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Description of some classes of Leibniz algebras

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ABSTRACT

In this paper we describe the isomorphism classes of finite-dimensional complex Leibniz algebras whose quotient algebra with respect to the ideal generated by squares is isomorphic to the direct sum of three-dimensional simple Lie algebra sl_2 and a three-dimensional solvable ideal. We choose a basis of the isomorphism classes' representatives and give explicit multiplication tables.

1. Introduction

Leibniz algebra is a generalization of Lie algebra. Leibniz algebras have been first introduced by Loday in [5] as a non-antisymmetric version of Lie algebras.

The classification problem of finite-dimensional Lie algebras is fundamental and a very difficult problem. It is split into three parts: (1) classification of nilpotent Lie algebras; (2) description of solvable Lie algebras with given nilradical; (3) description of Lie algebras with given radical. The third

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problem has been reduced to the description of semisimple subalgebras in the algebra of derivations of a given solvable algebra [6]. The classification of semisimple Lie algebras has been known ever since the works of Cartan and Killing. According to the Cartan–Killing theory the semisimple Lie algebras can be represented as a direct sum of the classical simple Lie algebras from series $A_n(n \geqslant 1),\ B_n(n \geqslant 2),\ C_n(n \geqslant 3),\ D_n(n \geqslant 4)$ and five exceptional simple Lie algebras $G_2,\ F_4,\ E_6,\ E_7,\ E_8$. The second problem has been reduced to the description of orbits of certain unipotent linear groups [7]. The first problem is most complicated. There a marked difference is noted between the structural theory of semisimple Lie algebras and the structural theory of solvable or nilpotent Lie algebras. Just recall that the classification of all complex Lie algebras is obtained in dimension up to 6, and nilpotent complex Lie algebras are classified only in dimension up to 7. In more higher dimensions there are only partial classifications as subclasses of nilpotent Lie algebras. It seems the same scheme as above occurs in Leibniz algebras case as well. The counterpart of the problem (1) has been studied in [1,8,10–15] and others. The problem (2) for Leibniz algebras is still remaining untouched. This paper presents a progress made in the problem (3). It deals with the description of some classes of semisimple complex Leibniz algebras.

All algebras considered are supposed to be over the field of complex numbers $\mathbb{C}$.

2. Preliminaries

This section contains necessary definitions and preliminary results.

Definition 1. An algebra L is called Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds true.

Every Lie algebra is a Leibniz algebra, but the bracket in a Leibniz algebra need not be skew-symmetric.

Let L be a Leibniz algebra and $I = \langle [x, x] \mid x \in L\rangle$ be the ideal of L generated by all squares. Then I is the minimal ideal with respect to the property that $G := L/I$ is a Lie algebra. The quotient mapping $\pi : L \longrightarrow G$ is a homomorphism of Leibniz algebras.

The definition of simplicity for Leibniz algebras has been suggested by Dzhumadil'daev in [3] as algebra L having the only ideals $\{0\},\ I$ and L. However, in order to eschew the solvability of L, the reasonable definition of the simplicity must be as follows.

Definition 2. Leibniz algebra L (with $[L, L] \neq I$) is said to be simple if the only ideals of L are $\{0\},\ I$ and L.

Obviously, in the case when the Leibniz algebra L is Lie, the ideal I is trivial and this definition agrees with the classical definition of simple Lie algebra.

There were two papers so far dealing with the classification of Leibniz algebras with the quotient algebra L/I to be a Lie algebra. In the first case in [9] the quotient algebra L/I was supposed to be isomorphic to sl_2 and in the second paper [2] the authors considered the case when L/I is isomorphic to the direct sum of sl_2 and a two-dimensional solvable Lie algebra.

We shall make use the result of [9], where simple Leibniz algebras with the quotient Lie algebra L/I isomorphic to the classic three dimensional simple algebra sl_2 have been classified. It is well-known that the algebra sl_2 has a basis $\{e, f, h\}$ with the multiplication table

$$[e, h] = 2e, \quad [f, h] = -2f, \quad [e, f] = h,$$

$$[h, e] = -2e, \quad [h, f] = 2f, \quad [f, e] = -h.$$

Here is the result of [9] which we make use in the paper.

Theorem 3. *Let L be a complex $(m+4)$-dimensional simple Leibniz algebra and let I be the ideal generated by squares in L. Assume that the quotient L/I is isomorphic to the simple Lie algebra sl_2. Then there exist a basis $\{e, f, h, x_0, x_1, \dots, x_m\}$ of L such that non-zero products of basis vectors in L are represented as follows:*

$$\begin{array}{lll} [e,h]=2e, & [h,f]=2f, & [e,f]=h, \\ [h,e]=-2e, & [f,h]=-2f, & [f,e]=-h, \\ [x_k,h]=(m-2k)x_k, & 0 \leqslant k \leqslant m, & \\ [x_k,f]=x_{k+1}, & 0 \leqslant k \leqslant m-1, & \\ [x_k,e]=-k(m+1-k)x_{k-1}, & 1 \leqslant k \leqslant m. & \end{array}$$

We remind that in the paper we study Leibniz algebras L with the condition $L/I \cong sl_2 \oplus R$, where R is a three dimensional solvable Lie algebra. Here we suppose that the ideal I is an irreducible sl_2-module. The case when R is a two dimensional solvable Lie algebra has been given in [2]. It is clear that this is the most reasonable way to get examples of semisimple Leibniz algebras.

To make the combination $sl_2 \oplus R$ we need the classification of three-dimensional solvable Lie algebras. Such a classification can be found in [4] and it is as follows.

Theorem 4. *Let R be a three dimensional solvable non-split Lie algebra. Then R is isomorphic to one of the following pairwise non-isomorphic Lie algebras*

$$R_1 : [u,w]=u, \quad [v,w]=\alpha v, \quad \alpha \neq 0,$$

$$R_2 : [u,w]=u+v, \quad [v,w]=v.$$

Remark 5. It is observed that two algebras from the class R_1 with parameters α and α' are not isomorphic unless $\alpha\alpha' = 1$.

Let L be a Leibniz algebra with condition $L/I \cong sl_2 \oplus R$. Then without losing generality we may assume that the vector space $L = sl_2 + I + R$ has a basis

$$\{e, h, f, x_0, x_1, \dots, x_m, y_1, y_2, \dots, y_n\},$$

where $\{e, h, f\}$ is basis in sl_2, $\{x_0, x_1, \dots, x_m\}$ is basis in I and $\{y_1, y_2, \dots, y_n\}$ is basis in R. In fact, if $L = sl_2 + I + R$ then $L/I \cong \overline{sl_2} \oplus \overline{R}$. Further, $L/I \cong sl_2 \oplus R$ stands for $L/I \cong \overline{sl_2} \oplus \overline{R}$. As I being an irreducible module according to Theorem 3 the products of the basis vectors $\{e, h, f\}$ and $\{x_0, x_1, \dots, x_m\}$ are represented as follows

$$\begin{array}{ll} [x_k,h]=(m-2k)x_k, & 0 \leqslant k \leqslant m, \\ [x_k,f]=x_{k+1}, & 0 \leqslant k \leqslant m-1, \\ [x_k,e]=-k(m+1-k)x_{k-1}, & 1 \leqslant k \leqslant m. \end{array}$$

As for the composition laws for the other basis vectors, introducing notations a_{kl}^u, α_{pr}, β_{st} and γ_{qv} for the structure constants, we write them as follows

$$\begin{array}{lll} [e,h]=2e+\sum\limits_{j=0}^{m} a_{eh}^j x_j, & [h,f]=2f+\sum\limits_{j=0}^{m} a_{hf}^j x_j, & [e,f]=h+\sum\limits_{j=0}^{m} a_{ef}^j x_j, \\ [h,e]=-2e+\sum\limits_{j=0}^{m} a_{he}^j x_j, & [f,h]=-2f+\sum\limits_{j=0}^{m} a_{fh}^j x_j, & [f,e]=-h+\sum\limits_{j=0}^{m} a_{fe}^j x_j, \\ [e,y_i]=\sum\limits_{j=0}^{m} \alpha_{ij} x_j, & [f,y_i]=\sum\limits_{j=0}^{m} \beta_{ij} x_j, & [h,y_i]=\sum\limits_{j=0}^{m} \gamma_{ij} x_j \end{array}$$

However, it is not difficult to see that, with a slight correction of the basis, the table of multiplication for the basis vectors $\{e, h, f\}$ can be written as follows (for the details we refer to [9]):

$$\begin{aligned}&[e, h] = 2e, \quad [h, f] = 2f, \quad [e, f] = h,\\ &[h, e] = -2e, \ [f, h] = -2f, \ [f, e] = -h.\end{aligned}$$

3. Main result

This section is devoted to the description of complex finite dimensional Leibniz algebras whose corresponding Lie algebra is isomorphic to $sl_2 \oplus R$, where R is a three dimensional solvable Lie algebra. Due to Theorem 4 the solvable part R can be taken from the list given there. An observation shows that one has to consider two alternative cases: $dimI \neq 3$ and $dimI = 3$.

