On Isomorphism Criteria for Leibniz Central Extensions of a Linear Deformation of μ

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ON ONE-DIMENSIONAL LEIBNIZ CENTRAL EXTENSIONS OF A FILIFORM LIE ALGEBRA

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Abstract

The paper deals with the classification of Leibniz central extensions of a filiform Lie algebra. We choose a basis with respect to which the multiplication table has a simple form. In low-dimensional cases isomorphism classes of the central extensions are given. In the case of parametric families of orbits, invariant functions (orbit functions) are provided.

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1. Introduction

Leibniz algebras were introduced by Loday [4]. A Leibniz algebra is a generalization of a Lie algebra: a skew-symmetric Leibniz algebra is a Lie algebra. The initial motivation of Loday to introduce this class of algebras was the search for an 'obstruction' to the periodicity in algebraic K-theory. Besides this purely algebraic motivation, certain relationships with classical geometry, noncommutative geometry and physics have recently been discovered. Leibniz algebras appear to be related in a natural way to several topics such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, quantum physics and so on, as a generalization of the corresponding applications of Lie algebras to these topics.

In 1891, Umlauf [7] initiated the study of the simplest nontrivial class of Lie algebras. In his thesis he presented a list of Lie algebras of dimension less than ten admitting a so-called adapted basis. (Lie algebras with this property have been called filiform Lie algebras.) Now it is well known that up to isomorphism there is only one such an algebra; the others are just linear deformations of it [3]. This is the filiform Lie algebra with the composition law $[\cdot, \cdot]$ given by

$$\mu_n : [e_i, e_0] = e_{i+1}, \quad 1 \leq i \leq n-2,$$

with respect to the adapted basis $\{e_0, e_1, \ldots, e_{n-1}\}$.

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The outline of the paper is as follows. Section 2 is a brief introduction to Leibniz algebras and central extensions. Section 3 describes the behaviour of parameters under the isomorphism action (adapted base change). Sections 3.1–3.5 contain the main results of the paper, consisting of complete lists of all one-dimensional Leibniz central extensions of μ_n, where $n = 4, \ldots, 8$. We distinguish the isomorphism classes and show that they exhaust all possible cases. For parametric family cases the corresponding invariant functions are provided. Since proofs from the five-dimensional cases can be carried over to higher-dimensional cases by minor changes, we have chosen to omit their proofs. All details of the omitted proofs are available from the authors.

2. Preliminaries

Let K be an algebraically closed field of characteristic 0. A Leibniz algebra L over K is a vector space equipped with a bilinear map $[\cdot, \cdot] : L \times L \to L$ satisfying the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

for all $x, y, z \in L$.

If, in addition, $[x, x] = 0$, for all $x \in L$, the Leibniz identity is equivalent to the Jacobi identity. In particular, a Lie algebra is an example of a Leibniz algebra.

The centre of a Leibniz algebra L is defined as $C(L) = \{x \in L \mid [x, L] = [L, x] = 0\}$.

Let L and $\hat{L}$ be two Leibniz algebras over a field K. The Leibniz algebra $\hat{L}$ is said to be a one-dimensional central extension of L if there is a Leibniz algebra exact sequence $0 \longrightarrow Kc \longrightarrow \hat{L} \longrightarrow L \longrightarrow 0$, where Kc is one-dimensional trivial Leibniz algebra and the image of Kc is contained in the centre of $\hat{L}$.

A Leibniz 2-cocycle on a Leibniz algebra L is a K-valued form θ satisfying the condition

$$\theta(x, [y, z]) = \theta([x, y], z) - \theta([x, z], y) \quad \text{for all } x, y, z \in L.$$

If a Leibniz 2-cocycle θ is also antisymmetric, then by definition, θ is a Lie 2-cocycle. As in the Lie algebra case, one-dimensional Leibniz central extensions of a Leibniz algebra L are uniquely determined by a Leibniz 2-cocycle on L. If a Leibniz 2-cocycle θ is induced by a linear map $\nu : L \to L$ (that is, $\theta(x, y) = \nu([x, y])$), then θ is said to be trivial (or a coboundary), while the corresponding one-dimensional Leibniz central extension is also a trivial extension; that is, it is isomorphic to the direct sum of L and K. Two Leibniz 2-cocycles θ and ϑ define the same central extension if their difference $\theta - \vartheta$ is a coboundary.

Given a Leibniz 2-cocycle θ on L, one can construct a one-dimensional Leibniz central extension $L_\theta = L \oplus Kc$ of L in a canonical way as follows:

$$[x + vc, y + wc]_{L_\theta} = [x, y]_L + \theta(x, y)c, \quad x, y \in L, v, w \in K,$$

where $[\cdot, \cdot]_L$ is the bracket on L. Every one-dimensional Leibniz central extension of L can be obtained in this way. The following result is known.

PROPOSITION 2.1 [5]. *There exists a one-to-one correspondence between the set of equivalent classes of one-dimensional Leibniz central extensions of L by K and the second Leibniz cohomology group $HL^2(L, K)$.*

In this paper we focus on one-dimensional Leibniz central extensions of the filiform Lie algebra μ_n denoted here by $CE(\mu_n)$.

Let L be a Leibniz algebra. Define

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1.$$

DEFINITION 2.2. A Leibniz algebra L is *nilpotent* if there exists an integer $s \in N$, such that

$$L^1 \supset L^2 \supset \cdots \supset L^s = \{0\}.$$

DEFINITION 2.3. A Leibniz algebra L is *filiform* if $\dim L^i = n - i$, where $n = \dim L$ and $2 \leq i \leq n$.

Obviously, a filiform Leibniz algebra is nilpotent. Throughout the paper all algebras are assumed to be over the field of complex numbers $\mathbb{C}$.

3. Simplifications in $CE(\mu_n)$

In this section we consider a subclass of the class of Leibniz algebras called truncated filiform Leibniz algebras in [6], which provided the motivation to study this case. The multiplication tables of the truncated filiform Leibniz algebras in the class $CE(\mu_n)$ can be represented as follows:

$$\begin{array}{ll}
[e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\
[e_0, e_i] = -e_{i+1}, & 2 \leq i \leq n-1, \\
[e_0, e_0] = b_{0,0} e_n, & \\
[e_0, e_1] = -e_2 + b_{0,1} e_n, & \\
[e_1, e_1] = b_{1,1} e_n, & \\
[e_i, e_j] = (-1)^{i-1} b_{1,i+j-1} e_n, & 1 \leq i < j \leq n-1, i+j \text{ odd}, \\
[e_i, e_j] = -[e_j, e_i], & 1 \leq i < j \leq n-1, \\
[e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^i b e_n, & 1 \leq i \leq n-1, \text{ where } b = 0 \text{ for even } n.
\end{array}$$

The basis $\{e_0, e_1, \ldots, e_n\}$ is said to be *adapted*. Here are a few remarks about the multiplication table above.

(1) The undefined brackets are assumed to be zero.

