) = B_1 e_1 + B_2 e_2 + B_3 e_3 + B_4 e_4. \end{aligned} \tag{14}$$

Then we get

$$\begin{aligned} e'_2 = f(e_2) &= [f(e_1), f(e_0)] = A_0 B_1 e_2 + A_0 B_2 e_3 \\ &\quad + \left(A_0 B_3 + A_1 B_1 b_{1,1} + (A_2 B_1 - A_1 B_2) b_{1,2}\right) e_4, \\ e'_3 = f(e_3) &= [f(e_2), f(e_0)] = A_0^2 B_1 e_3 + (A_0^2 B_2 - A_0 A_1 B_1 b_{1,2}) e_4, \\ e'_4 = f(e_4) &= [f(e_3), f(e_0)] = A_0^3 B_1 e_4. \end{aligned}$$

By using the table of multiplications, one finds the relation between the coefficients $b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}$ and $b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}$. First consider the equality $[f(e_0), f(e_0)] = b'_{0,0} f(e_4)$, we get Eq. (1) and from the equality $[f(e_1), f(e_0)] + [f(e_0), f(e_1)] = b'_{0,1} f(e_4)$, we have (2) and $[f(e_1), f(e_1)] = b'_{1,1} f(e_4)$ gives (3). Finally the equality (4) comes out from $[f(e_1), f(e_2)] = b'_{1,2} f(e_4)$.

"Only if" part.

Let Eqs. (1)–(4) hold. Then the base change (5) above is adapted, and it transforms $L(\alpha)$ into $L(\alpha')$. Indeed,

$$\begin{aligned} [e'_0, e'_0] &= \left[\sum_{i=0}^{4} A_i e_i, \sum_{i=0}^{4} A_i e_i\right] \\ &= A_0^2 [e_0, e_0] + A_0 A_1 [e_0, e_1] + A_0 A_1 [e_1, e_0] + A_1^2 [e_1, e_1] \\ &= \left(A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}\right) e_4 = b'_{0,0} A_0^3 B_1 e_4 = b'_{0,0} e'_4. \end{aligned}$$

$$\begin{aligned}
[e'_0, e'_1] &= \left[\sum_{i=0}^{4} A_i e_i, \sum_{i=1}^{4} B_i e_i\right] \\
&= -(A_0B_1e_2 + A_0B_2e_3 + (A_1B_1b_{1,1} + A_2B_1b_{1,2} - A_1B_2b_{1,2} + A_0B_3)e_4) \\
&\quad + B_1\left(B_{0,1}A_0 + 2\,A_1B_{1,1}\right)e_4 \\
&= -e'_2 + A_0^3B_1b'_{0,1}e_4 = -e'_2 + b'_{0,1}e'_4.
\end{aligned}$$

Using the same manner, one can prove that $[e'_1, e'_1] = b'_{1,1}e'_4$ and $[e'_1, e'_2] = b'_{1,2}\, e'_n$. □

Now we list all isomorphism classes of algebras from TLb_5.
Represent TLb_5 as a disjoint union of the following subsets:

$$\begin{aligned}
U_1 &= \{L(\alpha) \in TLb_5 : b_{1,1} \neq 0, b_{1,2} \neq 0\}, \\
U_2 &= \{L(\alpha) \in TLb_5 : b_{1,1} \neq 0, b_{1,2} = 0, \Delta \neq 0\}, \\
U_3 &= \{L(\alpha) \in TLb_5 : b_{1,1} \neq 0, b_{1,2} = \Delta = 0\}, \\
U_4 &= \{L(\alpha) \in TLb_5 : b_{1,1} = 0, b_{0,1} \neq 0, b_{1,2} \neq 0\}, \\
U_5 &= \{L(\alpha) \in TLb_5 : b_{1,1} = 0, b_{0,1} \neq 0, b_{1,2} = 0\}, \\
U_6 &= \{L(\alpha) \in TLb_5 : b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,2} \neq 0\}, \\
U_7 &= \{L(\alpha) \in TLb_5 : b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,2} = 0\}, \\
U_8 &= \{L(\alpha) \in TLb_5 : b_{1,1} = b_{0,1} = b_{0,0} = 0, b_{1,2} \neq 0\}, \\
U_9 &= \{L(\alpha) \in TLb_5 : b_{1,1} = b_{0,1} = b_{0,0} = b_{1,2} = 0\}.
\end{aligned}$$

Proposition 5.11.