3.1. Case: $dimI \neq 3$.

If $dimI \neq 3$, then one has the following result.

Lemma 6. *Let L be a Leibniz algebra corresponding to $L/I \cong sl_2 \oplus R$, where R is the solvable ideal and I generated by squares as an irreducible sl_2-module then*

$$[sl_2, R] = 0.$$

Proof. It is clear that it is sufficient to prove the statement for the basis vectors

$$\{e, f, h\} \in sl_2 \text{ and } \{y_1, y_2, \dots, y_n\} \in R.$$

Let us consider the Leibniz identity for the triplet $\{e, e, y_i\}$. Then we get

$$\begin{aligned}[e, [e, y_i]] &= [[e, e], y_i] - [[e, y_i], e] = -[[e, y_i], e] = -\sum_{j=0}^{m} \alpha_{ij}[x_j, e]\\ &= -\sum_{j=1}^{m}(-mj + j(j-1))\alpha_{ij}x_{j-1}.\end{aligned}$$

Since $[e, [e, y_i]] = [e, \sum\limits_{j=0}^{m} \alpha_{ij}x_j] = 0$, we have $\alpha_{ij} = 0$ for $1 \leqslant j \leqslant m$ and hence, $[e, y_i] = \alpha_{i0}x_0$.

Analogously, considering

$$0 = \left[e, \sum_{j=0}^{m} \beta_{ij}x_j \right] = [e, [f, y_i]] = [[e, f], y_i] - [[e, y_i], f] = [h, y_i] - \alpha_{i0}[x_0, f] = [h, y_i] - \alpha_{i0}x_1,$$

we obtain $[h, y_i] = \alpha_{i0}x_1$.

The equality

$$\begin{aligned}0 &= [e, [h, y_i]] = [[e, h], y_i] - [[e, y_i], h] = 2[e, y_i] - \alpha_{i0}[x_0, h] = 2\alpha_{i0}x_0 - m\alpha_{i0}x_0\\ &= \alpha_{i0}(2 - m)x_0,\end{aligned}$$

yields $\alpha_{i0} = 0$ for $m \neq 2$. Therefore, $[e, y_i] = [h, y_i] = 0$.

Consider the identity

$$0 = [f, [e, y_i]] = [[f, e], y_i] - [[f, y_i], e] = -[h, y_i] - [[f, y_i], e] = -[[f, y_i], e]$$

$$= -\sum_{j=0}^{m}\beta_{ij}[x_j, e] = -\sum_{j=1}^{m}(-mj + j(j-1))\beta_{ij}x_{j-1}.$$

This implies $\beta_{ij} = 0$ for $1 \leqslant j \leqslant m$, as a result we get $[f, y_i] = \beta_{i0}x_0$.

From the identity

$$0 = [f, [f, y_i]] = [[f, f], y_i] - [[f, y_i], f] = -[[f, y_i], f] = -\beta_{i0}[x_0, f] = -\beta_{i0}x_1,$$

we obtain $[f, y_i] = 0$. Therefore, $[e, y_i] = [f, y_i] = [h, y_i] = 0$. $\square$

The following proposition describes Leibniz algebra structures on vector space L with the conditions $L/I \cong sl_2 \oplus R$ and $R \cong R_1$.

Proposition 7. *There exist a basis $\{e, h, f, x_0, x_1, \ldots, x_m, u, v, w\}$ in L such that the non-zero Leibniz brackets on L are given as follows*

$$\begin{array}{lll}
[e, h] = 2e, & [h, f] = 2f, & [e, f] = h, \\
[h, e] = -2e, & [f, h] = -2f, & [f, e] = -h, \\
[x_k, h] = (m-2k)x_k, & 0 \leqslant k \leqslant m, & \\
[x_k, f] = x_{k+1}, & 0 \leqslant k \leqslant m-1, & \\
[x_k, e] = -k(m+1-k)x_{k-1}, & 1 \leqslant k \leqslant m, & \\
[w, e] = \sum\limits_{j=0}^{m} a_{we}^{j}x_j, & [w, f] = \sum\limits_{j=0}^{m} a_{wf}^{j}x_j, & [w, h] = \sum\limits_{j=0}^{m} a_{wh}^{j}x_j, \\
[u, w] = u, & [w, u] = -u, & [v, w] = \alpha v, \\
[w, v] = -\alpha v, & [w, w] = \sum\limits_{j=0}^{m} a_{w}^{j}x_j, & [x_i, w] = ax_i, 0 \leqslant i \leqslant m.
\end{array}$$

where $m = dimI - 1$ and the omitted products are zero.

Proof. As it was mentioned above the products in the subspaces $[sl_2, sl_2]$ and $[I, sl_2]$ have the form as the table of multiplications in Theorem 3. In addition, for the cases $[R, sl_2]$, $[I, R]$ and $[R, R]$ we put

$$\begin{array}{lll}
[u, e] = \sum\limits_{j=0}^{m} a_{ue}^{j}x_j, & [v, e] = \sum\limits_{j=0}^{m} a_{ve}^{j}x_j, & [w, e] = \sum\limits_{j=0}^{m} a_{we}^{j}x_j, \\
[u, f] = \sum\limits_{j=0}^{m} a_{uf}^{j}x_j, & [v, f] = \sum\limits_{j=0}^{m} a_{vf}^{j}x_j, & [w, f] = \sum\limits_{j=0}^{m} a_{wf}^{j}x_j, \\
[u, h] = \sum\limits_{j=0}^{m} a_{uh}^{j}x_j, & [v, h] = \sum\limits_{j=0}^{m} a_{vh}^{j}x_j, & [w, h] = \sum\limits_{j=0}^{m} a_{wh}^{j}x_j, \\
[x_i, u] = \sum\limits_{j=0}^{m} a_{ui}^{j}x_j, & [x_i, v] = \sum\limits_{j=0}^{m} a_{vi}^{j}x_j, & [x_i, w] = \sum\limits_{j=0}^{m} a_{wi}^{j}x_j, \\
[u, u] = \sum\limits_{j=0}^{m} a_{u}^{j}x_j, & [u, v] = \sum\limits_{j=0}^{m} a_{uv}^{j}x_j, & [u, w] = u + \sum\limits_{j=0}^{m} a_{uw}^{j}x_j, \\
[v, u] = \sum\limits_{j=0}^{m} a_{vu}^{j}x_j, & [v, v] = \sum\limits_{j=0}^{m} a_{v}^{j}x_j, & [v, w] = \alpha v + \sum\limits_{j=0}^{m} a_{vw}^{j}x_j, \\
[w, u] = -u + \sum\limits_{j=0}^{m} a_{wu}^{j}x_j, & [w, v] = -\alpha v + \sum\limits_{j=0}^{m} a_{wv}^{j}x_j, & [w, w] = \sum\limits_{j=0}^{m} a_{w}^{j}x_j.
\end{array}$$

