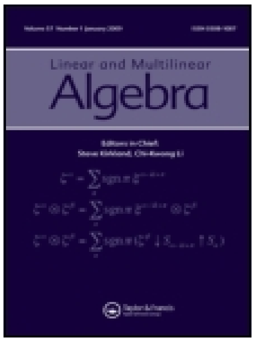

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Classification of a subclass of low-dimensional complex filiform Leibniz algebras

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We give a complete classification of a subclass of complex filiform Leibniz algebras obtained from naturally graded non-Lie filiform Leibniz algebras. The isomorphism criteria in terms of invariant functions are given.

Keywords: filiform Leibniz algebra; adapted basis; invariant; isomorphism

AMS Subject Classifications: 17A32; 17B30(primary); 13A50(secondary)

1. Introduction

Leibniz algebras were introduced by Loday [9]. A skew-symmetric Leibniz algebra is a Lie algebra. The main motivation of Loday to introduce this class of algebras was the search of an 'obstruction' to the periodicity in algebraic *K*-theory. Besides this purely algebraic motivation some relationships with classical geometry, non-commutative geometry and physics have been recently discovered.

The (co)homology theory, representations and related problems of Leibniz algebras were studied by Loday and Pirashvili [11], Frabetti [6] and others. A good survey of all these and related problems is [10].

The problems related to the group theoretical realizations of Leibniz algebras are studied by Kinyon and Weinstein [8] and others.

Deformation theory of Leibniz algebras and related physical applications of it is initiated by Fialowski et al. [5].

This article is devoted to the classification problem of filiform Leibniz algebras. The notion of filiform Leibniz algebra was introduced by Ayupov and Omirov [2]. According to Ayupov–Gomez–Omirov theorem, the class of all filiform Leibniz algebras is separated in four subclasses which are invariant with respect to the action of the linear group. One of these classes is the class of filiform Lie algebras. In this case there is a classification in small dimensions (Gomez–Khakimdjanov) and there is a classification of filiform Lie algebras admitting a non-trivial Malcev Torus (Goze–Khakimdjanov). Two of the other three classes come out from naturally graded non-Lie filiform Leibniz algebras. For this case in [3] and [4] a method of

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classification based on algebraic invariants was proposed. The paper [13] has dealt with the classification of one of these classes in low-dimensions based on results of [3]. This article is devoted to the classification of the second class in low dimensions. The third class, which comes out from naturally graded filiform Lie algebras, was treated in [12]. Notice that this class contains the class of all filiform Lie algebras.

The article is organized as follows. Section 2 is a brief introduction to filiform Leibniz algebras. In Section 3, the concepts of adapted basis and adapted transformation are given. Then we describe an action of the adapted transformations group on a filiform Leibniz algebra [7]. Section 4 contains main results of the article consisting of a complete classification of a subclass of low-dimensional filiform Leibniz algebras. Here, for 5- and 6-dimensional cases, we give only final results since the proofs in these cases are similar to those in 7-dimensional case (for the last instance we give a complete proof of a generic case in Section 4.3. In the discrete orbits cases (Proposition 4.10) we give a base change leading to appropriate canonical representative).

2. Preliminaries

Let L be a Leibniz algebra of dimension n. In some basis $\{e_1, e_2, \dots, e_n\}$, the structure of the Leibniz algebra is defined by the structure tensor $\gamma = \{\gamma_{ij}^k\}$, where $[e_i, e_j] = \gamma_{ij}^k e_k$. The components γ_{ij}^k satisfy the Leibniz identity:

$$\gamma_{jk}^l \gamma_{il}^m - \gamma_{ij}^l \gamma_{lk}^m + \gamma_{ik}^l \gamma_{lj}^m = 0, \quad i, j, k, m = 1, 2, \dots, n.$$

When passing to another basis in L, the structure constants are naturally transformed (in accordance with a tensor law). We denote by LB_n the set of all possible structure tensors $\gamma = \{\gamma_{ij}^k\}$ corresponding to all possible n-dimensional Leibniz algebras (over a fixed field K). It is clear that LB_n can be regarded as an algebraic subset of K^{n^3}. Consider the natural action of the group $GL_n(K)$ on LB_n. It is generated by the linear action of $GL_n(K)$ on the coordinates with respect to a basis $\{e_1, e_2, \dots, e_n\}$ of the Leibniz algebra L. The orbits of the action of $GL_n(K)$ on LB_n consist of all mutually isomorphic Leibniz algebras. The stationary subgroup of an arbitrary point under this action is naturally identified with the automorphism group $Aut(L)$ of the Leibniz algebra L corresponding to this point of the space of Leibniz algebras.

The descending central sequence of a Leibniz algebra L is defined as $\{C^i(L)\}$, $i \in N$, where $C^1(L) = L$ and $C^{i+1}(L) = [C^i(L), L]$.

Definition 2.1 A Leibniz algebra L is said to be filiform, if $\dim C^i(L) = n - i$, where $n = \dim L$ and $2 \leq i \leq n$.

Let $Leib_n$ denote the class of all n-dimensional filiform Leibniz algebras. It is clear that a filiform Leibniz algebra is nilpotent.

Let L be a nilpotent Leibniz algebra. Consider $L_i = C^i(L)/C^{i+1}(L)$, $1 \leq i \leq n-1$, and $grL = L_1 \oplus L_2 \oplus \cdots L_{n-1}$. Then $[L_i, L_j] \subseteqq L_{i+j}$ and we obtain the graded algebra grL.

Definition 2.2 A Leibniz algebra L' is said to be naturally graded if L' is isomorphic to grL, for some nilpotent Leibniz algebra L.

Later on, all algebras are supposed to be over the field of complex numbers $\mathbb{C}$ and omitted products of basis vectors are supposed to be zero.

The following theorem summarizes the results of [2,14].