(2) As a result of the Leibniz identity one has

$$b_{i+1,j} = -b_{i,j+1}, \quad 1 \leq i,\ j \leq n-1, i+j \neq n,$$

and

$$b_{1,2i+1} = 0, \quad 0 < i < \left[\frac{n-2}{2}\right].$$

As an immediate consequence of these relations we get

$$b_{i+2,i} = b_{i,i+2} = 0, \tag{3.1}$$

and

$$b_{i,j} = -b_{i-1,j+1} = b_{i-2,j+2} = \cdots = (-1)^{i-1} b_{i-(i-1),j+(i-1)} \\ = (-1)^{i-1} b_{1,j+i-1}. \tag{3.2}$$

(3) The centre of $L \in CE(\mu_n)$ is $\langle e_n \rangle$ and the quotient $L/\langle e_n \rangle$ is isomorphic to μ_n. Hence, one can choose a Leibniz cocycle θ on μ_n such that L is isomorphic to L_θ. (The procedure is the same as in the Lie algebras case (see [2]).)

Elements of $CE(\mu_n)$ represented by the table above are denoted by $L(\alpha)$, where

$$\alpha = \begin{cases} (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, \ldots, b_{1,n-2}) & \text{for even } n, \\ (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, \ldots, b_{1,n-3}, b) & \text{for odd } n. \end{cases}$$

Since a truncated filiform Leibniz algebra admits an adapted basis it is sufficient to consider only base changes sending adapted bases to adapted. This set is a subgroup G_{ad} of all base changes in $CE(\mu_n)$. A filiform Leibniz algebra is two-generated and so a base change on it can be given as:

$$f(e_0) = \sum_{i=0}^{n} A_i e_i, \quad f(e_1) = \sum_{i=0}^{n} B_i e_i.$$

Then one has the following proposition.

PROPOSITION 3.1. *Let $f \in G_{\text{ad}}$ and $L \in CE(\mu_n)$ then f has the following form:*

$$e'_0 = f(e_0) = \sum_{i=0}^{n} A_i e_i,$$
$$e'_i = f(e_i) = \sum_{k=i}^{n-1} A_0^{i-1} B_{k-i+1} e_i + (*) e_n, \quad 1 \le i \le n-1,$$
$$e'_n = f(e_n) = A_0^{n-2} B_1 (A_0 + A_1 b) e_n,$$

where $A_0 B_1 (A_0 + A_1 b) \neq 0$.

PROOF. It is easy to see that

$$e'_i = f(e_i) = [f(e_{i-1}), f(e_0)] \\ = \sum_{j=i}^{n-1} A_0^{i-2} (A_0 B_{j-i+1} - A_{j-i+1} B_0) e_j + (*) e_n, \quad 2 \le i \le n-1, \tag{3.3}$$

$$e'_n = f(e_n) = [f(e_{n-1}), f(e_0)] = A_0^{n-3} (A_0 B_1 - A_1 B_0)(A_0 + A_1 b) e_n. \tag{3.4}$$

Bearing in mind that f is adapted, we get

$$[f(e_1),\ f(e_2)] = b'_{1,2}e'_n = b'_{1,2}A_0^{n-3}(A_0B_1 - A_1B_0)(A_0 + A_1b)e_n$$
$$= B_0\sum_{i=3}^{n-1}(A_0B_{i-2} - A_{i-2}B_0)e_i + (*)e_n.$$

Therefore,

$$B_0(A_0B_{i-2} - A_{i-2}B_0) = 0, \quad 3 \le i \le n-1.$$

Since f is not singular, $A_0B_1 - A_1B_0 \neq 0$, and this implies that $B_0 = 0$. □

DEFINITION 3.2. The following types of adapted base change of $L \in CE(\mu_n)$ are said to be elementary:

$$\tau(a, b, c) = \begin{cases} \tau(e_0) = ae_0 + be_1, & \\ \tau(e_1) = ce_1, & ac \neq 0, \\ \tau(e_{i+1}) = [\tau(e_i),\ \tau(e_0)], & 1 \le i \le n-1, \end{cases}$$

$$\sigma(a, k) = \begin{cases} \sigma(e_0) = e_0 + ae_k, & 2 \le k \le n, \\ \sigma(e_1) = e_1, & \\ \sigma(e_{i+1}) = [\sigma(e_i),\ \sigma(e_0)], & 1 \le i \le n-1, \end{cases}$$

$$\phi(a, k) = \begin{cases} \phi(e_0) = e_0, & \\ \phi(e_1) = e_1 + ce_k, & 2 \le k \le n, \\ \phi(e_{i+1}) = [\phi(e_i),\ \phi(e_0)], & 1 \le i \le n-1, \end{cases}$$

where $a, b, c \in \mathbb{C}$.

PROPOSITION 3.3. *Let f be an adapted transformation of $L \in CE(\mu_n)$. Then it can be represented as the composition*

$$f = \phi(B_n, n) \circ \phi(B_{n-1}, n-1) \circ \cdots \circ \phi(B_2, 2) \circ \sigma(A_n, n) \circ \sigma(A_{n-1}, n-1) \circ \cdots$$
$$\circ\, \sigma(A_2, 2) \circ \tau(A_0, A_1, B_1).$$

PROOF. The proof is straightforward. □

PROPOSITION 3.4. *The adapted transformation*

$$g = \phi(B_n, n) \circ \phi(B_{n-1}, n-1) \circ \phi(B_{n-2}, n-2)$$
$$\circ\, \sigma(A_n, n) \circ \sigma(A_{n-1}, n-1) \circ \cdots \circ \sigma(A_2, 2),$$

for even n, and

$$g = \phi(B_n, n) \circ \phi(B_{n-1}, n-1) \circ \sigma(A_n, n) \circ \sigma(A_{n-1}, n-1) \circ \cdots \circ \sigma(A_2, 2),$$

for odd n, does not change the structure constants of $L \in CE(\mu_n)$.

PROOF. Consider the transformation

$$\sigma(A_k, k) = \begin{cases} \sigma(e_0) = e_0 + A_k e_k, & 2 \le k \le n, \\ \sigma(e_1) = e_1, & \\ \sigma(e_{i+1}) = [\sigma(e_i), \sigma(e_0)], & 1 \le i \le n-1. \end{cases}$$

First notice that

$$\sigma(e_i) = e_i + (-1)^i \delta(i, k) A_k b_{1,i+k-2} e_n, \quad 2 \le i \le n-1,$$

where

$$\delta(s, t) = \begin{cases} 1 & \text{if } s + t \text{ is even}, \\ -1 & \text{if } s = n-1, \\ 0 & \text{otherwise}. \end{cases}$$

Indeed,

$$\begin{aligned} [\sigma(e_i), \sigma(e_j)] &= [e_i + (-1)^i \delta(i, k) A_k b_{1,i+k-2} e_n, e_j + (-1)^j \delta(j, k) A_k b_{1,j+k-2} e_n] \\ &= [e_i, e_j]. \end{aligned}$$

The calculations for $\phi(B_n, n)$ and $\phi(B_{n-1}, n-1)$ are handled similarly. It is worth mentioning that for even n the transformation $\phi(B_{n-2}, n-2)$ does not change the structure constants, because in this case $b = 0$. □

The next lemma keeps track of the behaviour of the structure constants of algebras from $CE(\mu_n)$ under the adapted basis change.