Taking the following change of basis

$$u' = u - \sum_{j=0}^{m} a_{wu}^{j}x_j, \quad v' = \alpha v - \sum_{j=0}^{m} a_{wv}^{j}x_j,$$

we obtain

$$[u, w] = u + \sum_{j=0}^{m} a_{uw}^{j} x_j, \quad [w, u] = -u,$$
$$[v, w] = \alpha v + \sum_{j=0}^{m} a_{vw}^{j} x_j, \quad [w, v] = -\alpha v.$$

Consider the following equalities

$$\begin{aligned}[x_i, [h, u]] &= [[x_i, h], u] - [[x_i, u], h] = (m-2i)[x_i, u] - \sum_{k=0}^{m} a_{ui}^{k}[x_k, h] \\ &= (m-2i)\sum_{k=0}^{m} a_{ui}^{k} x_k - \sum_{k=0}^{m} a_{ui}^{k}(m-2k)x_k = \sum_{k=0}^{m} a_{ui}^{k}(m-2i-(m-2k))x_k \\ &= \sum_{k=0}^{m} 2a_{ui}^{k}(k-i)x_k.\end{aligned}$$

On the other hand, due to Lemma 6 we get $[x_i, [h, u]] = 0$. So, $a_{ui}^{k} = 0$ for $i \neq k$, which implies $[x_i, u] = a_{ui}^{i} x_i$. In order to simplify the notation we shall write a_{ui} instead of a_{ui}^{i}. By using the similar relations as above for $[x_i, [h, v]]$ and $[x_i, [h, w]]$ we get

$$[x_i, v] = a_{vi} x_i, \quad [x_i, w] = a_{wi} x_i.$$

In virtue of the identity

$$[x_i, [u, w]] = [[x_i, u], w] - [[x_i, w], u] = [a_{ui} x_i, w] - [a_{wi} x_i, u] = a_{ui} a_{wi} x_i - a_{wi} a_{ui} x_i = 0,$$

bearing in mind

$$[x_i, [u, w]] = \left[x_i, u + \sum_{k=0}^{m} a_{uw}^{k} x_k\right] = [x_i, u],$$

we obtain $[x_i, u] = 0$.

By applying the similar identity as above for $[x_i, [v, w]]$ we get $[x_i, v] = 0$.

The identity

$$\begin{aligned}0 = [x_i, [w, e]] &= [[x_i, w], e] - [[x_i, e], w] = a_{wi}[x_i, e] - (-mi + i(i-1))[x_{i-1}, w] \\ &= a_{wi}(-mi + i(i-1))x_{i-1} - a_{w,i-1}(-mi + i(i-1))x_{i-1} \\ &= (-mi + i(i-1))(a_{wi} - a_{w,i-1})x_{i-1},\end{aligned}$$

implies $a_{wi} = a_{w,i-1} = a$, that is $[x_i, w] = ax_i$, where $0 \leqslant i \leqslant m$. Therefore, we obtain the only non-zero products $[x_i, w] = ax_i$ for $[I, R]$.

Let us now treat $[R, R]$. Consider the identity

$$[[u, u], f] = [u, [u, f]] + [[u, f], u] = \left[u, \sum_{j=0}^{m} a_{uf}^{j} x_j\right] + \left[\sum_{j=0}^{m} a_{uf}^{j} x_j, u\right] = 0.$$

Due to

$$[[u, u], f] = \sum_{j=0}^{m} a_u^{j}[x_j, f] = \sum_{j=0}^{m-1} a_u^{j} x_{j+1},$$

along with $a_u^{j} = 0$ for $j \neq m$ we get $[u, u] = a_u^{m} x_m$.

Therefore, with regard for the identity

$$0=[u,[u,h]]+[[u,h],u]=[[u,u],h]=a_u^m[x_m,h]=-ma_u^m x_m,$$

we obtain $[u,u]=0$.

Analogously, by using the Leibniz identity for $[[v,v],f]$ and $[[v,v],h]$ as above it is easy to see that $[v,v]=0$.

Further, due to the identity

$$0=[v,[u,f]]+[[v,f],u]=[[v,u],f]=\sum_{j=0}^{m}a_{vu}^j[x_j,f]=\sum_{j=0}^{m-1}a_{vu}^j x_{j+1},$$

we get $a_{vu}^j=0$ for $j\neq m$, which gives $[v,u]=a_{vu}^m x_m$.

The identity

$$0=[v,[u,h]]+[[v,h],u]=[[v,u],h]=a_{vu}^m[x_m,h]=-ma_{vu}^m x_m,$$

gives $[v,u]=0$.

Applying again the Leibniz identity to $[[u,v],f]$ and $[[u,v],h]$ it is easy to see that $[u,v]=0$.

It is observed that, from the chain of the following equalities

$$\begin{aligned}
0&=[w,[u,e]]+[[w,e],u]=[[w,u],e]=-[u,e],\\
0&=[w,[u,f]]+[[w,f],u]=[[w,u],f]=-[u,f],\\
0&=[w,[u,h]]+[[w,h],u]=[[w,u],h]=-[u,h],\\
0&=[w,[v,e]]+[[w,e],v]=[[w,v],e]=-\alpha[v,e],\\
0&=[w,[v,f]]+[[w,f],v]=[[w,v],f]=-\alpha[v,f],\\
0&=[w,[v,h]]+[[w,h],v]=[[w,v],h]=-\alpha[v,h],
\end{aligned}$$

we obtain $[R,sl_2]$, as:

$$\begin{aligned}
&[u,e]=[u,f]=[u,h]=[v,e]=[v,f]=[v,h]=0,\\
&[w,e]=\sum_{j=0}^{m}a_{we}^j x_j,\quad [w,f]=\sum_{j=0}^{m}a_{wf}^j x_j,\quad [w,h]=\sum_{j=0}^{m}a_{wh}^j x_j,
\end{aligned}$$

To finish the proof of the proposition it is sufficient to derive the products

$$\begin{aligned}
&[u,w]=u,\qquad [w,u]=-u,\qquad [v,w]=\alpha v,\\
&[w,v]=-\alpha v,\ \ [w,w]=\sum_{j=0}^{m}a_w^j x_j.
\end{aligned}$$

Owing to the identity

$$0=[u,[w,f]]+[[u,f],w]=[[u,w],f]=[u,f]+\sum_{j=0}^{m}a_{uw}^j[x_j,f]=\sum_{j=0}^{m-1}a_{uw}^j x_{j+1},$$

we get $a_{uw}^j=0$ for $j\neq m$ and this implies that $[u,w]=u+a_{uw}^m x_m$.

The identity

$$0=[u,[w,h]]+[[u,h],w]=[[u,w],h]=[u+a_{uw}^m x_m,h]=-ma_{uw}^m x_m,$$

gives $[u,w]=u$.

Analogously, from $[[v,w],f]$ and $[[v,w],h]$ one gets $[v,w]=\alpha v$. $\square$

Let now R be isomorphic to R_2. Then one has

Proposition 8. *Let R be isomorphic to R_2 and $dimI \neq 3$, then there exists a basis $\{e, h, f, x_0, x_1, \dots, x_m, u, v, w\}$ of the vector space L such that the Leibniz algebra structure on L is defined as follows*

$$\begin{array}{lll}
[e, h] = 2e, & [h, f] = 2f, & [e, f] = h, \\
[h, e] = -2e, & [f, h] = -2f, & [f, e] = -h, \\
[x_k, h] = (m-2k)x_k, & 0 \leqslant k \leqslant m, & \\
[x_k, f] = x_{k+1}, & 0 \leqslant k \leqslant m-1, & \\
[x_k, e] = -k(m+1-k)x_{k-1}, & 1 \leqslant k \leqslant m, & \\
[w, e] = \sum\limits_{j=0}^{m} a_{we}^j x_j, & [w, f] = \sum\limits_{j=0}^{m} a_{wf}^j x_j, & [w, h] = \sum\limits_{j=0}^{m} a_{wh}^j x_j, \\
[u, w] = u + v, & [w, u] = -u - v, & [v, w] = v, \\
[w, v] = -v, & [w, w] = \sum\limits_{j=0}^{m} a_{w}^j x_j, & [x_i, w] = a x_i, \ 0 \leqslant i \leqslant m.
\end{array}$$

where $m = dimI - 1$ and the omitted products are zero.