1. *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_1 are isomorphic if and only if $\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta$.*
2. *For any λ from $\mathbb{C}$, there exists $L(\alpha) \in U_1 : \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta = \lambda$.*

Proof. 1. "⇒". Let $L(\alpha)$ and $L(\alpha')$ be isomorphic. Then due to Theorem 3.1, it is easy to see that $\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta$.

"⇐". Let the equality $\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta$ hold. Consider the base change (5) above with $A_0 = \frac{b_{1,1}}{b_{1,2}}$, $A_1 = -\frac{b_{0,1}}{2b_{1,2}}$, and $B_1 = \frac{b_{1,1}^2}{b_{1,2}^3}$. This changing leads $L(\alpha)$ into $L\left(\left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta, 0, 1, 1\right)$. An analogous base change transforms $L(\alpha')$ into $L\left(\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta', 0, 1, 1\right)$.

Since $\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta$ then $L(\alpha)$ is isomorphic to $L(\alpha')$.

2. Obvious. □

Proposition 5.12. *The subsets* $U_2, U_3, U_4, U_5, U_6, U_7, U_8,$ *and* U_9 *are single orbits with the representatives* $L(1,0,1,0), L(0,0,1,0), L(0,1,0,1), L(0,1,0,0), L(1,0,0,1), L(1,0,0,0), L(0,0,0,1),$ *and* $L(0,0,0,0)$, *respectively.*

Theorem 5.6. *Let* $L \in TLb_5$. *Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

1) $L(0, \lambda, \lambda)$:

$$L_5^t = \{[e_i, e_0] = e_{i+1}, 1 \leq i \leq 3, [e_0, e_i] = -e_{i+1}, 2 \leq i \leq 3\},$$
$$[e_0, e_0] = e_4, \quad [e_0, e_1] = -e_2, \quad [e_1, e_1] = \lambda e_4, \quad [e_2, e_1] = -[e_1, e_2] = \lambda e_4, \quad \lambda \in \mathbb{C}.$$

2) $L(0, 1, 0)$:

$$L_5^t, \quad [e_0, e_0] = e_4, \quad [e_0, e_1] = -e_2, \quad [e_1, e_1] = e_4.$$

3) $L(1, 0, 1)$:

$$L_5^t, \quad [e_0, e_1] = -e_2 + e_4, \quad [e_0, e_0] = e_4, \quad [e_2, e_1] = [e_1, e_2] = e_4.$$

4) $L(1, 0, 0)$:

$$L_5^t, \quad [e_0, e_1] = -e_2 + e_4, \quad [e_0, e_0] = e_4.$$

5) $L(2, 1, 1)$:

$$L_5^t, \quad [e_0, e_0] = e_4, \quad [e_0, e_1] = -e_2 + 4e_4, \quad [e_1, e_1] = 2e_4, \quad [e_2, e_1] = -[e_1, e_2] = e_4.$$

6) $L(2, 1, 0)$:

$$L_5^t, \quad [e_0, e_0] = e_4, \quad [e_0, e_1] = -e_2 + 4e_4, \quad [e_1, e_1] = 2e_4.$$

7) $L(0, 0, 1)$:

$$L_5^t, \quad [e_0, e_0] = e_4, \quad [e_2, e_1] = -[e_1, e_2] = e_4, [e_0, e_1] = -e_2.$$