Theorem 2.1 *Any complex* $(n+1)$*-dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non-isomorphic algebras*:

$$NGF_1 = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1 \end{cases},$$

$$NGF_2 = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \le i \le n-1, \end{cases}$$

$$NGF_3 = \begin{cases} [e_i, e_0] = -[e_0, e_i] = e_{i+1}, & 1 \le i \le n-1, \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = \alpha(-1)^{i+1} e_n, & 1 \le i \le n-1, \\ \alpha \in \{0,1\} \text{ for odd } n \text{ and } \alpha = 0 \text{ for even } n. \end{cases}$$

It is clear that NGF_3 is a Lie algebra, however neither NGF_1 nor NGF_2 is a Lie algebra. The above theorem means that the natural gradation of a Leibniz algebra may be an algebra from one of NGF_i for $i = 1, 2, 3$.

The following result of [2,7] describes the class of complex filiform Leibniz algebras whose natural gradation is one of NGF_i for $i = 1, 2, 3$.

Theorem 2.2 *Any* $(n+1)$*-dimensional complex non-Lie filiform Leibniz algebra can be included in one of the following three classes*:

$$FLeib_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1, \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + \cdots + \alpha_{n-1} e_{n-1} + \theta e_n, & \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + \cdots + \alpha_{n+1-j} e_n, & 1 \le j \le n-2, \\ \qquad \alpha_3, \alpha_4, \ldots, \alpha_n, \theta \in \mathbb{C}. & \end{cases}$$

$$SLeib_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \le i \le n-1, \\ [e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + \cdots + \beta_n e_n, & \\ [e_1, e_1] = \gamma e_n, & \\ [e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + \cdots + \beta_{n+1-j} e_n, & 2 \le j \le n-2, \\ \qquad \beta_3, \beta_4, \ldots, \beta_n, \gamma \in \mathbb{C}. & \end{cases}$$

$$TLeib_{n+1} = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le n-1, \\ [e_0, e_0] = b_{0,0} e_n, & \\ [e_0, e_1] = -e_2 + b_{0,1} e_n, & \\ [e_1, e_1] = b_{1,1} e_n, & \\ [e_i, e_j] = a^1_{i,j} e_{i+j+1} + \cdots + a^{n-(i+j+1)}_{i,j} e_{n-1} + b_{i,j} e_n, & 1 \le i < j \le n-2, \\ [e_i, e_j] = -[e_j, e_i], & 1 \le i < j \le n-1, \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^i b_{i,n-i} e_n, & 1 \le i \le n-1, \\ \text{where } a^k_{i,j}, b_{i,j} \in \mathbb{C}, \text{ and } b_{i,n-i} = b, \text{ whenever } 1 \le i \le n-1, & \\ \qquad b \in \{0,1\}, \text{ for odd } n \text{ and } b = 0, \text{ for even } n. & \end{cases}$$

A basis, leading to this representation, is said to be '*adapted*'.

These three classes have no non-trivial intersection and they are invariant with respect to the adapted base change. We have denoted the classes as $FLeib_{n+1}$, $SLeib_{n+1}$ and $TLeib_{n+1}$, respectively. Hence, the classification problem of $Leib_n$ has been reduced to the classification problem within each of the subclasses $FLeib_n$, $SLeib_n$ and $TLeib_n$.

Isomorphism criteria, classifications and invariants of $FLeib_n$ and $TLeib_n$ have been studied in [3,7,12,13].

In this article, we focus on the second class of algebras of the above theorem. Elements of $SLeib_{n+1}$ will be denoted by $L(\beta_3, \beta_4, \dots, \beta_n, \gamma)$, pointing out the dependence of them on parameters $\beta_3, \beta_4, \dots, \beta_n, \gamma$. The $(n+1)$-dimensional standard algebra $L(0, 0, 0, \dots, 0)$ is denoted by G^s_{n+1}.

3. Simplification of base change and isomorphism criterion for $SLeib_{n+1}$

Here we simplify the action of GL_n ('transport of structure') on $SLeib_n$. All the results of this section have appeared elsewhere, particularly in [4] and [7].

Let L be a Leibniz algebra defined on a vector space V and $\{e_0, e_1, \dots, e_n\}$ be an adapted basis of L.

Definition 3.1 A basis transformation $f \in GL(V)$ is said to be adapted for the structure of L, if the basis $\{f(e_0), f(e_1), \dots, f(e_n)\}$ is adapted.

The closed subgroup of $GL(V)$, spanned by the adapted transformations, is denoted by GL_{ad}.

Definition 3.2 The following types of basis transformations of $SLeib_{n+1}$ are said to be elementary:

$$\text{first type} - \ \sigma(b, n) = \begin{cases} f(e_0) = e_0 \\ f(e_1) = e_1 + be_n, \\ f(e_{i+1}) = [f(e_i), f(e_0)], \quad 2 \le i \le n-1, \\ f(e_2) = [f(e_0), f(e_0)] \end{cases}$$

$$\text{second type} - \eta(a, k) = \begin{cases} f(e_0) = e_0 + ae_k \\ f(e_1) = e_1 \\ f(e_{i+1}) = [f(e_i), f(e_0)], \quad 2 \le i \le n-1, 2 \le k \le n, \\ f(e_2) = [f(e_0), f(e_0)] \end{cases}$$

$$\text{third type} - \ \delta(a, b, d) = \begin{cases} f(e_0) = ae_0 + be_1 \\ f(e_1) = de_1 - \frac{bd\gamma}{a} e_{n-1}, \quad ad \neq 0 \\ f(e_{i+1}) = [f(e_i), f(e_0)], \quad 2 \le i \le n-1, \\ f(e_2) = [f(e_0), f(e_0)] \end{cases}$$

where a, b, $d \in \mathbb{C}$.