LEMMA 3.5. *Let $L(\alpha) \in CE(\mu_n)$ and $L(\alpha')$ be the image of $L(\alpha)$ under the action of G_{ad}. Then one has*

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^{n-2} B_1 (A_0 + A_1 b)}, \quad b'_{0,1} = \frac{A_0 b_{0,1} + 2 A_1 b_{1,1}}{A_0^{n-2} (A_0 + A_1 b)},$$

$$b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^{n-2} (A_0 + A_1 b)},$$

$$b'_{1,2j} = \frac{1}{A_0^{n-2j-1} B_1 (A_0 + A_1 b)} \times \Bigg(\sum_{k=1}^{n-1} \sum_{\substack{l=2j \\ l+k \ne n}}^{n-k-1} (-1)^{k-1} B_k B_{l-2j+1} b_{1,k+l-1} + \sum_{k=1}^{n-2} (-1)^k B_k B_{n-k-2j+1} b \Bigg),$$

$$b' = \frac{B_1 b}{A_0 + A_1 b}.$$

PROOF. Consider the product $[f(e_0),\ f(e_0)] = b'_{0,0} f(e_n)$. Equating coefficients of e_n in it, we get

$$A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1} = b'_{0,0} A_0^{n-2} B_1 (A_0 + A_1 b).$$

Then we find

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^{n-2} B_1 (A_0 + A_1 b)}.$$

The product $[f(e_1),\ f(e_1)] = b'_{1,1} f(e_n)$ yields $b'_{1,1} = B_1 b_{1,1} / A_0^{n-2}(A_0 + A_1 b)$. Analysing the equality

$$b'_{0,1} f(e_n) = [f(e_1),\ f(e_0)] + [f(e_0),\ f(e_1)],$$

we obtain $b'_{0,1} A_0^{n-2} B_1 (A_0 + A_1 b) = A_0 B_1 b_{0,1} + 2 A_1 B_1 b_{1,1}$, and this implies that

$$b'_{0,1} = \frac{A_0 b_{0,1} + 2 A_1 b_{1,1}}{A_0^{n-2}(A_0 + A_1 b)}.$$

From Propositions 3.3 and 3.4,

$$\begin{aligned}
e'_0 &= A_0 e_0 + A_1 e_1, \\
e'_1 &= B_1 e_1 + B_2 e_2 + \cdots + B_{n-2} e_{n-2}, \\
e'_i &= \sum_{k=i}^{n-1} A_0^{i-1} B_{k-i+1} e_i + (*) e_n, \quad 2 \le i \le n-1, \\
e'_n &= A_0^{n-2} B_1 (A_0 + A_1 b) e_n.
\end{aligned}$$

Then

$$\begin{aligned}
[e'_i, e'_j] &= \left[\sum_{k=i}^{n-1} A_0^{i-1} B_{k-i+1} e_k + (*) e_n, \sum_{l=j}^{n-1} A_0^{j-1} B_{l-j+1} e_l + (*) e_n \right] \\
&= \left[\sum_{k=i}^{n-1} A_0^{i-1} B_{k-i+1} e_k, \sum_{l=j}^{n-1} A_0^{j-1} B_{l-j+1} e_l \right] \\
&= \sum_{k=i}^{n-1} \sum_{l=j}^{n-1} A_0^{i+j-2} B_{k-i+1} B_{l-j+1} [e_k, e_l] \\
&= \sum_{k=i}^{n-1} \sum_{l=j}^{n-k} A_0^{i+j-2} B_{k-i+1} B_{l-j+1} b_{k,l} e_n.
\end{aligned}$$

Hence the equality $b'_{i,j} e'_n = [e'_i, e'_j]$ gives the relation

$$b'_{i,j} A_0^{n-2} B_1 (A_0 + A_1 b) = A_0^{i+j-2} \sum_{k=i}^{n-1} \sum_{l=j}^{n-k} B_{k-i+1} B_{l-j+1} b_{k,l},$$

and implies that

$$b'_{i,j} = \frac{1}{A_0^{n-i-j} B_1(A_0 + A_1 b)} \sum_{k=i}^{n-1} \sum_{l=j}^{n-k} B_{k-i+1} B_{l-j+1} b_{k,l}.$$

Now use (3.2) to obtain

$$b'_{1,2j} = \frac{1}{A_0^{n-2j-1} B_1(A_0 + A_1 b)} \times \Bigg(\sum_{k=1}^{n-1} \sum_{\substack{l=2j \\ l+k\neq n}}^{n-k-1} (-1)^{k-1} B_k B_{l-2j+1} b_{1,k+l-1} + \sum_{k=1}^{n-2} (-1)^k B_k B_{n-k-2j+1} b \Bigg).$$

Finally, the last equality comes from $[e'_{n-1}, e'_1] = b' e'_n$. □

The next sections deal with the applications of the results of this section to the classification problem of $CE(\mu_n)$ for $n = 4, \ldots, 8$. We remind the reader that the classification of all complex nilpotent Leibniz algebras in dimensions at most four has been given before in [1].

Here, to classify algebras from the class $CE(\mu_n)$ in each fixed dimension we represent $CE(\mu_n)$ as a disjoint union of its subsets. Some of these subsets are single orbits and others contain infinitely many orbits. In the last case we give invariant functions to distinguish the orbits.

To simplify calculations we introduce the following notation:

$$\Delta = b_{0,1}^2 - 4 b_{0,0} b_{1,1} \quad \text{and} \quad \Delta' = b'^2_{0,1} - 4 b'_{0,0} b'_{1,1}.$$

3.1. Isomorphism classes in $CE(\mu_4)$. In this section we describe the isomorphism classes of algebras from $CE(\mu_4)$. According to the notation introduced above, elements of $CE(\mu_4)$ will be denoted by $L(\alpha)$, where $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2})$. Note that in this case $b = 0$ because n is even (see the multiplication table of $CE(\mu_n)$).

THEOREM 3.6 (Isomorphism criterion for $CE(\mu_4)$). *Two filiform Leibniz algebras $L(\alpha)$ and $L(\alpha')$ from $CE(\mu_4)$ are isomorphic if and only if there exist $A_0, A_1, B_1 \in \mathbb{C}$ such that $A_0 B_1 \neq 0$ and the following equalities hold:*

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^3 B_1}, \tag{3.5}$$

$$b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^3}, \tag{3.6}$$

$$b'_{0,1} = \frac{A_0 b_{0,1} + 2 A_1 b_{1,1}}{A_0^3}, \tag{3.7}$$

$$b'_{1,2} = \frac{B_1 b_{1,2}}{A_0^2}. \tag{3.8}$$

PROOF. If $L(\alpha)$ and $L(\alpha')$ are isomorphic, then the equalities are a consequence of Lemma 3.5.