Proof. The compositions $[sl_2, sl_2]$ and $[I, sl_2]$ are obtained similarly to that of Proposition 7 and Theorem 3, respectively. To describe $[R, sl_2]$, $[I, R]$ and $[R, R]$ we suppose that

$$\begin{array}{lll}
[u, e] = \sum\limits_{j=0}^{m} a_{ue}^j x_j, & [v, e] = \sum\limits_{j=0}^{m} a_{ve}^j x_j, & [w, e] = \sum\limits_{j=0}^{m} a_{we}^j x_j, \\
[u, f] = \sum\limits_{j=0}^{m} a_{uf}^j x_j, & [v, f] = \sum\limits_{j=0}^{m} a_{vf}^j x_j, & [w, f] = \sum\limits_{j=0}^{m} a_{wf}^j x_j, \\
[u, h] = \sum\limits_{j=0}^{m} a_{uh}^j x_j, & [v, h] = \sum\limits_{j=0}^{m} a_{vh}^j x_j, & [w, h] = \sum\limits_{j=0}^{m} a_{wh}^j x_j, \\
[x_i, u] = \sum\limits_{j=0}^{m} a_{ui}^j x_j, & [x_i, v] = \sum\limits_{j=0}^{m} a_{vi}^j x_j, & [x_i, w] = \sum\limits_{j=0}^{m} a_{wi}^j x_j, \\
[u, u] = \sum\limits_{j=0}^{m} a_{u}^j x_j, & [u, v] = \sum\limits_{j=0}^{m} a_{uv}^j x_j, & [u, w] = u + v + \sum\limits_{j=0}^{m} a_{uw}^j x_j, \\
[v, u] = \sum\limits_{j=0}^{m} a_{vu}^j x_j, & [v, v] = \sum\limits_{j=0}^{m} a_{v}^j x_j, & [v, w] = v + \sum\limits_{j=0}^{m} a_{vw}^j x_j, \\
[w, u] = -u - v + \sum\limits_{j=0}^{m} a_{wu}^j x_j, & [w, v] = -v + \sum\limits_{j=0}^{m} a_{wv}^j x_j, & [w, w] = \sum\limits_{j=0}^{m} a_{w}^j x_j.
\end{array}$$

Taking the change of basis

$$u' = u - \sum_{j=0}^{m} a_{wu}^j x_j + \sum_{j=0}^{m} a_{wv}^j x_j, \quad v' = v - \sum_{j=0}^{m} a_{wv}^j x_j,$$

we derive

$$[w, u] = -u - v, \quad [w, v] = -v.$$

Analogously to the proof of Proposition 7 applying the Leibniz identity to $[x_i, [h, u]]$, $[x_i, [h, v]]$ and $[x_i, [h, w]]$ we get

$$[x_i, u] = a_{ui} x_i, \quad [x_i, v] = a_{vi} x_i, \quad [x_i, w] = a_{wi} x_i.$$

Due to the identities

$$[x_i,[v,w]]=[[x_i,v],w]-[[x_i,w],v]=a_{vi}[x_i,w]-a_{wi}[x_i,v]=a_{vi}a_{wi}x_i-a_{wi}a_{vi}x_i=0,$$
$$[x_i,[v,w]]=\left[x_i, v+\sum_{j=0}^{m}a_{uw}^{j}x_j\right],$$

we obtain $[x_i,v]=0$.

Consider the identity

$$[x_i,[u,w]]=[[x_i,u],w]-[[x_i,w],u]=[a_{ui}x_i,w]-[a_{wi}x_i,u]=a_{ui}a_{wi}x_i-a_{wi}a_{ui}x_i=0.$$

By virtue of the relation

$$[x_i,[u,w]]=\left[x_i, u+v+\sum_{k=0}^{m}a_{uw}^{k}x_k\right]=[x_i,u],$$

we obtain $[x_i,u]=0$.

In such a way as in the proof of Proposition 7 considering the Leibniz identity for $[x_i,[w,e]]$, $[u,[u,f]]$, $[u,[u,h]]$, $[v,[v,f]]$, $[v,[v,h]]$, $[u,[v,f]]$, $[u,[v,h]]$, $[v,[u,f]]$, $[v,[u,h]]$ we have

$$[x_i,w]=ax_i,\ [u,u]=0,\ [v,v]=0,\ [v,u]=0,\ [u,v]=0.$$

Applying the Leibniz identity for

$$[[w,u],e],\ [[w,u],f],\ [[w,u],h],\ [[w,v],e],\ [[w,v],f],\ [[w,v],h],$$

we obtain

$$[u,e]=[u,f]=[u,h]=[v,e]=[v,f]=[v,h]=0,$$
$$[w,e]=\sum_{j=0}^{m}a_{we}^{j}x_j,\quad [w,f]=\sum_{j=0}^{m}a_{wf}^{j}x_j,\quad [w,h]=\sum_{j=0}^{m}a_{wh}^{j}x_j,$$

To end the proof of the proposition it is required to derive the following products

$$[u,w]=u+v,\ \ [w,u]=-u-v,\ \ [v,w]=v,$$
$$[w,v]=-v,\qquad [w,w]=\sum_{j=0}^{m}a_{w}^{j}x_j.$$

From the identity

$$0=[u,[w,f]]+[[u,f],w]=[[u,w],f]=[u+v,f]+\sum_{j=0}^{m}a_{uw}^{j}[x_k,f]=\sum_{j=0}^{m-1}a_{uw}^{j}x_{j+1},$$

we have $a_{uw}^{j}=0$ for $j\neq m$ and so $[u,w]=u+v+a_{uw}^{m}x_m$.

The identity

$$0=[u,[w,h]]+[[u,h],w]=[[u,w],h]=[u+v+a_{uw}^{m}x_m,h]=-ma_{uw}^{m}x_m,$$

gives $[u,w]=u+v$.

Analogously from $[[v,w],f]$ and $[[v,w],h]$ we can get $[v,w]=-v$. $\square$

The following theorem describes the Leibniz algebras L whose quotient L/I is isomorphic to $sl_2\oplus R$, where R is a three dimensional solvable Lie algebra and $dimI\neq 3$.

Theorem 9. *Let $L/I \cong sl_2 \oplus R$, $dimR = 3$ and $dimI \neq 3$. Then L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$
\begin{array}{llll}
L_1(\alpha, a): & [e,h] = 2e, & [h,f] = 2f, & [e,f] = h, \\
& [h,e] = -2e, & [f,h] = -2f, & [f,e] = -h, \\
& [x_k, h] = (m-2k)x_k, & 0 \leqslant k \leqslant m, & \\
& [x_k, f] = x_{k+1}, & 0 \leqslant k \leqslant m-1, & \\
& [x_k, e] = -k(m+1-k)x_{k-1}, & 1 \leqslant k \leqslant m, & \\
& [u,w] = u, & [w,u] = -u, & [v,w] = \alpha v, \\
& [w,v] = -\alpha v, & [x_i, w] = a x_i, & 0 \leqslant i \leqslant m.
\end{array}
$$

$$
\begin{array}{llll}
L_2(a): & [e,h] = 2e, & [h,f] = 2f, & [e,f] = h, \\
& [h,e] = -2e, & [f,h] = -2f, & [f,e] = -h, \\
& [x_k, h] = (m-2k)x_k, & 0 \leqslant k \leqslant m, & \\
& [x_k, f] = x_{k+1}, & 0 \leqslant k \leqslant m-1, & \\
& [x_k, e] = -k(m+1-k)x_{k-1}, & 1 \leqslant k \leqslant m, & \\
& [u,w] = u+v, & [w,u] = -u-v, & [v,w] = v, \\
& [w,v] = -v, & [x_i, w] = a x_i, & 0 \leqslant i \leqslant m.
\end{array}
$$

where $m = dimI - 1$ and the omitted products are zero.

