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On isomorphism classes and invariants of a subclass of low-dimensional complex filiform Leibniz algebras

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The article aims to study the classification problem of low-dimensional complex filiform Leibniz algebras. It is known that filiform Leibniz algebras come out from two sources. The first source is a naturally graded non-Lie filiform Leibniz algebra, and another one is a naturally graded filiform Lie algebra. In this article, we classify a subclass of the class of filiform Leibniz algebras appearing from the naturally graded non-Lie filiform Leibniz algebra. We give complete classification and isomorphism criteria in dimensions 5–7. The method of classification is purely algorithmic. The isomorphism criteria are given in terms of invariant functions.

Keywords: filiform Leibniz algebra; invariant function; isomorphism

AMS Subject Classifications: 17A32; 17B30(primary); 13A50(secondary)

1. Introduction

In the early 1990s, Loday [10] introduced Leibniz algebra as a non-associative algebra with multiplication, satisfying the Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

This identity and the classical Jacobi identity are equivalent, when the multiplication is skewsymmetric. Leibniz algebras appear to be related, in a natural way, to several topics, such as differential geometry and homological algebra, classical algebraic topology, algebraic *K*-theory, loop spaces, noncommutative geometry and quantum physics, as a generalization of the corresponding applications of Lie algebras to these topics. Most papers deal with the homological problems of Leibniz algebras. In [11] Loday and Pirashvili have described the free Leibniz algebras, paper [13] by Mikhalev and Umirbaev is devoted to solution of the non-commutative analogue of the Jacobian conjecture in the affirmative for free Leibniz algebras, in the spirit of the corresponding result of Reutenauer [15], Shpilrain [16] and Umirbaev [17]. The problems concerning Cartan subalgebras and solvability were studied by Ayupov and Omirov [1]. The notion of simple Leibniz algebra was suggested by Dzhumadil'daev and Abdukassymova [7], who obtained some results concerning special cases of simple Leibniz algebras.

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Unfortunately, up to now, there is no paper that has included complete discussion on comparisons of structural theory of Lie and Leibniz algebras (one means results like Levi-Malcev decomposition theorem, Lie–Engel theorem, Malcev reduction theorem, the analogue of Killing form, Dinkin diagrams, root space decompositions, the Serre presentation, the theory of highest weight representations, the Weyl character formula and much more).

Papers [2,4,14] and preprints [5,6,9] concern the classification problem of filiform Leibniz algebras.

The organization of this article is as follows. Section 2 consists of some facts on filiform Leibniz algebras. They can be found in [4]. In Section 3, the reader will be reminded regarding adapted basis and adapted transformations. Then we describe the isomorphism action of the adapted transformations group. Section 4 deals with the classification problem of low-dimensional filiform Leibniz algebras. Here for 5- and 6-dimensional cases we give just final result since the proofs in these cases are similar to those of 7-dimensional case (for the last case in Section 4.3 we give complete proof only for generic case, since other cases can be managed by a minor adaption). In discrete orbits cases (Proposition 4.14) we give a base change, leading to appropriate canonical representative.

2. Preliminaries

Let V be a vector space of dimension n over an algebraically closed field K (char$K=0$). Bilinear maps $V\times V\to V$ form a vector space $\mathrm{Hom}(V\otimes V, V)$ of dimensional n^3, which can be considered together with its natural structure of an affine algebraic variety over K and denoted by $\mathrm{Alg}_n(K)\cong K^{n^3}$. An n-dimensional algebra structure L over K on V can be considered as an element $\lambda(L)$ of $\mathrm{Alg}_n(K)$ via the bilinear mapping $\lambda: V\otimes V\to V$ defining a binary algebraic operation on V: let $\{e_1, e_2, \ldots, e_n\}$ be a basis of the algebra L. Then, the table of multiplication of L is represented by point (γ_{ij}^k) of this affine space as follows:

$$\lambda(e_i, e_j) = \sum_{k=1}^{n} \gamma_{ij}^k e_k, \quad i,j = 1,2,\ldots,n.$$

Here γ_{ij}^k is said to be *structure constants* of L. The linear reductive group $GL_n(K)$ acts on $\mathrm{Alg}_n(K)$ by $(g*\lambda)(x,y)=g(\lambda(g^{-1}(x), g^{-1}(y)))$('transport of structure'). Two algebra structures λ_1 and λ_2 on V are isomorphic if and only if they belong to the same orbit under this action.