5.3.2. 6-Dimensional case. TLb_6 can be represented by the following table of multiplication:

$$TLb_6 = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \leq i \leq 4, \\ [e_0, e_i] = -e_{i+1}, & 2 \leq i \leq 4, \\ [e_0, e_0] = b_{0,0}e_5, [e_0, e_1] = -e_2 + b_{0,1}e_5, [e_1, e_1] = b_{1,1}e_5, & \\ [e_1, e_2] = -[e_2, e_1] = b_{1,2}e_4 + b_{1,3}e_5, & \\ [e_1, e_3] = -[e_3, e_1] = b_{1,2}e_5, & \\ [e_1, e_4] = -[e_4, e_1] = -[e_2, e_3] = [e_3, e_2] = -b_{2,3}e_n. & \end{cases}$$

Elements of TLb_6 will be denoted here by $L(\alpha)$, where $\alpha = (b_{0,1}, b_{1,1}, b_{1,2}, b_{1,3}, b_{2,3})$.

Theorem 5.7 (Isomorphism Criterion for TLb_6). *Two filiform Leibniz algebras $L(\alpha)$ and $L(\alpha')$ from TLb_6 are isomorphic iff there exist $A_0, A_1, B_1, B_2, B_3 \in \mathbb{C}$ such that $A_0B_1(A_0 + A_1\, b_{2,3}) \neq 0$, and the following equalities hold*:

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0A_1b_{0,1} + A_1^2 b_{1,1}}{A_0^3B_1(A_0 + A_1\, b_{2,3})},$$

$$b'_{0,1} = \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0^3(A_0 + A_1\, b_{2,3})},$$

$$b'_{1,1} = \frac{B_1b_{1,1}}{A_0^3(A_0 + A_1\, b_{2,3})},$$

$$b'_{1,2} = \frac{B_1\, b_{1,2}}{A_0^2},$$

$$b'_{1,3} = \frac{2\, A_0A_1B_1^2b_{1,2}^2 + A_0^2B_1^2b_{1,3} + \left(A_0^2\left(-2\, B_1B_3 + B_2^2\right) + A_1^2B_1^2b_{1,2}^2\right) b_{2,3}}{A_0^2B_1\left(A_0 + A_1b_{2,3}\right)},$$

$$b'_{2,3} = \frac{B_1\, b_{2,3}}{A_0 + A_1\, b_{2,3}}.$$

The proof is similar to that of Theorem 3.1.

Represent TLb_6 as a union of the following subsets:

$$U_1 = \{L(\alpha) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} \neq 0\},$$

$$U_2 = \{L(\alpha) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} = 0, b_{0,1} \neq 0\},$$

$$U_3 = \{L(\alpha) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} = b_{0,1} = 0, b_{1,2} \neq 0, b_{0,0} \neq 0\},$$

$$U_4 = \{L(\alpha) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} = b_{0,1} = 0, b_{1,2} \neq 0, b_{0,0} = 0\},$$

$$U_5 = \{L(\alpha) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} = b_{0,1} = b_{1,2} = 0, b_{0,0} \neq 0\},$$

$$U_6 = \{L(\alpha) \in TLb_6 : b_{2,3} \neq 0, b_{1,1} = b_{0,1} = b_{1,2} = b_{0,0} = 0\},$$

$$U_7 = \{L(\alpha) \in TLb_6 : b_{2,3} = 0, b_{1,2} \neq 0, b_{1,1} \neq 0\},$$

$$U_8 = \{L(\alpha) \in TLb_6 : b_{2,3} = 0, b_{1,2} \neq 0, b_{1,1} = 0, b_{0,1} \neq 0\},$$

$$U_9 = \{L(\alpha) \in TLb_6 : b_{2,3} = 0, b_{1,2} \neq 0, b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0\},$$

$$U_{10} = \{L(\alpha) \in TLb_6 : b_{2,3} = 0,\ b_{1,2} \neq 0, b_{1,1} = b_{0,1} = b_{0,0} = 0\},$$

$$U_{11} = \{L(\alpha) \in TLb_6 : b_{2,3} = b_{1,2} = 0, b_{1,1} \neq 0, b_{1,3} \neq 0\},$$