Conversely, suppose that the equalities (3.5)–(3.8) hold. Then the following base change is adapted and it transforms $L(\alpha)$ to $L(\alpha')$:

$$\begin{aligned}
e'_0 &= A_0e_0 + A_1e_1,\\
e'_1 &= B_1e_1,\\
e'_2 &= A_0B_1e_2 + A_1B_1b_{1,1}e_4,\\
e'_3 &= A_0^2B_1e_3 - A_1A_0B_1b_{1,2}e_4,\\
e'_4 &= A_0^3B_1e_4.
\end{aligned}$$

Indeed,

$$\begin{aligned}
[e'_0, e'_0] &= A_0^2b_{0,0}e_4 + A_0A_1(-e_2 + b_{0,1}e_4) + A_1A_0e_2 + A_1^2b_{1,1}e_4\\
&= \frac{{A_0}^2b_{0,0} + A_0A_1b_{0,1} + A_1^2b_{1,1}}{A_0^3B_1}A_0^3B_1e_4 = b'_{0,0}e'_4.
\end{aligned}$$

Similarly, one easily can see that

$$\begin{aligned}
[e'_0, e'_1] &= -A_0B_1e_2 + A_0B_1b_{0,1}e_4 + A_1B_1b_{1,1}e_4\\
&= -A_0B_1e_2 - A_1B_1b_{1,1}e_4 + A_0B_1b_{0,1}e_4 + 2A_1B_1b_{1,1}e_4\\
&= -e'_2 + B_1(A_0b_{0,1} + 2A_1b_{1,1})e_4\\
&= -e'_2 + b'_{0,1}A_0^3B_1e_4 = -e'_2 + b'_{0,1}e'_4\\
[e'_1, e'_1] &= B_1^2b_{1,1}e_4 = A_0^3B_1b'_{1,1}e_4 = b'_{1,1}e_4,\\
[e'_1, e'_2] &= B_1^2A_0b_{1,2}e_4 = A_0^3B_1b'_{1,2}e_4 = b'_{1,2}e_4.
\end{aligned}$$

The other products of the basis vectors $e'_0, e'_1, \ldots, e'_n$ are zero. □

In order to describe the isomorphism classes of algebras from $CE(\mu_4)$ we represent it as a disjoint union of the following subsets:

$$\begin{aligned}
U_4^1 &= \{L(\alpha) \in CE(\mu_4) \mid b_{1,1} \neq 0,\ b_{1,2} \neq 0\};\\
U_4^2 &= \{L(\alpha) \in CE(\mu_4) \mid b_{1,1} \neq 0,\ b_{1,2} = 0,\ \Delta \neq 0\};\\
U_4^3 &= \{L(\alpha) \in CE(\mu_4) \mid b_{1,1} \neq 0,\ b_{1,2} = \Delta = 0\};\\
U_4^4 &= \{L(\alpha) \in CE(\mu_4) \mid b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,2} \neq 0\};\\
U_4^5 &= \{L(\alpha) \in CE(\mu_4) \mid b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,2} = 0\};\\
U_4^6 &= \{L(\alpha) \in CE(\mu_4) \mid b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,2} \neq 0\};\\
U_4^7 &= \{L(\alpha) \in CE(\mu_4) \mid b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,2} = 0\};\\
U_4^8 &= \{L(\alpha) \in CE(\mu_4) \mid b_{1,1} = b_{0,1} = b_{0,0} = 0,\ b_{1,2} \neq 0\};\\
U_4^9 &= \{L(\alpha) \in CE(\mu_4) \mid b_{1,1} = b_{0,1} = b_{0,0} = b_{1,2} = 0\}.
\end{aligned}$$

Here, the subset U_4^1 turns out to be a union of infinitely many orbits. The following proposition is a description of U_4^1.

PROPOSITION 3.7.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_4^1 are isomorphic if and only if*

$$\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^4 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^4 \Delta.$$

(2) *For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_4^1$ such that $(b_{1,2}/b_{1,1})^4\Delta = \lambda$.*

PROOF. (1) Let $L(\alpha)$ and $L(\alpha')$ be isomorphic. Then from Theorem 3.6 there are complex numbers A_0, A_1 and B_1 such that $A_0B_1 \neq 0$ and the action of the adapted group G_{ad} is expressed by the following system of equalities:

$$\begin{aligned}
b'_{0,0} &= \frac{A_0^2 b_{0,0} + A_0A_1b_{0,1} + A_1^2 b_{1,1}}{A_0^3 B_1},\\
b'_{1,1} &= \frac{B_1 b_{1,1}}{A_0^3},\\
b'_{0,1} &= \frac{A_0 b_{0,1} + 2A_1 b_{1,1}}{A_0^3},\\
b'_{1,2} &= \frac{B_1}{A_0^2} b_{1,2}.
\end{aligned}$$

Then one can easily see that $(b'_{1,2}/b'_{1,1})^4\Delta' = (b_{1,2}/b_{1,1})^4\Delta$.

Conversely, suppose that that the equality $(b'_{1,2}/b'_{1,1})^4\Delta' = (b_{1,2}/b_{1,1})^4\Delta$ holds.

Let $\{e_0, e_1, e_2, e_3, e_4\}$ and $\{e'_0, e'_1, e'_2, e'_3, e'_4\}$ be adapted bases of $L(\alpha)$ and $L(\alpha')$, respectively. Then the adapted base change

$$\begin{aligned}
f_0 &= A_0e_0 + A_1e_1,\\
f_1 &= B_1e_1,\\
f_2 &= A_0B_1e_2 + A_1B_1b_{1,1}e_4,\\
f_3 &= A_0^2B_1e_3 - A_1A_0B_1b_{1,2}e_4,\\
f_4 &= A_0^3B_1e_4,
\end{aligned}$$

where $A_0 = b_{1,1}/b_{1,2}$, $A_1 = -b_{0,1}/2b_{1,2}$ and $B_1 = b_{1,1}^2/b_{1,2}^3$, transforms $L(\alpha)$ to $L((b/_{1,2}b_{1,1})^4\Delta, 0, 1, 1)$.

An analogous base change

$$\begin{aligned}
f'_0 &= A'_0e'_0 + A'_1e'_1,\\
f'_1 &= B'_1e'_1,\\
f'_2 &= A'_0B'_1e'_2 + A'_1B'_1b'_{1,1}e'_4,
\end{aligned}$$

$$f_3' = A_0'^2 B_1' e_3' - A_1' A_0' B_1' b_{1,2}' e_4',$$
$$f_4' = A_0'^3 B_1' e_4',$$

where $A_0' = b_{1,1}'/b_{1,2}'$, $A_1' = -b_{0,1}'/2b_{1,2}'$ and $B_1' = b/_{1,1}'^2 b_{1,2}'^3$, transforms $L(\alpha')$ to $L((b_{1,2}'/b_{1,1}')^4\Delta', 0, 1, 1)$. However, by hypothesis, $(b_{1,2}/b_{1,1})^4\Delta = (b_{1,2}'/b_{1,1}')^4\Delta'$. Hence $L(\alpha)$ and $L(\alpha')$ are isomorphic.

(2) This is obvious. □

As a consequence of this proposition the orbits in U_4^1 can be parameterized as $L(\lambda, 0, 1, 1)$, $\lambda \in \mathbb{C}$.

The next proposition is a description of single orbits.

PROPOSITION 3.8. *The subsets U_4^2, U_4^3, U_4^4, U_4^5, U_4^6, U_4^7, U_4^8 and U_4^9 are single orbits with representatives*

$$L(1, 0, 1, 0), \quad L(0, 0, 1, 0), \quad L(0, 1, 0, 1), \quad L(0, 1, 0, 0),$$
$$L(1, 0, 0, 1), \quad L(1, 0, 0, 0), \quad L(0, 0, 0, 1) \quad \textit{and} \quad L(0, 0, 0, 0),$$

respectively.

PROOF. For each of the subsets above we give the corresponding base change leading to the indicated representative.