PROPOSITION 3.1

(1) *An adapted transformation f of $SLeib_{n+1}$ can be represented in the form*

$$f = \sigma(b_n, n) \circ \eta(a_n, n) \circ \eta(a_{n-1}, n-2) \circ \cdots \circ \eta(a_2, 2) \circ \delta(a_0, a_1, b_1).$$

(2) *The transformations* $\sigma(b,n)$, $\eta(a,n)$ *and* $\eta(a,k)$, *where* $2 \le k \le n-2$, $a \in \mathbb{C}$ *preserve the structure constants of algebras from* $SLeib_{n+1}$.

Proof The proof is straightforward. ■

Since a composition of adapted transformations is adapted, the proposition above means that the transformation $\sigma(b_n,n) \circ \eta(a_n,n) \circ \eta(a_{n-1},n-2) \circ \cdots \circ \eta(a_2,2)$ does not change the structure constants of algebras from $SLeib_{n+1}$.

Thus, the action of GL_{ad} on $SLeib_{n+1}$ can be reduced to the action of elementary transformation of type three.

Let $R_a^m(x) := [[\ldots[x, \underbrace{a], a], \ldots, a]}_{m\text{-times}}$, and $R_a^0(x) := x$.

Now due to Proposition 3.1 it is easy to see that for $SLeib_{n+1}$ the adapted base change has the form [7]:

$$
\begin{aligned}
e'_0 &= Ae_0 + Be_1,\\
e'_1 &= De_1 - \frac{BD\gamma}{A}e_{n-1},\\
e'_2 &= A(A+B)e_2 + AB(\beta_3 e_3 + \cdots + \beta_{n-1}e_{n-1}) + B(A\beta_n + B\gamma)e_n,\\
e'_k &= A\left(\sum_{i=0}^{k-2} C_{k-1}^{k-1-i}A^{k-1-i}B^i R_{e_1}^i(e_{k-i}) + B^{k-1}R_{e_1}^{k-1}(e_0)\right),
\end{aligned}
$$

where $3 \le k \le n$ and $A, D \in \mathbb{C}$ such that $AD \neq 0$.

Now, we remind an isomorphism criterion for $SLeib_{n+1}$. Introduce the following series of functions:

$$
\begin{aligned}
\psi_t(y;z) &= \psi_t(y;z_3,z_4,\ldots,z_n,z_{n+1})\\
&= z_t - \sum_{k=3}^{t-1}\Bigg(C_{k-1}^{k-2}yz_{t+2-k} + C_{k-1}^{k-3}y^2\sum_{i_1=k+2}^{t} z_{t+3-i_1}\cdot z_{i_1+1-k}\\
&\quad + C_{k-1}^{k-4}y^3\sum_{i_2=k+3}^{t}\sum_{i_1=k+3}^{i_2} z_{t+3-i_2}\cdot z_{i_2+3-i_1}\cdot z_{i_1-k} + \cdots\\
&\quad + C_{k-1}^{1}y^{k-2}\sum_{i_{k-3}=2k-2}^{t}\sum_{i_{k-4}=2k-2}^{i_{k-3}}\cdots\sum_{i_1=2k-2}^{i_2} z_{t+3-i_{k-3}}\cdot z_{i_{k-3}+3-i_{k-4}}\\
&\quad \cdot\cdots\cdot z_{i_2+3-i_1}\cdot z_{i_1+5-2k} + y^{k-1}\sum_{i_{k-2}=2k-1}^{t}\sum_{i_{k-3}=2k-1}^{i_{k-2}}\cdots\sum_{i_1=2k-1}^{i_2} z_{t+3-i_{k-2}}\\
&\quad \cdot z_{i_{k-2}+3-i_{k-3}}\cdot\cdots\cdot z_{i_2+3-i_1}\cdot z_{i_1+4-2k}\Bigg)\cdot\psi_k(y;z), \quad \text{where } 3 \le t \le n,
\end{aligned}
$$

$$\psi_{n+1}(y;z) = z_{n+1}.$$

The isomorphism criterion for $SLeib_{n+1}$, first appeared in [7] and later on in [4], it was given in more invariant form as follows.

THEOREM 3.1 *Two algebras* $L(\beta)$ *and* $L(\beta')$ *from* $SLeib_{n+1}$, *where* $\beta = (\beta_3, \beta_4, \ldots, \beta_n, \gamma)$, *and* $\beta' = (\beta'_3, \beta'_4, \ldots, \beta'_n, \gamma')$, *are isomorphic, if and only if* *there exist complex numbers A, B and D, such that $AD \neq 0$ and the following conditions hold*:

$$\beta'_t = \frac{1}{A^{t-2}} \frac{D}{A} \psi_t\left(\frac{B}{A}; \beta\right), \quad 3 \leq t \leq n-1, \tag{1}$$

$$\beta'_n = \frac{1}{A^{n-2}} \frac{D}{A} \frac{B}{A} \gamma + \psi_n\left(\frac{B}{A}; \beta\right) \quad \textit{and} \tag{2}$$

$$\gamma' = \frac{1}{A^{n-2}} \left(\frac{D}{A}\right)^2 \psi_{n+1}\left(\frac{A}{B}; \beta\right). \tag{3}$$

Hereinafter, for the simplification purpose in the above case for transition from $(n+1)$-dimensional filiform Leibniz algebra $L(\beta)$ to $(n+1)$-dimensional filiform Leibniz algebra $L(\beta')$ we write $\beta' = \varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right)$:

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right) = \left(\varrho_1\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right), \varrho_2\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right), \ldots, \varrho_{n-1}\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right)\right),$$

where

$$\varrho_t(x, y, u; z) = x^{t-1} u \psi_{t+2}(y; z) \quad \text{for } 1 \leq t \leq n-2,$$

and

$$\varrho_{n-1}(x, y, u; z) = x^{n-5} u^2 \psi_{n+1}(y; z).$$

Here are the main properties of the operator ϱ, used in this article:

1^0. $\varrho(1, 0, 1; \cdot)$ is the identity operator.
2^0. $\varrho\left(\frac{1}{A_2}, \frac{B_2}{A_2}, \frac{D_2}{A_2}; \varrho\left(\frac{1}{A_1}, \frac{B_1}{A_1}, \frac{D_1}{A_1}; \beta\right)\right) = \varrho\left(\frac{1}{A_1 A_2}, \frac{B_1 A_2 + B_2 D_1}{A_1 A_2}, \frac{D_1 D_2}{A_1 A_2}; \beta\right)$
3^0. If $\beta' = \varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; \beta\right)$ then $\beta = \varrho\left(A, -\frac{B}{D}, \frac{A}{D}; \beta'\right)$.