Proof. Let L be an algebra satisfying the conditions of the theorem. According to Propositions 7 and 8 there are two classes such of algebras.
Case 1. Let R be isomorphic to R_1, then there exists a basis $\{e, h, f, x_0, x_1, \ldots, x_m, u, v, w\}$ of L such that the table of multiplications has the form:

$$
\begin{array}{lll}
[e,h] = 2e, & [h,f] = 2f, & [e,f] = h, \\
[h,e] = -2e, & [f,h] = -2f, & [f,e] = -h, \\
[x_k, h] = (m-2k)x_k, & 0 \leqslant k \leqslant m, & \\
[x_k, f] = x_{k+1}, & 0 \leqslant k \leqslant m-1, & \\
[x_k, e] = -k(m+1-k)x_{k-1}, & 1 \leqslant k \leqslant m, & \\
[w,e] = \sum\limits_{j=0}^{m} a_{we}^{j} x_j, & [w,f] = \sum\limits_{j=0}^{m} a_{wf}^{j} x_j, & [w,h] = \sum\limits_{j=0}^{m} a_{wh}^{j} x_j, \\
[u,w] = u, & [w,u] = -u, & [v,w] = \alpha v, \\
[w,v] = -\alpha v, & [w,w] = \sum\limits_{j=0}^{m} a_{w}^{j} x_j, & [x_i, w] = a x_i,\ 0 \leqslant i \leqslant m.
\end{array}
$$

where the omitted products are zero.

It is needed to show that $[w,e] = [w,f] = [w,h] = [w,w] = 0$.

Let $a \neq 0$, then taking the change of basis, $w' = w - \sum\limits_{j=0}^{m} \frac{a_w^j}{a} x_j$ it is not difficult to see that $[w', w'] = 0$.

By means of the identity

$$0 = [w, [w, h]] = [[w, w], h] - [[w, h], w] = -[[w, h], w] = -\sum_{j=0}^{m} a_{wh}^{j}[x_j, w] = -\sum_{j=0}^{m} a_{wh}^{j} a x_j,$$

we have $a_{wh}^{j} = 0$ for $0 \leqslant j \leqslant m$.

Analogously from the identities

$$0 = [w, [w, f]] = [[w, w], f] - [[w, f], w] = -\sum_{j=0}^{m} a_{wf}^{j}[x_j, w] = -\sum_{j=0}^{m} a_{wf}^{j} a x_j,$$

$$0 = [w, [w, e]] = [[w, w], e] - [[w, e], w] = -\sum_{j=0}^{m} a_{we}^{j}[x_j, w] = -\sum_{j=0}^{m} a_{we}^{j} a x_j,$$

we obtain $a_{wf}^{j} = 0$ and $a_{we}^{j} = 0$ for $0 \leqslant j \leqslant m$.

Therefore we get the restrictions

$$[w, e] = [w, f] = [w, h] = [w, w] = 0 \quad \text{for} \quad a \neq 0.$$

Let $a = 0$. Then from the identity

$$[w, [w, f]] = [[w, w], f] - [[w, f], w],$$

we have

$$0 = \sum_{j=0}^{m} a_{w}^{j}[x_j, f] = \sum_{j=0}^{m-1} a_{w}^{j} x_{j+1},$$

which implies $a_{w}^{j} = 0$, $0 \leqslant j \leqslant m-1$. Hence $[w, w] = a_{w}^{m} x_m$.

In accordance with the identity

$$0 = [w, [w, h]] = [[w, w], h] - [[w, h], w] = a_{w}^{m}[x_m, h] = -m a_{w}^{m} x_m,$$

we obtain $a_{w}^{m} = 0$, which implies $[w, w] = 0$.

Applying the change of basis

$$w' = w - \sum_{j=1}^{m} \frac{a_{we}^{j-1}}{-mj + j(j-1)} x_j,$$

we obtain

$$\begin{aligned}
[w', e] &= [w, e] - \sum_{j=1}^{m} \frac{a_{we}^{j-1}}{-mj + j(j-1)} [x_j, e] \\
&= [w, e] - \sum_{j=1}^{m} \frac{a_{we}^{j-1}}{-mj + j(j-1)} (-mj + j(j-1)) x_{j-1} \\
&= \sum_{j=0}^{m} a_{we}^{j} x_j - \sum_{j=1}^{m} a_{we}^{j-1} x_{j-1} = \sum_{j=0}^{m} a_{we}^{j} x_j - \sum_{j=0}^{m-1} a_{we}^{j} x_j = a_{we}^{m} x_m.
\end{aligned}$$

So, we can suppose that

$$[w, e] = a_{we}^{m} x_m, \; [w, h] = \sum_{j=0}^{m} a_{wh}^{j} x_j, \; [w, f] = \sum_{j=0}^{m} a_{wf}^{j} x_j.$$

Considering the identity

$$[w, [e, h]] = [[w, e], h] - [[w, h], e] = a_{we}^m[x_m, h] - \sum_{j=0}^{m} a_{wh}^j[x_j, e]$$
$$= -ma_{we}^m x_m - \sum_{j=1}^{m} a_{wh}^j(-mj + j(j-1))x_{j-1}$$

and taking into account $[w, [e, h]] = 2[w, e] = 2a_{we}^m x_m$, we find $a_{we}^m = 0$ and $a_{wh}^j = 0$, for $j \neq 0$.

Hence, $[w, e] = 0$, $[w, f] = \sum\limits_{j=0}^{m} a_{wf}^j x_j$, $[w, h] = a_{wh}^0 x_0$.

The identity

$$[w, [e, f]] = [[w, e], f] - [[w, f], e] = -\sum_{j=0}^{m} a_{wf}^j[x_j, e] = -\sum_{j=1}^{m} a_{wf}^j(-mj + j(j-1))x_{j-1}$$
$$= ma_{wf}^1 x_0 - \sum_{j=2}^{m} a_{wf}^j(-mj + j(j-1))x_{j-1},$$

along with $[w, [e, f]] = [w, h] = a_{wh}^0 x_0$ gives $a_{wh}^0 = ma_{wf}^1$ and $a_{wf}^j = 0$ where $j \geqslant 2$. Therefore,

$$[w, f] = a_{wf}^0 x_0 + a_{wf}^1 x_1.$$

The identity

$$-2[w, f] = [w, [f, h]] = [[w, f], h] - [[w, h], f] = [a_{wf}^0 x_0 + a_{wf}^1 x_1, h] - ma_{wf}^1[x_0, f],$$

implies

$$-2a_{wf}^0 x_0 - 2a_{wf}^1 x_1 = ma_{wf}^0 x_0 + a_{wf}^1(m-2)x_1 - ma_{wf}^1 x_1,$$

$$(m+2)a_{wf}^0 x_0 = 0 \Rightarrow a_{wf}^0 = 0.$$

Hence, $[w, f] = a_{wf}^1 x_1$ and $[w, h] = ma_{wf}^1 x_0$.

It is not difficult to see that applying the change of basis

$$w' = w - a_{wf}^1 x_0,$$

we get

$$[w', f] = [w, f] - a_{wf}^1[x_0, f] = a_{wf}^1 x_1 - a_{wf}^1 x_1 = 0,$$
$$[w', h] = [w, h] - a_{wf}^1[x_0, h] = ma_{wf}^1 x_0 - ma_{wf}^1 x_0 = 0.$$

Hence,

$$[w, e] = [w, f] = [w, h] = [w, w] = 0.$$

Case 2. Let R be isomorphic to R_2, then according to Proposition 8 there exists a basis $\{e, h, f, x_0, x_1, \dots, x_m, u, v, w\}$ in L with the following table of multiplications

$$\begin{array}{lll}
[e, h] = 2e, & [h, f] = 2f, & [e, f] = h, \\
[h, e] = -2e, & [f, h] = -2f, & [f, e] = -h, \\
[x_k, h] = (m-2k)x_k, & 1 \leqslant k \leqslant m, &
\end{array}$$

$$\begin{array}{lll}
[x_k, f] = x_{k+1}, & 0 \leqslant k \leqslant m-1, & \\
[x_k, e] = -k(m+1-k)x_{k-1}, & 1 \leqslant k \leqslant m, & \\
[w, e] = \sum\limits_{j=0}^{m} a_{we}^{j} x_j, & [w, f] = \sum\limits_{j=0}^{m} a_{wf}^{j} x_j, & [w, h] = \sum\limits_{j=0}^{m} a_{wh}^{j} x_j \\
[u, w] = u + v, & [w, u] = -u - v, & [v, w] = v, \\
[w, v] = -v, & [w, w] = \sum\limits_{j=0}^{m} a_{w}^{j} x_j, & [x_i, w] = a x_i, \ 0 \leqslant i \leqslant m.
\end{array}$$

where the omitted products are zero.