Definition 2.1 *An algebra L over a field K is called a Leibniz algebra, if its bilinear operation* $[\cdot,\cdot]$ *satisfies the following Leibniz identity*:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

Let $\mathrm{Leib}_n(K)$ be the subvariety of $\mathrm{Alg}_n(K)$, consisting of all n-dimensional Leibniz algebras over K. It is invariant under the above mentioned action of $GL_n(K)$. As a subset of $\mathrm{Alg}_n(K)$ the set $\mathrm{Leib}_n(K)$ is specified by a system of equations with respect to structure constants γ_{ij}^k:

$$\sum (\gamma_{jk}^l \gamma_{il}^m - \gamma_{ij}^l \gamma_{lk}^m + \gamma_{ik}^l \gamma_{lj}^m) = 0, \quad i,j,k = 1,2,\ldots,n.$$

It is easy to see that, if $[\cdot,\cdot]$ in Leibniz algebra is anticommutative, then it is a Lie algebra. Therefore, Leibniz algebra is a 'noncommutative' generalization of Lie algebra. For the Lie algebras case, several classifications of low-dimensional cases have been given. Except for simple Lie algebras, the classifications problem of all Lie algebras in common remains a big problem. Malcev [12] reduced the classification of solvable Lie algebras to the classification of nilpotent Lie algebras. Apparently, the first non-trivial classification of some classes of low-dimensional nilpotent Lie algebras are due to Umlauf. In his thesis [18], he presented the redundant list of nilpotent Lie algebras of dimension less than seven. He also gave the list of nilpotent Lie algebras of dimension less than 10 admitting a so-called adapted basis (now, the nilpotent Lie algebra with this property is called filiform Lie algebra). The importance of filiform Lie algebras in the study of the variety of nilpotent Lie algebras laws was shown by Vergne [19]. Paper [8] concerns the classification problem of low-dimensional filiform Lie algebras.

In [5,6], a method of classification of fixed dimensional filiform Leibniz algebras has been proposed. This article is an implementation of the method in low-dimensional cases.

Throughout the following sections, all algebras are assumed to be over the field of complex numbers $\mathbb{C}$.

Let L be a Leibniz algebra. We put

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \in \mathbb{N}.$$

Definition 2.2 *A Leibniz algebra L is said to be nilpotent, if there exists an integer $s \in \mathbb{N}$, such that*

$$L^s = \{0\}.$$

The smallest integer s for that $L^s = 0$ is called *the nilindex* of L.

Definition 2.3 *An n-dimensional Leibniz algebra L is said to be filiform, if*

$$\dim L^i = n - i, \text{ where } 2 \le i \le n.$$

It is clear that a filiform Leibniz algebra is nilpotent.

The class of filiform Leibniz algebras in dimension n is denoted by Leib_n.

In [9], Gómez and Omirov split Leib_n into three subclasses as follows.

THEOREM 2.1 *Any $(n+1)$-dimensional complex filiform Leibniz algebra admits a basis $\{e_0, e_1, \dots, e_n\}$ called adapted, such that the table of multiplication of the algebra has one of the following forms, where omitted products of basis vectors are supposed to be zero:*

$$FLeib_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1, \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + \cdots + \alpha_{n-1} e_{n-1} + \theta e_n, & \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + \cdots + \alpha_{n+1-j} e_n, & 1 \le j \le n-2, \\ \quad \alpha_3, \alpha_4, \dots, \alpha_n, \quad \theta \in \mathbb{C}. & \end{cases}$$

$$S\text{Leib}_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \le i \le n-1, \\ [e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + \cdots + \beta_n e_n, & \\ [e_1, e_1] = \gamma e_n, & \\ [e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + \cdots + \beta_{n+1-j} e_n, & 2 \le j \le n-2, \\ \quad \beta_3, \beta_4, \ldots, \beta_n, \gamma \in \mathbb{C}. & \end{cases}$$