From here on, we assume that $n \geq 4$ since there are complete classifications of complex nilpotent Leibniz algebras in dimensions at most four [1].

In this article we proceed from the viewpoint of [4]. Let N_n stand for the adapted number of isomorphism classes in $SLeib_n$. Later on, if no confusion is possible, we write xy for $[x, y]$ as well.

4. Classification

For the simplification purpose, we establish the following notations:

$$\Lambda_1 = 4\beta_3\beta_5 - 5\beta_4^2, \quad \Lambda_2 = 4\beta_3^2\beta_6 - 7\beta_4^3,$$

and

$$\Lambda'_1 = 4\beta'_3\beta'_5 - 5\beta'^2_4, \quad \Lambda'_2 = 4\beta'^2_3\beta'_6 - 7\beta'^3_4.$$

4.1. *Dimension 5*

The class $SLeib_5$ is represented as a disjoint union of its subsets as follows:

$$SLeib_5 = \bigcup_{i=1}^{6} U_i,$$

where

$$U_1 = \{L(\beta) \in SLeib_5\colon\ \beta_3 \neq 0, \gamma - 2\beta_3^2 \neq 0\},$$
$$U_2 = \{L(\beta) \in SLeib_5\colon\ \beta_3 \neq 0, \gamma - 2\beta_3^2 = 0, \beta_4 \neq 0\},$$
$$U_3 = \{L(\beta) \in SLeib_5\colon\ \beta_3 = 0, \gamma \neq 0\},$$
$$U_4 = \{L(\beta) \in SLeib_5\colon\ \beta_3 = 0, \gamma = 0, \beta_4 = 0\},$$
$$U_5 = \{L(\alpha) \in SLeib_5\colon\ \beta_3 \neq 0, \gamma - 2\beta_3^2 = 0, \beta_4 = 0\},$$
$$U_6 = \{L(\beta) \in SLeib_5\colon\ \beta_3 = 0, \gamma = 0, \beta_4 \neq 0\}.$$

Now we consider the isomorphism problem for each of these subsets separately.

Proposition 4.1

(i) *Two algebras $L(\beta)$ and $L(\beta^{'})$ from U_1 are isomorphic, if and only if*

$$\frac{\gamma}{\beta_3^2} = \frac{\gamma'}{\beta_3'^2}. \tag{4}$$

(ii) *Orbits in U_1 can be parametrized as $L(1, 0, \lambda)$, $\lambda \in \mathbb{C} \setminus \{2\}$.*

Proposition 4.2 *The subsets U_2, U_3, U_4, U_5 and U_6 are single orbits under the action of G_{ad} with the representatives $L(1, 1, 2)$, $L(0, 0, 1)$, $L(0, 1, 0)$, $L(1, 0, 2)$ and $L(0, 0, 0)$, respectively.*

Theorem 4.1 *Let L be an element of $SLeib_5$. Then, it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

(1) $L(0,0,0) = G_5^s$: $e_0e_0 = e_2$, $e_ie_0 = e_{i+1}$, $2 \leq i \leq 3$.
(2) $L(0,1,0)$: G_5^s, $e_0e_1 = e_4$.
(3) $L(0,0,1)$: G_5^s, $e_1e_1 = e_4$.
(4) $L(1,1,2)$: G_5^s, $e_0e_1 = e_3 + e_4$, $e_1e_1 = 2e_4$, $e_2e_1 = e_4$.
(5) $L(1,0,\lambda)$: G_5^s, $e_0e_1 = e_3$, $e_1e_1 = \lambda e_4$, $e_2e_1 = e_4$, $\lambda \in \mathbb{C}$.

Note 4.1 The orbit U_5 with the representative $L(1, 0, 2)$ can be included in the parametric family of orbits $L(1, 0, \lambda)$ at $\lambda = 2$.

The adapted number of isomorphism classes N_5=5.

4.2. *Dimension 6*

This section concerns 6-dimensional case. The set $SLeib_6$ can be represented as a disjoint union of its subsets as follows:

$$SLeib_6 = \bigcup_{i=1}^{9} U_i,$$

where

$$\begin{aligned}
U_1 &= \{L(\beta) \in SLeib_6\colon \beta_3 \neq 0, \gamma \neq 0\},\\
U_2 &= \{L(\beta) \in SLeib_6\colon \beta_3 \neq 0, \gamma = 0, \Lambda_1 \neq 0\},\\
U_3 &= \{L(\beta) \in SLeib_6\colon \beta_3 \neq 0, \gamma = 0, \Lambda_1 = 0\},\\
U_4 &= \{L(\beta) \in SLeib_6\colon \beta_3 = 0, \beta_4 \neq 0, \gamma \neq 0\},\\
U_5 &= \{L(\beta) \in SLeib_6\colon \beta_3 = 0, \beta_4 \neq 0, \gamma = 0, \beta_5 \neq 0\},\\
U_6 &= \{L(\beta) \in SLeib_6\colon \beta_3 = 0, \beta_4 \neq 0, \gamma = 0, \beta_5 = 0\},\\
U_7 &= \{L(\beta) \in SLeib_6\colon \beta_3 = 0, \beta_4 = 0, \gamma \neq 0\},\\
U_8 &= \{L(\alpha) \in SLeib_6\colon \beta_3 = 0, \beta_4 = 0, \gamma = 0, \beta_5 \neq 0\},\\
U_9 &= \{L(\alpha) \in SLeib_6\colon \beta_3 = 0, \beta_4 = 0, \gamma = 0, \beta_5 = 0\}.
\end{aligned}$$

Proposition 4.3

(i) *Two algebras* $L(\beta)$ *and* $L(\beta^{'})$ *from* U_1 *are isomorphic, if and only if*

$$\frac{2\beta_3\beta_4\gamma + \beta_3^2\Lambda_1}{\gamma^2} = \frac{2\beta_3'\beta_4'\gamma' + \beta_3'^2\Lambda_1'}{\gamma'^2}. \tag{5}$$

(ii) *Orbits in* U_1 *can be parametrized as* $L(1, 0, \lambda, 1)$, $\lambda \in \mathbb{C}$.