For U_4^2:

$$e_0' = A_0 e_0 + A_1 e_1, \quad e_1' = B_1 e_1, \quad e_2' = A_0 B_1 e_2 + A_1 B_1 b_{1,1} e_4,$$
$$e_3' = A_0^2 B_1 e_3, \quad e_4' = A_0^3 B_1 e_4,$$

where

$$A_0^4 = \frac{\Delta}{4}, \quad A_1^4 = \frac{b_{0,1}^4 \Delta}{64 b_{1,1}^4} \quad \text{and} \quad B_1^4 = \frac{\Delta^3}{64 b_{1,1}^4}.$$

For U_4^3:

$$e_0' = A_0 e_0 + A_1 e_1, \quad e_1' = B_1 e_1, \quad e_2' = A_0 B_1 e_2 + A_1 B_1 b_{1,1} e_4,$$
$$e_3' = A_0^2 B_1 e_3, \quad e_4' = A_0^3 B_1 e_4,$$

where

$$A_0 \in \mathbb{C}^*, \quad A_1 = \frac{-A_0 b_{0,1}}{2 b_{1,1}} \quad \text{and} \quad B_1 = \frac{A_0^3}{b_{1,1}}.$$

For U_4^4:

$$e_0' = A_0 e_0 + A_1 e_1, \quad e_1' = B_1 e_1, \quad e_2' = A_0 B_1 e_2,$$
$$e_3' = A_0^2 B_1 e_3 - A_0 A_1 B_1 b_{1,2} e_4, \quad e_4' = A_0^3 B_1 e_4,$$

where

$$A_0^2 = b_{0,1}, \quad A_1^2 = \frac{b_{0,0}^2}{b_{0,1}} \quad \text{and} \quad B_1 = \frac{b_{0,1}}{b_{1,2}}.$$

For U_4^5:

$$e'_0 = A_0e_0 + A_1e_1, \quad e'_1 = B_1e_1, \quad e'_2 = A_0B_1e_2, \quad e'_3 = A_0^2B_1e_3, \quad e'_4 = A_0^3B_1e_4,$$

where

$$A_0^2 = b_{0,1}, \quad A_1^2 = \frac{b_{0,0}^2}{b_{0,1}} \quad \text{and} \quad B_1 \in \mathbb{C}^*.$$

For U_4^6:

$$e'_0 = A_0e_0 + A_1e_1, \quad e'_1 = B_1e_1, \quad e'_2 = A_0B_1e_2,$$
$$e'_3 = A_0^2B_1e_3 - A_0A_1B_1b_{1,2}e_4, \quad e'_4 = A_0^3B_1e_4,$$

where

$$A_0^3 = b_{0,0}b_{1,2}, \quad A_1 \in \mathbb{C} \quad \text{and} \quad B_1^3 = \frac{b_{0,0}^3}{b_{0,0}b_{1,2}}.$$

For U_4^7:

$$e'_0 = A_0e_0 + A_1e_1, \quad e'_1 = B_1e_1, \quad e'_2 = A_0B_1e_2, \quad e'_3 = A_0^2B_1e_3, \quad e'_4 = A_0^3B_1e_4,$$

where

$$A_0 \in \mathbb{C}^*, \quad A_1 \in \mathbb{C} \quad \text{and} \quad B_1 = \frac{b_{0,0}}{A_0}.$$

For U_4^8:

$$e'_0 = A_0e_0 + A_1e_1, \quad e'_1 = B_1e_1, \quad e'_2 = A_0B_1e_2,$$
$$e'_3 = A_0^2B_1e_3 - A_0A_1B_1b_{1,2}e_4, \quad e'_4 = A_0^3B_1e_4,$$

where

$$A_0 \in \mathbb{C}^*, \quad A_1 \in \mathbb{C} \quad \text{and} \quad B_1 = \frac{A_0^2}{b_{1,2}}.$$

For U_4^9:

$$e'_0 = A_0e_0 + A_1e_1, \quad e'_1 = B_1e_1, \quad e'_2 = A_0B_1e_2, \quad e'_3 = A_0^2B_1e_3, \quad e'_4 = A_0^3B_1e_4,$$

where

$$A_0, B_1 \in \mathbb{C}^*, \quad A_1 \in \mathbb{C}.$$

This concludes the proof. □

3.2. Isomorphism classes in $CE(\mu_5)$. Notice that in this case $b = b_{2,3} = b_{1,4}$. Hence, an element of $CE(\mu_5)$ is $L(\alpha)$, with $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b)$.

THEOREM 3.9 (Isomorphism criterion for $CE(\mu_5)$). *Two filiform Leibniz algebras $L(\alpha)$ and $L(\alpha')$ from $CE(\mu_5)$ are isomorphic if and only if there exist $A_0, A_1, B_1, B_2, B_3 \in \mathbb{C}$ such that $A_0 B_1 (A_0 + A_1 b) \neq 0$, and the following equalities hold:*

$$\begin{aligned}
b'_{0,0} &= \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^3 B_1 (A_0 + A_1 b)},\\
b'_{0,1} &= \frac{A_0 b_{0,1} + 2A_1 b_{1,1}}{A_0^3 (A_0 + A_1 b)},\\
b'_{1,1} &= \frac{B_1 b_{1,1}}{A_0^3 (A_0 + A_1 b)},\\
b'_{1,2} &= \frac{B_1^2 b_{1,2} + (B_2^2 - 2B_1 B_3) b}{A_0^2 B_1 (A_0 + A_1 b)},\\
b' &= \frac{B_1 b}{A_0 + A_1 b}.
\end{aligned}$$

To find the isomorphism classes in $CE(\mu_5)$ we represent it as a union of the following subsets:

$$\begin{aligned}
U_5^1 &= \{L(\alpha) \in CE(\mu_5) \mid b \neq 0,\ b_{1,1} \neq 0\};\\
U_5^2 &= \{L(\alpha) \in CE(\mu_5) \mid b \neq 0,\ b_{1,1} = 0,\ b_{0,1} \neq 0\};\\
U_5^3 &= \{L(\alpha) \in CE(\mu_5) \mid b \neq 0,\ b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0\};\\
U_5^4 &= \{L(\alpha) \in CE(\mu_5) \mid b \neq 0,\ b_{1,1} = b_{0,1} = b_{0,0} = 0\};\\
U_5^5 &= \{L(\alpha) \in CE(\mu_5) \mid b = 0,\ b_{1,1} \neq 0,\ b_{1,2} \neq 0\};\\
U_5^6 &= \{L(\alpha) \in CE(\mu_5) \mid b = 0,\ b_{1,1} \neq 0,\ b_{1,2} = 0,\ \Delta \neq 0\};\\
U_5^7 &= \{L(\alpha) \in CE(\mu_5) \mid b = 0,\ b_{1,1} \neq 0,\ b_{1,2} = \Delta = 0\};\\
U_5^8 &= \{L(\alpha) \in CE(\mu_5) \mid b = b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,2} \neq 0\};\\
U_5^9 &= \{L(\alpha) \in CE(\mu_5) \mid b = b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,2} = 0\};\\
U_5^{10} &= \{L(\alpha) \in CE(\mu_5) \mid b = b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,2} \neq 0\};\\
U_5^{11} &= \{L(\alpha) \in CE(\mu_5) \mid b = b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,2} = 0\};\\
U_5^{12} &= \{L(\alpha) \in CE(\mu_5) \mid b = b_{1,1} = b_{0,1} = b_{0,0} = 0,\ b_{1,1} \neq 0\};\\
U_5^{13} &= \{L(\alpha) \in CE(\mu_5) \mid b = b_{1,1} = b_{0,1} = b_{0,0} = b_{1,1} = 0\}.
\end{aligned}$$

Here, the subsets U_5^1 and U_5^5 turn out to be a union of infinitely many orbits. The following propositions are descriptions of them.