$$T\text{Leib}_{n+1} = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le n-1, \\ [e_0, e_0] = b_{0,0} e_n, & \\ [e_0, e_1] = -e_2 + b_{0,1} e_n, & \\ [e_1, e_1] = b_{1,1} e_n, & \\ [e_i, e_j] = -[e_j, e_i] \in \text{span}_{\mathbb{C}}\{e_{i+j+1}, e_{i+j+2}, \ldots, e_n \mid & \\ \quad 1 \le i \le n-3,\ 2 \le j \le n-1-i\}, & \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^i b_{i,n-i} e_n, & \\ \quad \textit{where } a_{i,j}^k, b_{i,j} \in \mathbb{C} \textit{ and } b_{i,n-i} = b & \\ \quad \textit{whenever } 1 \le i \le n-1, \textit{ and } b = 0 \textit{ for even } n. & \end{cases}$$

In [9], the base change for this kind algebra has been reduced to the so-called adapted base change. The subclasses are stable with respect to the adapted base change. Hence, the isomorphisms problem inside of each class can be studied separately. This article deals with $F\text{Leib}_{n+1}$. Results on $SLeib_{n+1}$ and $TLeib_{n+1}$ have appeared elsewhere, particularly in [6,14].

Elements of $F\text{Leib}_{n+1}$ will be denoted by $L(\alpha_3, \alpha_4, \ldots, \alpha_n, \theta)$, which means that they are defined by the parameters $\alpha_3, \alpha_4, \ldots, \alpha_n, \theta$.

3. On adapted changing of basis and isomorphism criterion for $F\text{Leib}_{n+1}$

In this section, we simplify the isomorphic action of GL_n ('transport of structure') on $F\text{Leib}_n$. All the details can be found in [5,9].

Let L be an element of $F\text{Leib}_{n+1}$, V be the underlying vector space and $\{e_0, e_1, \ldots, e_n\}$ be the adapted basis of L.

Definition 3.1 The basis transformation $f \in GL(V)$ is said to be adapted for the structure of L, if the basis $\{f(e_0), f(e_1), \ldots, f(e_n)\}$ is adapted.

The closed subgroup of $GL(V)$ spanned by the adapted transformations is denoted by GL_{ad}. In GL_{ad} we consider the following types of basis transformations of $F\text{Leib}_{n+1}$ called elementary:

$$\text{first type, } - \tau(a, b, k) = \begin{cases} f(e_0) = e_0 + a e_k, & \\ f(e_1) = e_1 + b e_k, & \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \le i \le n-1, \quad 2 \le k \le n, \\ f(e_2) = [f(e_0), f(e_0)]. & \end{cases}$$

$$\text{second type} - \nu(a, b) = \begin{cases} f(e_0) = a e_0 + b e_1, & \\ f(e_1) = (a+b) e_1 + b(\theta - \alpha_n) e_{n-1}, & a(a+b) \ne 0, \\ f(e_{i+1}) = [f(e_i), f(e_0)], & 1 \le i \le n-1, \\ f(e_2) = [f(e_0), f(e_0)]. & \end{cases}$$

The proof of the following proposition is straightforward.

Proposition 3.1

(a) *If f is an adapted transformation of $F\text{Leib}_{n+1}$ then*

$$f = \tau(a_n, b_n, n) \circ \tau(a_{n-1}, a_{n-1}, n-1) \circ \cdots \circ \tau(a_2, a_2, 2) \circ \nu(a_0, a_1).$$

(b) *Transformations of the form* $\tau(a,b,n)$, $\tau(a,a,n)$ *and* $\tau(a,a,k)$, *where* $2\le k\le n-2$, $a\in\mathbb{C}$ *preserve the structure constants of* $FLeib_{n+1}$.