Proposition 4.4 *The subsets* U_2, U_3, U_4, U_5, U_6, U_7, U_8 *and* U_9 *are single orbits under the action of* G_{ad} *with the representatives* $L(1, 0, 1, 0)$, $L(1, 0, 0, 0)$, $L(0, 1, 0, 1)$, $L(0, 1, 1, 0)$, $L(0, 1, 0, 0)$, $L(0, 0, 0, 1)$, $L(0, 0, 1, 0)$ *and* $L(0, 0, 0, 0)$, *respectively*.

Theorem 4.2 *Let L be an element of* $SLeib_6$. *Then, it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

1) $L(0,0,0,0) = G_6^s : e_0e_0 = e_2,\ e_ie_0 = e_{i+1},\ 2 \leq i \leq 4.$
2) $L(0,0,1,0)$: $G_6^s,\ e_0e_1 = e_5.$
3) $L(0,0,0,1)$: $G_6^s,\ e_1e_1 = e_5.$
4) $L(0,1,0,0)$: $G_6^s,\ e_0e_1 = e_4,\ e_2e_1 = e_5.$
5) $L(0,1,1,0)$: $G_6^s,\ e_0e_1 = e_4 + e_5,\ e_2e_1 = e_5.$
6) $L(0,1,0,1)$: $G_6^s,\ e_0e_1 = e_4,\ e_1e_1 = e_5,\ e_2e_1 = e_5.$
7) $L(1,0,0,0)$: $G_6^s,\ e_0e_1 = e_3,\ e_2e_1 = e_4,\ e_3e_1 = e_5.$
8) $L(1,0,1,0)$: $G_6^s,\ e_0e_1 = e_3 + e_5,\ e_2e_1 = e_4,\ e_3e_1 = e_5.$
9) $L(1,0,\lambda,1)$: $G_6^s,\ e_0e_1 = e_3 + \lambda e_5,\ e_1e_1 = e_5,\ e_2e_1 = e_4,\ e_3e_1 = e_5,\quad \lambda \in \mathbb{C}.$

The adapted number of isomorphism classes N_6=9.

4.3. *Dimension 7*

In this section one considers $SLeib_7$. The following is a representation of $SLeib_7$ as a disjoint union of its subsets:

$$SLeib_7 = \bigcup_{i=1}^{18} U_i,$$

where

$$U_1 = \{L(\beta) \in SLeib_7\colon \beta_3 \neq 0, \Lambda_1 \neq 0, \Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma \neq 0\},$$
$$U_2 = \{L(\beta) \in SLeib_7\colon \beta_3 \neq 0, \Lambda_1 \neq 0, \Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma = 0\},$$
$$U_3 = \{L(\beta) \in SLeib_7\colon \beta_3 \neq 0, \Lambda_1 = 0, \gamma \neq 0, \Lambda_2 + 2\beta_4\gamma \neq 0\},$$
$$U_4 = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 \neq 0, \beta_5 \neq 0\},$$
$$U_5 = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 \neq 0, \beta_5 = 0, \gamma - 3\beta_4^2 \neq 0\},$$
$$U_6 = \{L(\beta) \in SLeib_7\colon \beta_3 \neq 0, \Lambda_1 = 0, \gamma \neq 0, \Lambda_2 + 2\beta_4\gamma = 0\},$$
$$U_7 = \{L(\beta) \in SLeib_7\colon \beta_3 \neq 0, \Lambda_1 = 0, \gamma = 0, \Lambda_2 \neq 0\},$$
$$U_8 = \{L(\beta) \in SLeib_7\colon \beta_3 \neq 0, \Lambda_1 = 0, \gamma = 0, \Lambda_2 = 0\},$$
$$U_9 = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 \neq 0, \beta_5 = 0, \gamma - 3\beta_4^2 = 0, \beta_6 \neq 0\},$$
$$U_{10} = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 \neq 0, \beta_5 = 0, \gamma - 3\beta_4^2 = 0, \beta_6 = 0\},$$
$$U_{11} = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 = 0, \beta_5 \neq 0, \beta_6 \neq 0, \gamma \neq 0\},$$
$$U_{12} = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 = 0, \beta_5 \neq 0, \beta_6 \neq 0, \gamma = 0\},$$
$$U_{13} = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 = 0, \beta_5 \neq 0, \beta_6 = 0, \gamma \neq 0\},$$
$$U_{14} = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 = 0, \beta_5 \neq 0, \beta_6 = 0, \gamma = 0\},$$
$$U_{15} = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 = 0, \beta_5 = 0, \beta_6 \neq 0, \gamma \neq 0\},$$
$$U_{16} = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 = 0, \beta_5 = 0, \beta_6 \neq 0, \gamma = 0\},$$
$$U_{17} = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 = 0, \beta_5 = 0, \beta_6 = 0, \gamma \neq 0\},$$
$$U_{18} = \{L(\beta) \in SLeib_7\colon \beta_3 = 0, \beta_4 = 0, \beta_5 = 0, \beta_6 = 0, \gamma = 0\}.$$