PROPOSITION 3.10.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_5^1 are isomorphic if and only if*

$$\frac{\Delta' b'^2}{\left(b'_{0,1}b' - 2b'_{1,1}\right)^2} = \frac{\Delta b^2}{\left(b_{0,1}b - 2b_{1,1}\right)^2}.$$

(2) *For any λ from $\mathbb{C}$, there exists $L(\alpha) \in U_5^1$ such that $\Delta b^2/(b_{0,1}b - 2b_{1,1})^2 = \lambda$.*

The orbits in U_5^1 are parameterized as $L(\lambda, 0, 1, 0, 1)$, $\lambda \in \mathbb{C}$.

PROPOSITION 3.11.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_5^5 are isomorphic if and only if*

$$\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^6 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^6 \Delta.$$

(2) *For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_5^5$ such that $(b_{1,2}/4b_{1,1})^6 \Delta = \lambda$.*

The orbits in U_5^5 are parameterized as $L(\lambda, 0, 1, 1, 0)$, $\lambda \in \mathbb{C}$.

PROPOSITION 3.12. *The subsets U_5^2, U_5^3, U_5^4, U_5^6, U_5^7, U_5^8, U_5^9, U_5^{10}, U_5^{11}, U_5^{12}, and U_5^{13} are single orbits with representatives*

$$L(0,1,0,0,1),\quad L(1,0,0,0,1),\quad L(0,0,0,0,1),\quad L(1,0,1,0,0),$$
$$L(0,0,1,0,0),\quad L(0,1,0,1,0),\quad L(0,1,0,0,0),\quad L(1,0,0,1,0),$$
$$L(1,0,0,0,0),\quad L(0,0,0,1,0)\quad \textit{and}\quad L(0,0,0,0,0),$$

respectively.

3.3. Isomorphism classes in $CE(\mu_6)$. The section deals with the classification of $CE(\mu_6)$. It is easy to see here that $b_{2,3} = -b_{1,4}$.

THEOREM 3.13 (Isomorphism criterion for $CE(\mu_6)$). *Two filiform Leibniz algebras $L(\alpha)$ and $L(\alpha')$, $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4})$ and $\alpha' = (b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}, b'_{1,4})$, from $CE(\mu_6)$ are isomorphic if and only if there exist $A_0, A_1, B_1, B_2, B_3 \in \mathbb{C}$ such that $A_0 B_1 \neq 0$ and the following equalities hold:*

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^5 B_1},$$
$$b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^5},$$
$$b'_{0,1} = \frac{A_0 b_{0,1} + 2A_1 b_{1,1}}{A_0^5},$$

$$b'_{1,2} = \frac{1}{A_0^4 B_1}(B_1^2 b_{1,2} + (2B_1B_3 - B_2^2)b_{1,4}),$$
$$b'_{1,4} = \frac{B_1}{A_0^2} b_{1,4}.$$

The set $CE(\mu_6)$ is represented as a union of the following subsets

$U_6^1 = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} \neq 0,\ b_{1,4} \neq 0\};$

$U_6^2 = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} \neq 0,\ b_{1,4} = 0,\ b_{1,2} \neq 0\};$

$U_6^3 = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} \neq 0,\ b_{1,4} = b_{1,2} = 0,\ \Delta \neq 0\};$

$U_6^4 = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} \neq 0,\ b_{1,4} = b_{1,2} = \Delta = 0\};$

$U_6^5 = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,4} \neq 0\};$

$U_6^6 = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,4} = 0,\ b_{1,2} \neq 0\};$

$U_6^7 = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} = 0,\ b_{0,1} \neq 0,\ b_{1,4} = b_{1,2} = 0\};$

$U_6^8 = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,4} \neq 0\};$

$U_6^9 = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,4} = 0,\ b_{1,2} \neq 0\};$

$U_6^{10} = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0,\ b_{1,4} = b_{1,2} = 0\};$

$U_6^{11} = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} = b_{0,1} = b_{0,0} = 0,\ b_{1,4} \neq 0\};$

$U_6^{12} = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} = b_{0,1} = b_{0,0} = b_{1,4} = 0,\ b_{1,2} \neq 0\};$

$U_6^{13} = \{L(\alpha) \in CE(\mu_6) \mid b_{1,1} = b_{0,1} = b_{0,0} = b_{1,4} = b_{1,2} = 0\}.$

PROPOSITION 3.14.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_6^1 are isomorphic if and only if*

$$\left(\frac{b'_{1,4}}{b'_{1,1}}\right)^8 \Delta'^3 = \left(\frac{b_{1,4}}{b_{1,1}}\right)^8 \Delta^3.$$

(2) *For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_6^1$ such that $(b_{1,4}/b_{1,1})^8\Delta^3 = \lambda$.*

The orbits in U_6^1 can be parameterized as $L(\lambda, 0, 1, 0, 1)$, $\lambda \in \mathbb{C}$.

PROPOSITION 3.15.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_6^2 are isomorphic if and only if*

$$\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^8 \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^8 \Delta.$$

(2) *For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_6^2$ such that $(b_{1,2}/b_{1,1})^8\Delta = \lambda$.*

Then orbits in U_6^2 can be parameterized as $L(\lambda, 0, 1, 1, 0)$, $\lambda \in \mathbb{C}$.

PROPOSITION 3.16. *The subsets U_6^3, U_6^4, U_6^5, U_6^6, U_6^7, U_6^8, U_6^9, U_6^{10}, U_6^{11}, U_6^{12}, and U_6^{13} are single orbits with representatives*

$$L(1,0,1,0,0),\quad L(0,0,1,0,0),\quad L(0,1,0,0,1),\quad L(0,1,0,1,0),$$
$$L(0,1,0,0,0),\quad L(1,0,0,0,1),\quad L(1,0,0,1,0),\quad L(1,0,0,0,0),$$
$$L(0,0,0,0,1),\quad L(0,0,0,1,0)\quad \textit{and}\quad L(0,0,0,0,0),$$

respectively.

3.4. Isomorphism classes in $CE(\mu_7)$. From the interrelations between $b_{i,j}$ (see the multiplication table of $CE(\mu_n)$) we get $b_{2,5} = -b_{3,4} = b$, $b_{2,3} = -b_{1,4}$. Hence $L(\alpha)$, where $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b)$, designates an element of $CE(\mu_7)$.

THEOREM 3.17 (Isomorphism criterion for $CE(\mu_7)$). *Two filiform Leibniz algebras $L(\alpha)$ and $L(\alpha')$ from $CE(\mu_7)$ are isomorphic if and only if there exist A_0, A_1, $B_i \in \mathbb{C}$, $1 \le i \le 5$, such that $A_0 B_1(A_0 + A_1 b) \neq 0$, and the following equalities hold:*

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^5 B_1 (A_0 + A_1 b)},$$
$$b'_{0,1} = \frac{A_0 b_{0,1} + 2A_1 b_{1,1}}{A_0^5 (A_0 + A_1 b)},$$
$$b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^5 (A_0 + A_1 b)},$$
$$b'_{1,2} = \frac{B_1^2 b_{1,2} + (2B_1 B_3 - B_2^2) b_{1,4} + (2B_2 B_4 - 2B_1 B_5 - B_3^3) b}{2A_0^4 B_1 (A_0 + A_1 b)},$$
$$b'_{1,4} = -\frac{B_1 b_{1,4} + (2B_1 B_3 - B_2^2) b}{A_0^2 B_1 (A_0 + A_1 b)},$$
$$b' = \frac{Bb}{A_0 + A_1 b}.$$

The class $CE(\mu_7)$ is represented as a union of the following subsets.