The details are similar to that of **Case 1.** As a result we get the class L_2. It is easy to see that for the different values of the parameter a one gets non-isomorphic to each other algebras. □

Remark 10. Two algebras from the class L_1 with parameters α and α' are not isomorphic unless $\alpha\alpha' = 1$.

3.2. Case: $dimI = 3$.

Let $L/I \cong sl_2 \oplus R$, where $dimR = 3$ and $dimI = 3$.

Again we distinguish two cases:

Let first R be isomorphic to R_1. Considering the Leibniz identity we obtain the following table of multiplication

$$\begin{array}{lll}
[e, h] = 2e, & [h, f] = 2f, & [e, f] = h, \\
[h, e] = -2e, & [f, h] = -2f, & [f, e] = -h, \\
[x_1, e] = -2x_0, & [x_2, e] = -2x_1, & [x_0, f] = x_1, \\
[x_1, f] = x_2, & [x_0, h] = 2x_0, & [x_2, h] = -2x_2, \\
[e, u] = \gamma_1 x_0, & [f, u] = \frac{1}{2}\gamma_1 x_2, & [h, u] = \gamma_1 x_1, \\
[e, v] = \gamma_2 x_0, & [f, v] = \frac{1}{2}\gamma_2 x_2, & [h, v] = \gamma_2 x_1, \\
[e, w] = \gamma_3 x_0, & [f, w] = \frac{1}{2}\gamma_3 x_2, & [h, w] = \gamma_3 x_1, \\
[u, w] = u, & [v, w] = \alpha v, & \\
[w, u] = -u, & [w, v] = -\alpha v, & \\
[x_0, w] = \theta x_0, & [x_1, w] = \theta x_1, & [x_2, w] = \theta x_2,
\end{array} \tag{1}$$

with the constraints

$$\gamma_1(1-\theta) = 0, \quad \gamma_2(\alpha - \theta) = 0.$$

Denote this class by $L(\gamma_1, \gamma_2, \gamma_3, \theta, \alpha)$. Then one has

Theorem 11. *Let $L/I \cong sl_2 \oplus R$, $dimI = 3$, $dimR = 3$ and R be isomorphic to R_1. Then L is isomorphic to one of the pairwise non-isomorphic algebras:*

$$L_1(1, 0, 0, 1, \alpha), \quad L_2(0, 0, 0, \theta, \alpha), \quad L_3(0, 0, 1, \theta, \alpha).$$

Proof. Let L be an algebra satisfying the conditions of the theorem. Then as has been mentioned above the table of multiplication of L is written as (1) with constraints

$$\gamma_1(1-\theta)=0, \quad \gamma_2(\alpha-\theta)=0.$$

Case 1. Let $\gamma_1 \neq 0$, then $\theta=1$. Taking the change of basis

$$u'=\frac{1}{\gamma_1}u, \quad v'=-\frac{\gamma_2}{\gamma_1}u+v, \quad w'=-\frac{\gamma_3}{\gamma_1}u+w,$$

we obtain $\gamma_2=\gamma_3=0$ and $\gamma_1=1$. Therefore, we get $L_1(1,0,0,1,\alpha)$.
Case 2. Let $\gamma_1=0$, then we have $\gamma_2(\alpha-\theta)=0$.
Case 2.1. Let $\gamma_2 \neq 0$, then $\theta=\alpha$. Applying the change of basis

$$u'=\frac{1}{\gamma_2}v, \quad v'=u, \quad w'=-\frac{\gamma_3}{\alpha\gamma_2}v+\frac{1}{\alpha}w,$$

one can get

$$\begin{aligned}
&[e,u']=x_0, && [f,u']=\tfrac{1}{2}x_2, && [h,u']=x_1,\\
&[e,v']=0, && [f,v']=0, && [h,v']=0,\\
&[e,w']=0, && [f,w']=0, && [h,w']=0,\\
&[x_0,w']=x_0, && [x_1,w']=x_1, && [x_2,w']=x_2.
\end{aligned}$$

Therefore, in this case L is written as $L_1(1,0,0,1,\alpha)$.
Case 2.2. Let $\gamma_2=0$. If $\gamma_3=0$, then we have $L_2(0,0,0,\theta,\alpha)$.
If $\gamma_3 \neq 0$, then taking the change of basis

$$x_0'=\gamma_3x_0, \quad x_1'=\gamma_3x_1, \quad x_2'=\gamma_3x_2,$$

we obtain $L_3(0,0,1,\theta,\alpha)$.
The conditions

$$\begin{aligned}
&L_1(1,0,0,1,\alpha): [sl_2,R]\neq 0,\ [sl_2,R^2]\neq 0,\\
&L_2(0,0,0,\theta,\alpha): [sl_2,R]=0,\ [sl_2,R^2]=0,\\
&L_3(0,0,1,\theta,\alpha): [sl_2,R]\neq 0,\ [sl_2,R^2]=0.
\end{aligned}$$

show that the algebras L_1-L_3 are not pairwise isomorphic. □

From Remark 5 it follows that if $\alpha\alpha'=1$, and $\theta'\alpha=\theta$ then the corresponding two algebras from the class L_2 are isomorphic. The analogous condition is true for algebras from the class L_3.
Let us now consider the case when R is isomorphic to R_2 and $dimI=3$.
Using the Leibniz identity we find

$$\begin{aligned}
&[e,h]=2e, && [h,f]=2f, && [e,f]=h,\\
&[h,e]=-2e, && [f,h]=-2f, && [f,e]=-h,\\
&[x_1,e]=-2x_0, && [x_2,e]=-2x_1, && [x_0,f]=x_1,\\
&[x_1,f]=x_2, && [x_0,h]=2x_0, && [x_2,h]=-2x_2,\\
&[e,u]=\alpha_1x_0, && [f,u]=\tfrac{1}{2}\alpha_1x_2, && [h,u]=\alpha_1x_1,\\
&[e,v]=\alpha_2x_0, && [f,v]=\tfrac{1}{2}\alpha_2x_2, && [h,v]=\alpha_2x_1,\\
&[e,w]=\alpha_3x_0, && [f,w]=\tfrac{1}{2}\alpha_3x_2, && [h,w]=\alpha_3x_1,\\
&[x_0,w]=\theta x_0, && [x_1,w]=\theta x_1, && [x_2,w]=\theta x_2,\\
&[u,e]=\beta_1x_0+\beta_2x_1, && [u,f]=-\tfrac{1}{2}\beta_1x_2, && [u,h]=\beta_2x_2,
\end{aligned}$$

$$
\begin{array}{lll}
[v,e]=\beta_3x_0+\beta_4x_1, & [v,f]=-\frac{1}{2}\beta_3x_2, & [v,h]=\beta_4x_2,\\
[w,e]=\beta_5x_0+\beta_2x_1, & [w,f]=-\frac{1}{2}\beta_5x_2, & [w,h]=\beta_2x_2,\\
[u,w]=u+v+\gamma_1x_1+\gamma_2x_2, & [v,w]=v+\gamma_3x_1+\gamma_4x_2, & [w,w]=\gamma_7x_1+\gamma_8x_2\\
[w,u]=-u-v+\gamma_5x_1+\gamma_6x_2, & [w,v]=-v+\gamma_9x_1+\gamma_{10}x_2, &
\end{array}
$$

Applying the change of basis $u'=u+\frac{1}{2}\beta_1x_1+\frac{1}{2}\beta_2x_2$ we find

$$
\begin{aligned}
[u',e]&=\left[u+\frac{1}{2}\beta_1x_1+\frac{1}{2}\beta_2x_2,e\right]=[u,e]+\frac{1}{2}\beta_1[x_1,e]+\frac{1}{2}\beta_2[x_2,e]\\
&=\beta_1x_0+\beta_2x_1-\beta_1x_0-\beta_2x_1=0,\\
[u',f]&=[u,f]+\frac{1}{2}\beta_1[x_1,f]+\frac{1}{2}\beta_2[x_2,f]=-\frac{1}{2}\beta_1x_2+\frac{1}{2}\beta_1x_2=0,\\
[u',h]&=[u,h]+\frac{1}{2}\beta_1[x_1,h]+\frac{1}{2}\beta_2[x_2,h]=\beta_2x_2-\beta_2x_2=0.
\end{aligned}
$$

Hence $[u',e]=[u',f]=[u',h]=0$.