Proposition 4.5

(i) *Two algebras $L(\beta)$ and $L(\beta')$ from U_1 are isomorphic, if and only if*

$$\frac{\Lambda_1^3}{(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)^2} = \frac{\Lambda_1'^3}{(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')^2}, \tag{6}$$

and

$$\frac{\gamma\Lambda_1^2}{(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)^2} = \frac{\gamma'\Lambda_1'^2}{(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')^2}. \tag{7}$$

(ii) *The subset U_1 is a union of orbits with representatives $L(1, 0, \lambda_1, \lambda_1, \lambda_2)$, $\lambda_1 \in \mathbb{C}^*$, $\lambda_2 \in \mathbb{C}$.*

Proof

(i) $\Rightarrow$: Let $L(\beta)$ and $L(\beta')$ be isomorphic. Then, due to Theorem 3.1, there are complex numbers A, B and D: $AD \neq 0$, such that the action of the adapted group G_{ad}

can be expressed by the following system of equalities:

$$\beta_3' = \frac{1}{A}\frac{D}{A}\beta_3, \tag{8}$$

$$\beta_4' = \frac{1}{A^2}\frac{D}{A}\left(\beta_4 - 2\frac{B}{A}\beta_3^2\right), \tag{9}$$

$$\beta_5' = \frac{1}{A^3}\frac{D}{A}\left(\beta_5 - 5\frac{B}{A}\beta_3\beta_4 + 5\left(\frac{B}{A}\right)^2\beta_3^3\right), \tag{10}$$

$$\beta_6' = \frac{1}{A^4}\frac{D}{A}\left(\frac{B}{A}\gamma + \beta_6 - 6\frac{B}{A}\beta_3\beta_5 + 21\left(\frac{B}{A}\right)^2\beta_3^2\beta_4 - 3\frac{B}{A}\beta_4^2 - 14\left(\frac{B}{A}\right)^3\beta_3^4\right), \tag{11}$$

$$\gamma' = \frac{1}{A^4}\left(\frac{D}{A}\right)^2\gamma. \tag{12}$$

Then, it is easy to see that $\Lambda_1' = \frac{1}{A^4}\left(\frac{D}{A}\right)^2\Lambda_1$ and $\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma' = \frac{1}{A^6}\left(\frac{D}{A}\right)^3(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)$. Now we need just substitute it into $\frac{\Lambda_1'^3}{(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')^2}$ and $\frac{\gamma'\Lambda_1'^2}{(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')^2}$ to get the required equalities.

$\Leftarrow$: Let the equalities (6) and (7) hold. We put

$$A_0 = \frac{\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma}{\beta_3\Lambda_1}, \tag{13}$$

$$B_0 = \frac{\beta_4(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)}{2\beta_3^3\Lambda_1}, \tag{14}$$

$$D_0 = \frac{(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)^2}{\beta_3^3\Lambda_1^2}, \tag{15}$$

and

$$A_0' = \frac{\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma'}{\beta_3'\Lambda_1'}, \tag{16}$$

$$B_0' = \frac{\beta_4'(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')}{2\beta_3'^3\Lambda_1'^2}, \tag{17}$$

$$D_0' = \frac{(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')^2}{\beta_3'^3\Lambda_1'^2}. \tag{18}$$

Then, $\beta_0 = \varrho(\frac{1}{A_0}, \frac{B_0}{A_0}, \frac{D_0}{A_0}, \beta)$ and $\beta_0' = \varrho(\frac{1}{A_0'}, \frac{B_0'}{A_0'}, \frac{D_0'}{A_0'}, \beta')$ (see the convention in Section 3), where

$$\beta_0 = L\left(1, 0, \frac{\Lambda_1^3}{(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)^2}, \frac{\gamma\Lambda_1^2}{(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)^2}\right),$$

and

$$\beta_0' = L\left(1, 0, \frac{\Lambda_1'^3}{(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')^2}, \frac{\gamma'\Lambda_1'^2}{(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')^2}\right).$$

Then, the equalities (6) and (7) imply that $\beta_0 = \beta_0'$. Now we make use of the properties $1^0 - 3^0$ of ϱ and find the complex numbers A, B and D: $AD \neq 0$:

$$A = \frac{A_0}{A_0'}, \tag{19}$$

$$B = \frac{B_0 D_0' - B_0' D_0}{A_0' D_0'} \tag{20}$$

and

$$D = \frac{D_0}{D_0'}. \tag{21}$$

Thus we get

$$A = \frac{\beta_3'\Lambda_1'(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)}{\beta_3\Lambda_1(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')}, \tag{22}$$

$$B = \frac{\beta_3'\Lambda_1'^2(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)}{2\beta_3^3\Lambda_1(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')^2}\left(\frac{\beta_4(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')}{\Lambda_1'} - \frac{\beta_4'(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)}{\Lambda_1}\right) \tag{23}$$

and

$$D = \frac{\beta_3'^3\Lambda_1'^2(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)^2}{\beta_3^3\Lambda_1^2(\Lambda_2' - 3\beta_4'\Lambda_1' + 2\beta_4'\gamma')^2}. \tag{24}$$

An easy computation shows that, for the A, B and D found above, we get the corresponding system of equalities (8)–(12):

$$\frac{1}{A}\frac{D}{A}\beta_3 = \beta_3',$$

$$\frac{1}{A^2}\frac{D}{A}\left(\beta_4 - 2\frac{B}{A}\beta_3^2\right) = \beta_4',$$

$$\frac{1}{A^3}\frac{D}{A}\left(\beta_5 - 5\frac{B}{A}\beta_3\beta_4 + 5\left(\frac{B}{A}\right)^2\beta_3^3\right) = \beta_5',$$

$$\frac{1}{A^4}\frac{D}{A}\left(\frac{B}{A}\gamma + \beta_6 - 6\frac{B}{A}\beta_3\beta_5 + 21\left(\frac{B}{A}\right)^2\beta_3^2\beta_4 - 3\frac{B}{A}\beta_4^2 - 14\left(\frac{B}{A}\right)^3\beta_3^4\right) = \beta_6',$$

$$\frac{1}{A^4}\left(\frac{D}{A}\right)^2\gamma = \gamma',$$

meaning that $L(\beta)$ and $L(\beta')$ are isomorphic.