$$U_{12} = \{L(\alpha) \in TLb_6 : b_{2,3} = b_{1,2} = 0, b_{1,1} \neq 0, b_{1,3} = 0, \Delta \neq 0\},$$

$$U_{13} = \{L(\alpha) \in TLb_6 : b_{2,3} = b_{1,2} = 0, b_{1,1} \neq 0, b_{1,3} = \Delta = 0\},$$

$$U_{14} = \{L(\alpha) \in TLb_6 : b_{2,3} = b_{1,2} = b_{1,1} = 0, b_{0,1} \neq 0, b_{1,3} \neq 0\},$$

$$U_{15} = \{L(\alpha) \in TLb_6 : b_{2,3} = b_{1,2} = b_{1,1} = 0, b_{0,1} \neq 0, b_{1,3} = 0\},$$

$$U_{16} = \{L(\alpha) \in TLb_6 : b_{2,3} = b_{1,2} = b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,3} \neq 0\},$$
$$U_{17} = \{L(\alpha) \in TLb_6 : b_{2,3} = b_{1,2} = b_{1,1} = b_{0,1} = 0, b_{0,0} \neq 0, b_{1,3} = 0\},$$
$$U_{18} = \{L(\alpha) \in TLb_6 : b_{2,3} = b_{1,2} = b_{1,1} = b_{0,1} = b_{0,0} = 0, b_{1,3} \neq 0\},$$
$$U_{19} = \{L(\alpha) \in TLb_6 : b_{2,3} = b_{1,2} = b_{1,1} = b_{0,1} = b_{0,0} = b_{1,3} = 0\}.$$

Proposition 5.13.

1. *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_1 are isomorphic if and only if*

$$\left(\frac{b'_{2,3}}{2b'_{1,1} - b'_{0,1}b'_{2,3}}\right)^2 \Delta' = \left(\frac{b_{2,3}}{2\,b_{1,1} - b_{0,1}b_{2,3}}\right)^2 \Delta,$$

and

$$\frac{\left(2b'_{1,1} - b'_{2,3}b'_{0,1}\right)^3 b'^3_{1,2}}{b'^2_{2,3}b'^4_{1,1}} = \frac{\left(2b_{1,1} - b_{2,3}b_{0,1}\right)^3 b^3_{1,2}}{b^2_{2,3}b^4_{1,1}}.$$

2. *For any $\lambda_1, \lambda_2 \in \mathbb{C}$, there exists $L(\alpha) \in U_1$:*

$$\left(\frac{b_{2,3}}{2b_{1,1} - b_{0,1}b_{2,3}}\right)^2 \Delta = \lambda_1, \qquad \frac{\left(2b_{1,1} - b_{2,3}b_{0,1}\right)^3 b^3_{1,2}}{b^2_{2,3}b^4_{1,1}} = \lambda_2.$$

Then orbits from the set U_1 can be parameterized as $L(\lambda_1, 0, 1, \lambda_2, 0, 1)$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

Proposition 5.14.

1. *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_2 are isomorphic if and only if*

$$\frac{\left(b'_{0,1} - b'_{2,3}b'_{0,0}\right)^4 b'^3_{1,2}}{b'^3_{2,3}b'^5_{0,1}} = \frac{\left(b_{0,1} - b_{2,3}b_{0,0}\right)^4 b^3_{1,2}}{b^3_{2,3}b^5_{0,1}}.$$

2. *For any $\lambda \in \mathbb{C}$, there exists $L(\alpha) \in U_2$: $\frac{\left(b_{0,1}-b_{2,3}b_{0,0}\right)^4 b^3_{1,2}}{b^3_{2,3}b^5_{0,1}} = \lambda$.*

Therefore, orbits from U_2 can be parameterized as $L(0, 1, 0, \lambda, 0, 1)$, $\lambda \in \mathbb{C}$.

Proposition 5.15.