Similarly taking the change of basis

$$v'=v+\frac{1}{2}\beta_3x_1+\frac{1}{2}\beta_4x_2,\quad w'=w+\frac{1}{2}\beta_5x_1+\frac{1}{2}\beta_2x_2,$$

we obtain $[v',e]=[v',f]=[v',h]=[w',e]=[w',f]=[w',h]=0$.

Using the Leibniz identity we get

$$
\begin{aligned}
&[u,[e,w]]=[[u,e],w]-[[u,w],e] \quad\Rightarrow\quad \gamma_1=\gamma_2=0\\
&[v,[e,w]]=[[v,e],w]-[[v,w],e] \quad\Rightarrow\quad \gamma_3=\gamma_4=0\\
&[w,[e,u]]=[[w,e],u]-[[w,u],e] \quad\Rightarrow\quad \gamma_5=\gamma_6=0\\
&[w,[e,v]]=[[w,e],v]-[[w,v],e] \quad\Rightarrow\quad \gamma_7=\gamma_8=0\\
&[w,[e,w]]=[[w,e],w]-[[w,w],e] \quad\Rightarrow\quad \gamma_9=\gamma_{10}=0
\end{aligned}
$$

Therefore, we obtain the class of algebras with the table of multiplications as follows

$$
\begin{array}{lll}
[e,h]=2e, & [h,f]=2f, & [e,f]=h,\\
[h,e]=-2e, & [f,h]=-2f, & [f,e]=-h,\\
[x_1,e]=-2x_0, & [x_2,e]=-2x_1, & [x_0,f]=x_1,\\
[x_1,f]=x_2 & [x_0,h]=2x_0, & [x_2,h]=-2x_2,\\
[e,u]=\alpha_1x_0, & [f,u]=\frac{1}{2}\alpha_1x_2, & [h,u]=\alpha_1x_1,\\
[e,v]=\alpha_2x_0, & [f,v]=\frac{1}{2}\alpha_2x_2, & [h,v]=\alpha_2x_1,\\
[e,w]=\alpha_3x_0, & [f,w]=\frac{1}{2}\alpha_3x_2, & [h,w]=\alpha_3x_1,
\end{array}
$$

$$
\begin{array}{lll}
[x_0,w]=\theta x_0, & [x_1,w]=\theta x_1, & [x_2,w]=\theta x_2,\\
[u,w]=u+v, & [v,w]=v, & \\
[w,u]=-u-v, & [w,v]=-v, &
\end{array}
$$

where

$$\alpha_1(1-\theta)=0,\quad \alpha_1+\alpha_2-\alpha_1\theta=0. \tag{2}$$

This class of algebras is denoted by $G(\alpha_1, \alpha_2, \alpha_3, \theta)$.

It is obvious, that due to the constraints (2), we get $\alpha_2 = 0$ and $\alpha_1(1-\theta) = 0$.

Theorem 12. *Let $L/I \cong sl_2 \oplus R$, $dimI = 3$, $dimR = 3$ and R be isomorphic to R_2. Then L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$G_1(1,0,0,1), \quad G_2(0,0,0,\theta), \quad G_3(0,0,1,\theta).$$

Proof. *Case 1.* Let $\alpha_1 \neq 0$ then $\theta = 1$. Taking the change of basis

$$u' = \frac{1}{\alpha_1}u, \quad v' = v, \quad w' = w - \frac{\alpha_3}{\alpha_1}u,$$

we get $\alpha_1 = 1, \alpha_3 = 0$, i.e., it is $G_1(1,0,0,1)$.

Case 2. Let $\alpha_1 = 0$. If $\alpha_3 = 0$, then we have $G_2(0,0,0,\theta)$.

If $\alpha_3 \neq 0$, then taking the change of basis

$$x_0' = \alpha_3 x_0, \quad x_1' = \alpha_3 x_1, \quad x_2' = \alpha_3 x_2,$$

we get $G_3(0,0,1,\theta)$.

The conditions

$$G_1(1,0,0,1): \ [sl_2, R] \neq 0, \ [sl_2, R^2] \neq 0,$$
$$G_2(0,0,0,\theta): \ [sl_2, R] = 0, \ [sl_2, R^2] = 0,$$
$$G_3(0,0,1,\theta): \ [sl_2, R] \neq 0, \ [sl_2, R^2] = 0.$$

imply that the algebras $G_1 - G_3$ are not pairwise isomorphic. $\square$

Table 1
List of Isomorphism classes with representatives.

	$dim\ I = 3$	$dim\ I \neq 3$
R_1	$L_1(1,0,0,1,\alpha)$ $L_2(0,0,0,\theta,\alpha)$ $L_3(0,0,1,\theta,\alpha)$	$L_1(\alpha, a)$
R_2	$G_1(1,0,0,1)$ $G_2(0,0,0,\theta)$ $G_3(0,0,1,\theta)$	$L_2(a)$

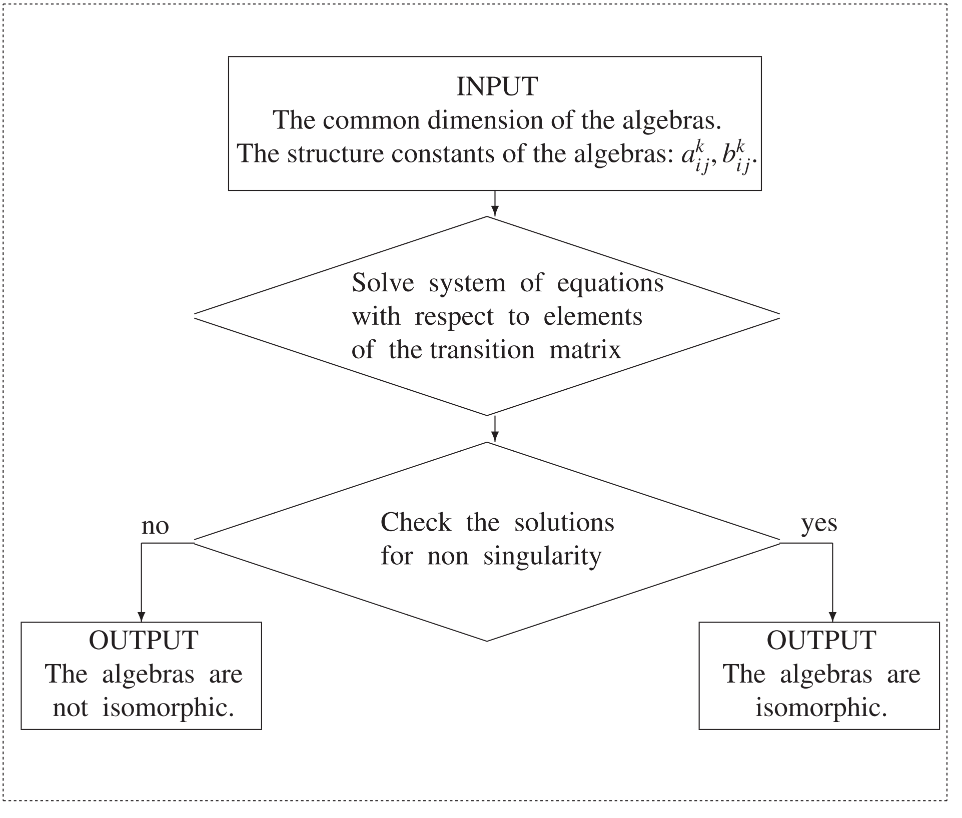
Table 1 presents the list of isomorphism classes of complex Leibniz algebras with L/I isomorphic to the direct sum of sl_2 and a three-dimensional solvable Lie algebra.