Since a composition of adapted transformations is adapted, the proposition above means that the transformation $\tau(a_n,b_n,n)\circ\tau(a_{n-1},a_{n-1},n-1)\circ\cdots\circ\tau(a_2,a_2,2)$ does not change the structure constants of $F\mathrm{Leib}_{n+1}$ and thus the action of GL_{ad} on $F\mathrm{Leib}_{n+1}$ can be reduced to the action of elementary transformation of the second type.

Let $R_a^m(x):=[[\dots[x,\underbrace{a],a],\dots,a]}_{m\text{-times}}$, and $R_a^0(x):=x$.

Now, due to Proposition 3.1(b), it is easy to see that, for $F\mathrm{Leib}_{n+1}$, the adapted change of basis has the following forms:

$$
\begin{aligned}
e_0' &= Ae_0+Be_1,\\
e_1' &= (A+B)e_1-B(\theta-\alpha_n)e_{n-1},\\
e_2' &= A(A+B)e_2+A(A+B)(\alpha_3e_3+\cdots+\alpha_{n-1}e_{n-1})+B(A\alpha_n+B\theta)e_n,\\
e_k' &= (A+B)\left(\sum_{i=0}^{k-2}C_{k-1}^{k-1-i}A^{k-1-i}B^iR_{e_1}^i(e_{k-i})+B^{k-1}R_{e_1}^{k-1}(e_0)\right),
\end{aligned}
$$

where $3\le k\le n$ and $A,B\in\mathbb{C}$ such that $A(A+B)\neq 0$.

Now we restate the isomorphism criterion for $F\mathrm{Leib}_{n+1}$. First, we introduce the following series of functions:

$$
\begin{aligned}
\varphi_t(y;z) &= \varphi_t(y;z_3,z_4,\dots,z_n,z_{n+1})\\
&= \Bigg((1+y)z_t-\sum_{k=3}^{t-1}(C_{k-1}^{k-2}yz_{t+2-k}+C_{k-1}^{k-3}y^2\sum_{i_1=k+2}^{t}z_{t+3-i_1}\cdot z_{i_1+1-k}\\
&\quad+C_{k-1}^{k-4}y^3\sum_{i_2=k+3}^{t}\sum_{i_1=k+3}^{i_2}z_{t+3-i_2}\cdot z_{i_2+3-i_1}\cdot z_{i_1-k}+\cdots\\
&\quad+C_{k-1}^{1}y^{k-2}\sum_{i_{k-3}=2k-2}^{t}\sum_{i_{k-4}=2k-2}^{i_{k-3}}\cdots\sum_{i_1=2k-2}^{i_2}z_{t+3-i_{k-3}}\cdot z_{i_{k-3}+3-i_{k-4}}\cdot\cdots\\
&\quad\cdot z_{i_2+3-i_1}\cdot z_{i_1+5-2k}+y^{k-1}\sum_{i_{k-2}=2k-1}^{t}\sum_{i_{k-3}=2k-1}^{i_{k-2}}\cdots\sum_{i_1=2k-1}^{i_2}z_{t+3-i_{k-2}}\\
&\quad\cdot z_{i_{k-2}+3-i_{k-3}}\cdot\cdots\cdot z_{i_2+3-i_1}\cdot z_{i_1+4-2k})\cdot\varphi_k(y;z)\Bigg),\quad\text{for } 3\le t\le n.
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{n+1}(y;z) &= \varphi_{n+1}(y;z_3,z_4,\dots,z_n,z_{n+1})\\
&= \Bigg(z_{n+1}+yz_n-(1+y)\sum_{k=3}^{n-1}(C_{k-1}^{k-2}yz_{n+2-k}+C_{k-1}^{k-3}y^2\\
&\quad\times\sum_{i_1=k+2}^{n}z_{n+3-i_1}\cdot z_{i_1+1-k}+C_{k-1}^{k-4}y^3\\
&\quad\times\sum_{i_2=k+3}^{n}\sum_{i_1=k+3}^{i_2}z_{n+3-i_2}\cdot z_{i_2+3-i_1}\cdot z_{i_1-k}+\cdots\\
&\quad+C_{k-1}^{1}y^{k-2}\sum_{i_{k-3}=2k-2}^{t}\sum_{i_{k-4}=2k-2}^{i_{k-3}}\cdots\sum_{i_1=2k-2}^{i_2}z_{n+3-i_{k-3}}\cdot z_{i_{k-3}+3-i_{k-4}}\cdot\cdots\\
&\quad\cdot z_{i_2+3-i_1}\cdot z_{i_1+5-2k}+y^{k-1}\sum_{i_{k-2}=2k-1}^{n}\sum_{i_{k-3}=2k-1}^{i_{k-2}}\cdots\sum_{i_1=2k-1}^{i_2}z_{n+3-i_{k-2}}\\
&\quad\cdot z_{i_{k-2}+3-i_{k-3}}\cdot\cdots\cdot z_{i_2+3-i_1}\cdot z_{i_1+4-2k})\cdot\varphi_k(y;z)\Bigg).
\end{aligned}
$$