$U_7^1 = \{L(\alpha) \in CE(\mu_7) \mid b \neq 0,\ b_{1,1} \neq 0\}$;

$U_7^2 = \{L(\alpha) \in CE(\mu_7) \mid b \neq 0,\ b_{1,1} = 0,\ b_{0,1} \neq 0\}$;

$U_7^3 = \{L(\alpha) \in CE(\mu_7) \mid b \neq 0,\ b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0\}$;

$U_7^4 = \{L(\alpha) \in CE(\mu_7) \mid b \neq 0,\ b_{1,1} = b_{0,1} = b_{0,0} = 0\}$;

$U_7^5 = \{L(\alpha) \in CE(\mu_7) \mid b = 0,\ b_{1,4} \neq 0,\ b_{1,1} \neq 0\}$;

$U_7^6 = \{L(\alpha) \in CE(\mu_7) \mid b = 0,\ b_{1,4} \neq 0,\ b_{1,1} = 0,\ b_{0,1} \neq 0\}$;

$U_7^7 = \{L(\alpha) \in CE(\mu_7) \mid b = 0,\ b_{1,4} \neq 0,\ b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0\}$;

$U_7^8 = \{L(\alpha) \in CE(\mu_7) \mid b = 0,\ b_{1,4} \neq 0,\ b_{1,1} = b_{0,1} = b_{0,0} = 0\}$;

$$U_7^9 = \{L(\alpha) \in CE(\mu_7) \mid b = b_{1,4} = 0,\ b_{1,2} \neq 0,\ b_{1,1} \neq 0\};$$
$$U_7^{10} = \{L(\alpha) \in CE(\mu_7) \mid b = b_{1,4} = 0,\ b_{1,2} \neq 0,\ b_{1,1} = 0,\ b_{0,1} \neq 0\};$$
$$U_7^{11} = \{L(\alpha) \in CE(\mu_7) \mid b = b_{1,4} = 0,\ b_{1,2} \neq 0,\ b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0\};$$
$$U_7^{12} = \{L(\alpha) \in CE(\mu_7) \mid b = b_{1,4} = 0,\ b_{1,2} \neq 0,\ b_{1,1} = b_{0,1} = b_{0,0} = 0\};$$
$$U_7^{13} = \{L(\alpha) \in CE(\mu_7) \mid b = b_{1,4} = b_{1,2} = 0,\ b_{1,1} \neq 0,\ \Delta \neq 0\};$$
$$U_7^{14} = \{L(\alpha) \in CE(\mu_7) \mid b = b_{1,4} = b_{1,2} = 0,\ b_{1,1} \neq 0,\ \Delta = 0\};$$
$$U_7^{15} = \{L(\alpha) \in CE(\mu_7) \mid b = b_{1,4} = b_{1,2} = b_{1,1} = 0,\ b_{0,1} \neq 0\};$$
$$U_7^{16} = \{L(\alpha) \in CE(\mu_7) \mid b = b_{1,4} = b_{1,2} = b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0\};$$
$$U_7^{17} = \{L(\alpha) \in CE(\mu_7) \mid b = b_{1,4} = b_{1,2} = b_{1,1} = b_{0,1} = b_{0,0} = 0\}.$$

PROPOSITION 3.18.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_7^1 are isomorphic if and only if*

$$\left(\frac{b'}{2b'_{1,1} - b'_{0,1}b'}\right)^2 \Delta' = \left(\frac{b}{2b_{1,1} - b_{0,1}b}\right)^2 \Delta.$$

(2) *For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_7^1$ such that $(b/(2b_{1,1} - b_{0,1}b))^2\Delta = \lambda$.*

Then orbits in U_7^1 can be parameterized as $L(\lambda, 0, 1, 0, 0, 1)$, $\lambda \in \mathbb{C}$.

PROPOSITION 3.19.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_7^5 are isomorphic if and only if*

$$\left(\frac{b'_{1,4}}{b'_{1,1}}\right)^{10} \Delta'^3 = \left(\frac{b_{1,4}}{b_{1,1}}\right)^{10} \Delta^3.$$

(2) *For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_7^5$ such that $(b_{1,4}/b_{1,1})^{10}\Delta^3 = \lambda$.*

Then orbits in U_7^5 can be parameterized as $L(\lambda, 0, 1, 0, 1, 0)$, $\lambda \in \mathbb{C}$.

PROPOSITION 3.20.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_7^9 are isomorphic if and only if*

$$\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^{10} \Delta'^3 = \left(\frac{b_{1,2}}{b_{1,1}}\right)^{10} \Delta^3.$$

(2) *For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_7^9$ such that $(b_{1,2}/b_{1,1})^{10}\Delta^3 = \lambda$.*

Then orbits in U_7^9 can be parameterized as $L(\lambda, 0, 1, 1, 0, 0)$, $\lambda \in \mathbb{C}$.

PROPOSITION 3.21. *The subsets U_7^2, U_7^3, U_7^4, U_7^6, U_7^7, U_7^8, U_7^{10}, U_7^{11}, U_7^{12}, U_7^{13}, U_7^{14}, U_7^{15}, U_7^{16}, and U_7^{17} are single orbits with representatives*

$$L(0,1,0,0,0,1),\quad L(1,0,0,0,0,1),\quad L(0,0,0,0,0,1),\quad L(0,1,0,0,1,0),$$
$$L(1,0,0,0,1,0),\quad L(0,0,0,0,1,0),\quad L(0,1,0,1,0,0),\quad L(1,0,0,1,0,0),$$
$$L(0,0,0,1,0,0),\quad L(1,0,1,0,0,0),\quad L(0,0,1,0,0,0),\quad L(0,1,0,0,0,0),$$
$$L(1,0,0,0,0,0)\quad \textit{and} \quad L(0,0,0,0,0,0),$$

respectively.

3.5. Isomorphism classes in $CE(\mu_8)$. It is easy to see that $b_{1,4} = -b_{3,2}$, $b_{1,6} = b_{3,4} = b_{5,2}$ and $b = 0$. An element of $CE(\mu_8)$ is denoted by $L(\alpha)$, where $\alpha = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b_{1,6})$.

THEOREM 3.22 (Isomorphism criterion for $CE(\mu_8)$). *Two filiform Leibniz algebras $L(\alpha)$ and $L(\alpha')$ from $CE(\mu_8)$ are isomorphic if and only if there exist A_0, A_1, $B_i \in \mathbb{C}$, $1 \le i \le 5$, such that $A_0B_1 \neq 0$ and the following equalities hold:*

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0A_1b_{0,1} + A_1^2 b_{1,1}}{A_0^7 B_1},$$
$$b'_{0,1} = \frac{2A_1b_{1,1} + A_0b_{0,1}}{A_0^7},$$
$$b'_{1,1} = \frac{B_1b_{1,1}}{A_0^7},$$
$$b'_{1,2} = \frac{B_1^2b_{1,2} + (2B_1B_3 - B_2^2)b_{1,4} + (2B_1B_5 - 2B_2B_4 + B_3^2)b_{1,6}}{A_0^6B_1},$$
$$b'_{1,4} = \frac{B_1^2b_{1,4} + (2B_1B_3 - B_2^2)b_{1,6}}{A_0^4B_1},$$
$$b'_{1,6} = \frac{B_1b_{1,6}}{A_0^2}.$$

The class $CE(\mu_8)$ is represented as a union of the following subsets.