Remark 13. For some computations in the paper we used the Mathematica software.

- We have double checked by using the Mathematica software that the algebras from Theorem 9 in low dimensional cases for different values of the parameter a are not isomorphic to each other.
- In finding the table of multiplications (1) we used the Mathematica software.
- On the page 2223 to show that $\gamma_i = 0$, for $i = 1, 2, \ldots, 10$ also a computer program has been used.

3.3. *The computer program*

In this section we provide a computer program that checks if two algebras from Theorem 9 are isomorphic. The structure constants of the given algebras are denoted by $a[i, j, k]$ and $b[i, j, k]$, respectively. Here is a block scheme for the program which is followed by the program in Wolfram Mathematica 7.




```
dim = 8; (*Dimension of the algebra*)
changebasis = Table[c[i, j], {i, 1, dim}, {j, 1, dim}];
firstalgebra =
  Table[a[i, j, k], {i, 1, dim}, {j, 1, dim}, {k, 1, dim}];
secondalgebra =
  Table[b[i, j, k], {i, 1, dim}, {j, 1, dim}, {k, 1, dim}];

(*First Algebra*)
For[i = 1, i \[LessSlantEqual] dim, i++,
  For[j = 1, j <= dim, j++,
   For[k = 1, k \[LessSlantEqual] dim, k++,
    a[i, j, k] := 0]]];

a[1, 3, 1] = 2; a[3, 2, 2] = 2; a[1, 2, 3] = 1; a[3, 1, 1] = -2;
a[2, 3, 2] = -2; a[2, 1, 3] = -1; a[7, 3, 7] = 1; a[8, 3, 8] = -1;
```

```
a[7, 2, 8] = 1; a[8, 1, 7] = -1; a[4, 6, 4] = 1; a[4, 6, 5] = 1;
a[6, 4, 4] = -1; a[6, 4, 5] = -1; a[5, 6, 5] = 1; a[6, 5, 5] = -1;
a[7, 6, 7] = 1; a[8, 6, 8] = 1;

(*Second Algebra*)
For[i = 1, i \[LessSlantEqual] dim, i++,
  For[j = 1, j <= dim, j++,
   For[k = 1, k \[LessSlantEqual] dim, k++,
    b[i, j, k] := 0]]];

b[1, 3, 1] = 2; b[3, 2, 2] = 2; b[1, 2, 3] = 1; b[3, 1, 1] = -2;
b[2, 3, 2] = -2; b[2, 1, 3] = -1; b[7, 3, 7] = 1; b[8, 3, 8] = -1;
b[7, 2, 8] = 1; b[8, 1, 7] = -1; b[4, 6, 4] = 1; b[4, 6, 5] = 1;
b[6, 4, 4] = -1; b[6, 4, 5] = -1; b[5, 6, 5] = 1; b[6, 5, 5] = -1;
b[7, 6, 7] = 2; b[8, 6, 8] = 2;

For[i = 1, i <= dim, i++,
 For[j = 1, j <= dim, j++,
  For[k = 1, k <= dim, k++,
   expression[i, j, k] := \!\(
\*UnderoverscriptBox[\(\[Sum]\), \(s = 1\), \(dim\)]\(
\*UnderoverscriptBox[\(\[Sum]\), \(m = 1\), \(dim\)]\((c[i, m]*
         c[j, s]*a[m, s, k])\)\)\) - \!\(
\*UnderoverscriptBox[\(\[Sum]\), \(p = 1\), \(dim\)]\(c[p, k]*
       b[i, j, p]\)\)]]]

(*solve system*)
system =
  Union[Select[
    Flatten[Table[
      expression[i, j, k], {i, 1, dim}, {j, 1, dim}, {k, 1, dim}]], !
      NumberQ[#] &]];
equations =
  Map[(# == 0) &, GroebnerBasis[system, Flatten[changebasis]]];
Off[Solve::"svars"]; Off[Solve::"verif"];
listsol =
  Union[Solve[
    Reduce[Union[equations, {c[1, 1] \neq 0}], Flatten[changebasis]],
    Flatten[changebasis]]];

(*we compute the determinant of all solutions*)
For[u = 1, u <= Length[listsol], u++,
  matriz[u_Integer] :=
   Simplify[Table[c[i, j], {i, 1, dim}, {j, 1, dim}] /. listsol[[u]]]
  ];

For[u = 1, u <= Length[listsol], u++,
  determinant[u_Integer] := Det[matriz[u]]];

For[u = 1, u <= Length[listsol], u++,
 If[! NumberQ[determinant[u]] || determinant[u] \neq 0,
  Print["Isomorphic Algebras"], Print["NonIsomorphic Algebras"]]]
```

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References

[1] S. Albeverio, B.A. Omirov, I.S. Rakhimov, Varieties of nilpotent complex Leibniz algebras of dimension less than five, Comm. Algebra 33 (2005) 1575–1585.
[2] L.M. Camacho, S. Gomez-Vidal, B.A. Omirov, Leibniz algebras with associated Lie algebras $sl_2 \dot{+} R$ ($dimR = 2$), 2011, arXiv:1111.4631v1 [math.RA].
[3] A.S. Dzhumadil'daev, S.A. Abdykassymova, Leibniz algebras in characteristic p, C. R. Acad. Sci. Paris Sér. I Math. 332 (12) (2001) 1047–1052.
[4] N. Jacobson, Lie Algebras, Interscience Publishers, Wiley, New York, 1962.
[5] J.-L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, Enseign. Math. 39 (1993) 269–293.
[6] A.I. Malcev, On semisimple subgroups of Lie groups, Izv. AN SSSR Ser. Matem. 8 (4) (1944) 143–174.
[7] A.I. Malcev, On solvable Lie algebras, Izv. AN SSSR Ser. Matem. 9 (5) (1945) 329–356.
[8] B.A. Omirov, I.S. Rakhimov, On Lie-like complex filiform Leibniz algebras, Bull. Austral. Math. Soc. 79 (2009) 391–404.
[9] I.S. Rakhimov, B.A. Omirov, R.M. Turdibaev, On description of Leibniz algebras corresponding to sl_2, Algebr. Represent. Theory, in press.
[10] I.S. Rakhimov, U.D. Bekbaev, On isomorphism classes and invariants of finite dimensional complex filiform Leibniz algebras, Comm. Algebra 38 (12) (2010) 4705–4738.
[11] I.S. Rakhimov, M.A. Hassan, On one-dimensional central extension of a filiform Lie algebra, Bull. Austral. Math. Soc. 83 (3) (2011) 205–224.
[12] I.S. Rakhimov, M.A. Hassan, On isomorphism criteria for Leibniz central extensions of a linear deformation of μ_n, Internat. J. Algebra Comput. 21 (5) (2011) 715–729.
[13] I.S. Rakhimov, S.K. Said Husain, On isomorphism classes and invariants of low dimensional complex filiform Leibniz algebras, Linear and Multilinear Algebra 59 (2) (2011) 205–220.
[14] I.S. Rakhimov, S.K. Said Husain, Classification of a subclass of nilpotent Leibniz algebras, Linear and Multilinear algebra 59 (3) (2011) 339–354.
[15] I.S. Rakhimov, M.A. Hassan, On low-dimensional filiform Leibniz algebras and their invariants, Bull. Malaysian Math. Sci. Soc. 34 (3) (2011) 475–485.