1. *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_7 are isomorphic if and only if*

$$\frac{4\,b'_{0,0}b'^4_{1,2} - 2\,b'_{1,3}b'_{0,1}b'^2_{1,2} + b'^2_{1,3}b'_{1,1}}{b'_{1,2}b'^2_{1,1}} = \frac{4\,b_{0,0}b^4_{1,2} - 2\,b_{1,3}b_{0,1}b^2_{1,2} + b^2_{1,3}b_{1,1}}{b_{1,2}b^2_{1,1}}$$

and

$$\frac{\left(b'_{0,1}b'^2_{1,2} - b'_{1,3}b'_{1,1}\right)^2}{b'_{1,2}b'^3_{1,1}} = \frac{\left(b_{0,1}b^2_{1,2} - b_{1,3}b_{1,1}\right)^2}{b_{1,2}b^3_{1,1}}.$$

2. *For any $\lambda_1, \lambda_2 \in \mathbb{C}$, there exists $L(\alpha) \in U_7$:*

$$\frac{4\, b_{0,0}b_{1,2}^4 - 2\, b_{1,3}b_{0,1}b_{1,2}^2 + b_{1,3}^2 b_{1,1}}{b_{1,2}b_{1,1}^2} = \lambda_1, \quad \frac{\left(b_{0,1}b_{1,2}^2 - b_{1,3}b_{1,1}\right)^2}{b_{1,2}b_{1,1}^3} = \lambda_2.$$

Orbits from U_7 can be parameterized as $L(\lambda_1, \lambda_2, 1, 1, 0, 0)$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

Proposition 5.16.

1. *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_8 are isomorphic if and only if*

$$\frac{\left(2\, b'_{0,0}b'^2_{1,2} - b'_{1,3}b'_{0,1}\right)^3}{b'^3_{1,2}b'^4_{0,1}} = \frac{\left(2\, b_{0,0}b_{1,2}^2 - b_{1,3}b_{0,1}\right)^3}{b_{1,2}^3 b_{0,1}^4}.$$

2. *For any $\lambda \in \mathbb{C}$, there exists $L(\alpha) \in U_8$:*

$$\frac{\left(2\, b_{0,0}b_{1,2}^2 - b_{1,3}b_{0,1}\right)^3}{b_{1,2}^3 b_{0,1}^4} = \lambda.$$

The orbits from the set U_8 can be parameterized as $L(\lambda, 1, 0, 1, 0, 0)$, $\lambda \in \mathbb{C}$.

Proposition 5.17.

1. *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_{11} are isomorphic if and only if*

$$\left(\frac{b'_{1,3}}{b'_{1,1}}\right)^6 \Delta' = \left(\frac{b_{1,3}}{b_{1,1}}\right)^6 \Delta.$$

2. *For any $\lambda \in \mathbb{C}$, there exists $L(\alpha) \in U_{11}$:*

$$\left(\frac{b_{1,3}}{b_{1,1}}\right)^6 \Delta = \lambda.$$

The orbits from U_{11} can be parameterized as $L(\lambda, 0, 1, 0, 1, 0)$, $\lambda \in \mathbb{C}$.

Proposition 5.18. *The subsets $U_3, U_4, U_5, U_6, U_9, U_{10}, U_{12}, U_{13}, U_{14}, U_{15}, U_{16}, U_{18}$, and U_{19} are single orbits with the representatives $L(1, 0, 0, 1, 0, 1)$, $L(0, 0, 0, 1, 0, 1)$, $L(1, 0, 0, 0, 0, 1)$, $L(0, 0, 0, 0, 0, 1)$, $L(1, 0, 0, 1, 0, 0)$, $L(0, 0, 0, 1, 0, 0)$, $L(1, 0, 1, 0, 0, 0)$, $L(0, 0, 1, 0, 0, 0)$, $L(0, 1, 0, 0, 1, 0)$, $L(0, 1, 0, 0, 0, 0)$, $L(1, 0, 0, 0, 1, 0)$, $L(1, 0, 0, 0, 0, 0)$, $L(0, 0, 0, 0, 1, 0)$, and $L(0, 0, 0, 0, 0, 0)$, respectively.*

Theorem 5.8. *Let L be a complex filiform Leibniz algebra in TLb_6. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