$U_8^1 = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} \neq 0,\ b_{1,1} \neq 0\}$;

$U_8^2 = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} \neq 0,\ b_{1,1} = 0,\ b_{0,1} \neq 0\}$;

$U_8^3 = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} \neq 0,\ b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0\}$;

$U_8^4 = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} \neq 0,\ b_{1,1} = b_{0,1} = b_{0,0} = 0\}$;

$U_8^5 = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = 0,\ b_{1,4} \neq 0,\ b_{1,1} \neq 0\}$;

$U_8^6 = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = 0,\ b_{1,4} \neq 0,\ b_{1,1} = 0,\ b_{0,1} \neq 0\}$;

$U_8^7 = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = 0,\ b_{1,4} \neq 0,\ b_{1,1} = b_{0,1} = 0,\ b_{0,0} \neq 0\}$;

$$U_8^8 = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = 0,\, b_{1,4} \neq 0,\, b_{1,1} = b_{0,1} = b_{0,0} = 0\};$$
$$U_8^9 = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = b_{1,4} = 0,\, b_{1,2} \neq 0,\, b_{1,1} \neq 0\};$$
$$U_8^{10} = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = b_{1,4} = 0,\, b_{1,2} \neq 0,\, b_{1,1} = 0,\, b_{0,1} \neq 0\};$$
$$U_8^{11} = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = b_{1,4} = 0,\, b_{1,2} \neq 0,\; b_{1,1} = b_{0,1} = 0,\; b_{0,0} \neq 0\};$$
$$U_8^{12} = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = b_{1,4} = 0,\, b_{1,2} \neq 0,\, b_{1,1} = b_{0,1} = b_{0,0} = 0\};$$
$$U_8^{13} = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = b_{1,4} = b_{1,2} = 0,\, b_{1,1} \neq 0,\, \Delta \neq 0\};$$
$$U_8^{14} = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = b_{1,4} = b_{1,2} = 0,\, b_{1,1} \neq 0,\, \Delta = 0\};$$
$$U_8^{15} = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = b_{1,4} = b_{1,2} = b_{1,1} = 0,\; b_{0,1} \neq 0\};$$
$$U_8^{16} = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = b_{1,4} = b_{1,2} = b_{1,1} = b_{0,1} = 0,\, b_{0,0} \neq 0\};$$
$$U_8^{17} = \{L(\alpha) \in CE(\mu_8) \mid b_{1,6} = b_{1,4} = b_{1,2} = b_{1,1} = b_{0,1} = b_{0,0} = 0\}.$$

The following propositions describe U_8^1, U_8^5 and U_8^9.

PROPOSITION 3.23.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_8^1 are isomorphic if and only if*

$$\left(\frac{b'_{1,6}}{b'_{1,1}}\right)^{12} \Delta'^5 = \left(\frac{b_{1,6}}{b_{1,1}}\right)^{12} \Delta^5.$$

(2) *For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_8^1$ such that $(b_{1,6}/b_{1,1})^{12}\Delta^5 = \lambda$.*

Orbits in U_8^1 can be parameterized as $L(\lambda, 0, 1, 0, 0, 1)$, $\lambda \in \mathbb{C}$.

PROPOSITION 3.24.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_8^5 are isomorphic if and only if*

$$\left(\frac{b'_{1,1}}{b'_{1,4}}\right)^{4} \frac{1}{\Delta'} = \left(\frac{b_{1,1}}{b_{1,4}}\right)^{4} \frac{1}{\Delta}.$$

(2) *For any λ from $\mathbb{C}$ there exists $L(\alpha) \in U_8^5$ such that $(b_{1,1}/b_{1,4})^4(1/\Delta) = \lambda$.*

The parametrization of orbits in U_8^5 can be viewed as $L(\lambda, 0, 1, 0, 1, 0)$, $\lambda \in \mathbb{C}$.

PROPOSITION 3.25.

(1) *Two algebras $L(\alpha)$ and $L(\alpha')$ from U_8^9 are isomorphic if and only if*

$$\left(\frac{b'_{1,2}}{b'_{1,1}}\right)^{12} \Delta' = \left(\frac{b_{1,2}}{b_{1,1}}\right)^{12} \Delta.$$

(2) *For any $\lambda \in \mathbb{C}$ there exists $L(\alpha) \in U_8^9$ such that $(b_{1,2}/b_{1,1})^{12}\Delta = \lambda$.*

The set of orbits in U_8^9 can be parameterized as $L(\lambda, 0, 1, 1, 0, 0)$, $\lambda \in \mathbb{C}$.

PROPOSITION 3.26. *The subsets* U_8^2, U_8^3, U_8^4, U_8^6, U_8^7, U_8^8, U_8^{10}, U_8^{11}, U_8^{12}, U_8^{13}, U_8^{14}, U_8^{15}, U_8^{16}, *and* U_8^{17} *are single orbits with representatives*

$$L(0, 1, 0, 0, 0, 1),\quad L(1, 0, 0, 0, 0, 1),\quad L(0, 0, 0, 0, 0, 1),\quad L(0, 1, 0, 0, 1, 0),$$
$$L(1, 0, 0, 0, 1, 0),\quad L(0, 0, 0, 0, 1, 0),\quad L(0, 1, 0, 1, 0, 0),\quad L(1, 0, 0, 1, 0, 0),$$
$$L(0, 0, 0, 1, 0, 0),\quad L(1, 0, 1, 0, 0, 0),\quad L(0, 0, 1, 0, 0, 0),\quad L(0, 1, 0, 0, 0, 0),$$
$$L(1, 0, 0, 0, 0, 0)\quad \textit{and}\quad L(0, 0, 0, 0, 0, 0),$$

respectively.

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References

[1] S. Albeverio, B. A. Omirov and I. S. Rakhimov, 'Classification of four-dimensional nilpotent complex Leibniz algebras', *Extracta Math.* **21**(3) (2006), 197–210.

[2] J. R. Gomez, A. Jimenez-Merchan and Y. Khakimdjanov, 'Low-dimensional filiform Lie algebras', *J. Pure Appl. Algebra* **130** (1998), 133–158.

[3] You. B. Hakimjanov (Khakimdjanov), 'Variété des lois d'algèbres de Lie nilpotentes', *Geom. Dedicata* **40** (1991), 229–295.

[4] J.-L. Loday, 'Une version noncommutative des algèbres de Lie: les algèbres de Leibniz', *Enseign. Math.* **39** (1993), 269–293.

[5] J.-L. Loday and T. Pirashvili, 'Universal enveloping algebras of Leibniz algebras and (co)homology', *Math. Ann.* **296** (1993), 139–158.

[6] B. A. Omirov and I. S. Rakhimov, 'On Lie-like complex filiform Leibniz algebras', *Bull. Aust. Math. Soc.* **79** (2009), 391–404.

[7] K. A. Umlauf, *Über die Zusammensetzung der endlichen continuierlichen Transformationsgruppen, insbesondere der Gruppen vom Range Null*, Inaugural-Dissertation, Universität Leipzig (Von Breitkopf & Härtel, Leipzig, 1891).

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