(ii) Evidently, for any $\lambda_1, \lambda_2 \in \mathbb{C}$: $\lambda_2 \neq 0$ there exists an algebra $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)$ from U_1 such that

$$\lambda_1 = \frac{\Lambda_1^3}{(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)^2} \quad \text{and} \quad \lambda_2 = \frac{\gamma\Lambda_1^2}{(\Lambda_2 - 3\beta_4\Lambda_1 + 2\beta_4\gamma)^2}.$$

The proof is complete. ■

Proposition 4.6

(i) *Two algebras* $L(\beta)$ *and* $L(\beta')$ *from* U_2 *are isomorphic, if and only if*

$$\frac{\gamma}{\Lambda_1} = \frac{\gamma'}{\Lambda_1'}. \tag{25}$$

(ii) *The subset* U_2 *is a union of continuous family of orbits with representatives* $L(1, 0, 1, 0, \lambda)$, $\lambda \in \mathbb{C}$.

Proposition 4.7

(i) *Two algebras* $L(\beta)$ *and* $L(\beta')$ *from* U_3 *are isomorphic, if and only if*

$$\frac{\gamma^3}{(\Lambda_2 + 2\beta_4\gamma)^2} = \frac{\gamma'^3}{(\Lambda_2' + 2\beta_4'\gamma')^2}. \tag{26}$$

(ii) U_3 *can be represented as a union of orbits and the orbits are parametrized as* $L(1, 0, 0, \lambda, \lambda)$, $\lambda \in \mathbb{C}^*$.

Proposition 4.8

(i) *Two algebras* $L(\beta)$ *and* $L(\beta')$ *from* U_4 *are isomorphic, if and only if*

$$\frac{\gamma}{\beta_4^2} = \frac{\gamma'}{\beta_4'^2}. \tag{27}$$

(ii) *Orbits in* U_4 *can be parametrized as* $L(0, 1, 1, 0, \lambda)$, $\lambda \in \mathbb{C}$.

Proposition 4.9

(i) *Two algebras* $L(\beta)$ *and* $L(\beta')$ *from* U_5 *are isomorphic, if and only if*

$$\frac{\gamma}{\beta_4^2} = \frac{\gamma'}{\beta_4'^2}. \tag{28}$$

(ii) *Orbits in* U_5 *can be parametrized as* $L(0, 1, 0, 0, \lambda)$, $\lambda \in \mathbb{C} \setminus \{3\}$.

Proposition 4.10 *The subsets* $U_6, U_7, U_8, U_9, U_{10}, U_{11}, U_{12}, U_{13}, U_{14}, U_{15}, U_{16}, U_{17}$, *and* U_{18} *are single orbits under the action of* G_{ad} *with the representatives* $L(1, 0, 0, 0, 1)$, $L(1, 0, 0, 1, 0)$, $L(1, 0, 0, 0, 0)$, $L(0, 1, 0, 1, 3)$, $L(0, 1, 0, 0, 3)$, $L(0, 0, 1, 0, 1)$, $L(0, 0, 1, 1, 0)$, $L(0, 0, 1, 1, 1)$, $L(0, 0, 1, 0, 0)$, $L(0, 0, 0, 0, 1)$, $L(0, 0, 0, 1, 0)$, $L(0, 0, 0, 1, 1)$ *and* $L(0, 0, 0, 0, 0)$, *respectively*.

Proof The subsets $U_6, \ldots, U_{18}$ can be represented as orbits with respect to the action of G_{ad}. Below the corresponding actions leading to the canonical representatives are indicated:

U_6 : For $L(\beta_3,\ \beta_4,\ \beta_5,\ \beta_6,\ \gamma) \in U_6$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(1, 0, 0, 0, 1), \tag{29}$$

where $A = \sqrt{\frac{\beta_3^2}{\gamma}}, B = \frac{\beta_4}{2\beta_3^2}\sqrt{\frac{\beta_3^2}{\gamma}}$ and $D = \frac{\beta_3}{\gamma}$.

U_7 : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_7$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(1, 0, 0, 1, 0), \tag{30}$$

where $A = \sqrt[3]{\frac{\Delta_2}{4\beta_3^3}}, B = \frac{\beta_4}{2\beta_3^2}\sqrt[3]{\frac{\Delta_2}{4\beta_3^3}}$ and $D = \frac{\Delta_2}{4\beta_3^4}\sqrt{\frac{\Delta_2}{4\beta_3^3}}$.

U_8 : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_8$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(1, 0, 0, 0, 0), \tag{31}$$

where A is a nonzero complex number, $B = \frac{\beta_4}{2\beta_3^2}A$ and $D = \frac{1}{\beta_3}A^2$.

U_9 : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_9$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(0, 1, 0, 1, 3), \tag{32}$$

where $A = \sqrt{\frac{\beta_6}{\beta_4}}$, B is any complex number and $D = \frac{\beta_6}{\beta_4^2}\sqrt{\frac{\beta_6}{\beta_4}}$.

U_{10} : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_{10}$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(0, 1, 0, 0, 3), \tag{33}$$

where A is a nonzero complex number, B is any complex number and $D = \frac{A^3}{\beta_4}$.

U_{11} : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_{11}$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(0, 0, 1, 0, 1), \tag{34}$$

where $A = \frac{\beta_5}{\sqrt{\gamma}}$, $B = -\frac{\beta_6}{\gamma}$ and $D = \frac{\beta_5^3}{\gamma^2}$.