1) $L(0, \lambda_1, \lambda_2, \lambda_1)$:

$$L_6^t = \{[e_i, e_0] = e_{i+1}, 1 \le i \le 4, [e_0, e_i] = -e_{i+1}, 2 \le i \le 4\},$$

$$[e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2, \quad [e_1, e_1] = \lambda_1 e_5, \quad [e_2, e_1] = -[e_1, e_2] = \lambda_2 e_4 + \lambda_1 e_5, \quad [e_3, e_1] = -[e_1, e_3] = \lambda_2 e_5, \quad \lambda_1, \lambda_2 \in \mathbb{C}, \quad \lambda_1 \neq 0.$$

2) $L(0, \lambda, \lambda, 0)$:

$$L_6^t, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2, \quad [e_2, e_1] = -[e_1, e_2] = \lambda e_4, \quad [e_1, e_1] = [e_3, e_1] = -[e_1, e_3] = \lambda e_5, \lambda \in \mathbb{C}.$$

3) $L(0, 1, 0, 0)$:

$$L_6^t, \quad [e_0, e_0] = [e_1, e_1] = e_5, \quad [e_0, e_1] = -e_2.$$

4) $L(\lambda, 0, \lambda, 0)$:

$$L_6^t, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + \lambda e_5, \quad [e_2, e_1] = -[e_1, e_2] = \lambda e_4, \quad [e_3, e_1] = -[e_1, e_3] = \lambda e_5, \quad \lambda \in \mathbb{C}^*.$$

5) $L(1, 0, 1, -2)$:

$$L_6^t, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + e_5, \quad [e_2, e_1] = -[e_1, e_2] = e_4 - 2e_5, \quad [e_3, e_1] = -[e_1, e_3] = e_5.$$

6) $L(1, 0, 0, 1)$:

$$L_6^t, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + e_5, \quad [e_2, e_1] = -[e_1, e_2] = e_5.$$

7) $L(1, 0, 0, 0)$:

$$L_6^t, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + e_5.$$

8) $L(\lambda, \lambda^2, \lambda, 0)$:

$$L_6^t, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + \lambda e_5, \quad [e_1, e_1] = \lambda^2 e_5, \quad [e_2, e_1] = -[e_1, e_2] = \lambda e_4, \quad [e_3, e_1] = -[e_1, e_3] = \lambda e_5, \quad \lambda \in \mathbb{C}^*.$$

9) $L(4, 4, 0, 1)$:

$$L_6^t, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + 4e_5, \quad [e_1, e_1] = 4e_5, \quad [e_2, e_1] = -[e_1, e_2] = e_5.$$

10) $L(4, 4, 0, 0)$:

$$L_6^t, \quad [e_0, e_0] = e_5, \quad [e_0, e_1] = -e_2 + 4e_5, \quad [e_1, e_1] = 4e_5.$$

11) $L(0, 0, 1, 0)$:

$$L_6^t, \quad [e_0, e_0] = [e_3, e_1] = -[e_1, e_3] = e_5, \quad [e_0, e_1] = -e_2, [e_2, e_1] = -[e_1, e_2] = e_4.$$

12) $L(0, 0, 0, 1)$:

$$L_6^t, \quad [e_0, e_0] = [e_2, e_1] = -[e_1, e_2] = e_5, \quad [e_0, e_1] = -e_2.$$

As we have mentioned before the class TLb_n contains n-dimensional filiform Lie algebras. The sets U_8, U_9 in the 5-dimensional case and the sets $U_4, U_6, U_{10}, U_{18}, U_{19}$ in the 6-dimensional case represent Lie cases. Our classification here agreed with the classification of 5- and 6-dimensional filiform Lie algebras in Gómez et al. [9].

CONCLUSION

The methods and algorithms of this article are applicable to any fixed dimensional case. They have been implemented in dimensions at most 9, and complete lists of all isomorphism types of algebras from Lb_n ($n = 5, 6, 7, 8, 9$) are obtained.

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