U_{12} : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_{12}$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(0, 0, 1, 1, 0), \tag{35}$$

where $A = \frac{\beta_6}{\beta_5}$, B is any complex number and $D = \frac{\beta_6^4}{\beta_5^5}$.

U_{13} : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_{13}$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(0, 0, 1, 1, 1), \tag{36}$$

where A, $B = D$ are nonzero complex numbers.

U_{14} : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_{14}$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(0, 0, 1, 0, 0), \tag{37}$$

where A is any nonzero complex number, $D = \frac{1}{\beta_5}A^4$, and B is any complex number.

U_{15} : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_{15}$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(0,0,0,0,1), \tag{38}$$

where A is any nonzero complex number, $B = -\frac{\beta_6}{\gamma} A$ and $D = \sqrt{\frac{A^6}{\gamma}}$.

U_{16} : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_{16}$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(0,0,0,1,0), \tag{39}$$

where A is any nonzero complex number, $D = \frac{A^5}{\beta_6}$ and $B \in \mathbb{C}$.

U_{17} : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_{17}$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(0,0,0,1,1), \tag{40}$$

where A, $B = D$ are any nonzero complex numbers.

U_{18} : For $L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma) \in U_{18}$

$$\varrho\left(\frac{1}{A}, \frac{B}{A}, \frac{D}{A}; L(\beta_3, \beta_4, \beta_5, \beta_6, \gamma)\right) = L(0,0,0,0,0), \tag{41}$$

where A, D are any nonzero complex numbers and $B \in \mathbb{C}$. ■

Note that in the basis changing of the subsets U_6, U_7, U_9, U_{11} and U_{15} above, the value of the roots can be taken an arbitrary.

We summarize the previous results in the following classification theorem.

THEOREM 4.3 *Let L be an element of* $SLeib_7$. *Then, it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

(1) $L(0,0,0,0,0) = G_7^s : e_0e_0 = e_2,\ e_ie_0 = e_{i+1},\ 2 \le i \le 5.$
(2) $L(0,0,0,1,1)$: $G_7^s,\ e_0e_1 = e_6,\ e_1e_1 = e_6.$
(3) $L(0,0,0,1,0)$: $G_7^s,\ e_0e_1 = e_6.$
(4) $L(0,0,0,0,1)$: $G_7^s,\ e_1e_1 = e_6.$
(5) $L(0,0,1,0,0)$: $G_7^s,\ e_0e_1 = e_5,\ e_2e_1 = e_6.$
(6) $L(0,0,1,1,1)$: $G_7^s,\ e_0e_1 = e_5 + e_6,\ e_1e_1 = e_6,\ e_2e_1 = e_6.$
(7) $L(0,0,1,1,0)$: $G_7^s,\ e_0e_1 = e_5 + e_6,\ e_2e_1 = e_6.$
(8) $L(0,0,1,0,1)$: $G_7^s,\ e_0e_1 = e_5,\ e_1e_1 = e_6,\ e_2e_1 = e_6.$
(9) $L(0,1,0,1,3)$: $G_7^s,\ e_0e_1 = e_4 + e_6,\ e_1e_1 = 3e_6,\ e_2e_1 = e_5,\ e_3e_1 = e_6.$
(10) $L(1,0,0,1,0)$: $G_7^s,\ e_0e_1 = e_3 + e_4,\ e_2e_1 = e_4,\ e_3e_1 = e_6,\ e_4e_1 = e_6.$
(11) $L(0,1,0,0,\lambda)$:
$G_7^s,\ e_0e_1 = e_4 + e_5 + e_6,\ e_1e_1 = \lambda e_6,\ e_2e_1 = e_5 + e_6,\ e_3e_1 = e_6,\ \lambda \in \mathbb{C}.$
(12) $L(0,1,1,0,\lambda)$:
$G_7^s,\ e_0e_1 = e_4 + e_5,\ e_1e_1 = \lambda e_6,\ e_2e_1 = e_5 + e_6,\ e_3e_1 = e_6,\ \lambda \in \mathbb{C}.$
(13) $L(1,0,0,\lambda,\lambda)$:
$G_7^s,\ e_0e_1 = e_3 + \lambda e_6,\ e_1e_1 = \lambda e_6,\ e_2e_1 = e_4,\ e_3e_1 = e_5,\ e_4e_1 = e_6,\ \lambda \in \mathbb{C}.$
(14) $L(1,0,1,0,\lambda)$:
$G_7^s,\ e_0e_1 = e_3 + e_5,\ e_1e_1 = \lambda e_6,\ e_2e_1 = e_4 + e_6,\ e_3e_1 = e_5,$
$e_4e_1 = e_6,\ \lambda \in \mathbb{C}.$

(15) $L(1,0,\lambda_1,\lambda_1,\lambda_2)$:
G_7^s, $e_0e_1 = e_3 + \lambda_1 e_5 + \lambda_1 e_6$, $e_1e_1 = \lambda_2 e_6$, $e_2e_1 = e_4 + \lambda_1 e_6$, $e_3e_1 = e_5$, $e_4e_1 = e_6$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

The adapted number of isomorphism classes $N_7 = 15$.

Note 4.2 The orbits U_6, U_8 and U_{10} can be included in parametric family of orbits with the representatives $L(1,0,\lambda_1,\lambda_1,\lambda_2)$, $L(1,0,0,\lambda,\lambda)$, and $L(0,1,0,0,\lambda)$ at the value of parameters $\{\lambda_1 = 0, \lambda_2 = 1\}$, $\lambda = 0$ and $\lambda = 3$, respectively.

5. Conclusion

To classify $SLeib_n$, we split it into its subsets and then classify algebras from each of these subsets. In this procedure, some of the subsets turn out to be a union of infinitely many orbits and other to be just a single orbit. In each case, we indicate the corresponding canonical representatives of the orbits. We expect the formula $N_n = n^2 - 7n + 15$ in dimension n for the number of isomorphism classes. The formula has been confirmed up to $n = 9$.

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