The isomorphism criterion for $FLieb_{n+1}$ proven in [9] is then spelled out as follows.

THEOREM 3.1 [5] *Two algebras* $L(\alpha)$ *and* $L(\alpha')$ *from* $F\mathrm{Leib}_{n+1}$, *where* $\alpha=(\alpha_3, \alpha_4,\dots,\alpha_n,\theta)$ *and* $\alpha'=(\alpha_3{}',\alpha_4{}',\dots,\alpha_n{}',\theta')$, *are isomorphic, if and only if there exist complex numbers A and B, such that* $A(A+B)\neq 0$ *and the following conditions hold*:

$$\alpha_t' = \frac{1}{A^{t-2}}\varphi_t\left(\frac{B}{A};\alpha\right), \quad 3\le t\le n, \tag{1}$$

$$\theta = \frac{1}{A^{n-2}}\varphi_{n+1}\left(\frac{B}{A};\alpha\right). \tag{2}$$

To simplify notations, in the above case for transition from $(n+1)$-dimensional filiform Leibniz algebra $L(\alpha)$ to $(n+1)$-dimensional filiform Leibniz algebra $L(\alpha')$ we write $\alpha'=\rho(\frac{1}{A},\frac{B}{A};\alpha))$:

$$\rho\left(\frac{1}{A},\frac{B}{A};\alpha\right) = \left(\rho_1\left(\frac{1}{A},\frac{B}{A};\alpha\right),\rho_2\left(\frac{1}{A},\frac{B}{A};\alpha\right),\dots,\rho_{n-1}\left(\frac{1}{A},\frac{B}{A};\alpha\right)\right),$$

where

$$\rho_t(x,y;z) = x^t\varphi_{t+2}(y;z) \quad \text{for } 1\le t\le n-2,$$

and

$$\rho_{n-1}(x,y;z) = x^{n-2}\varphi_{n+1}(y;z).$$

Here are main properties of the operator ρ used in this article:

1′. $\rho(1,0,1;\cdot))$ is the identity operator.
2′. $\rho(\frac{1}{A_2},\frac{B_2}{A_2};\rho(\frac{1}{A_1},\frac{B_1}{A_1};\alpha)) = \rho(\frac{1}{A_1A_2},\frac{A_1B_2+A_2B_1+B_1B_2}{A_1A_2};\alpha)$.
3′ If $\alpha'=\rho(\frac{1}{A},\frac{B}{A};\alpha)$ then $\alpha=\rho(A,-\frac{B}{A+B};\alpha')$.

From here on, n is a positive integer. We assume that $n\ge 4$, since there are complete classifications of complex nilpotent Leibniz algebras of dimension of at most four [3].

In our study, we proceed from the viewpoint of [5]. Later on, if no confusion is possible, we write xy for $[x,y]$ as well.

4. Classification

In this section we classify algebras from $F\mathrm{Leib}_{n+1}$ for $n=4,5,6$. For the purpose of simplification, we establish the following notations and conventions:

$$\Delta_3=\alpha_3,\ \Delta_4=\alpha_4+2\alpha_3^2,\ \Delta_5=\alpha_5-5\alpha_3^3,\ \ \Delta_6=\alpha_6+14\alpha_3^4,\ \Theta_i=\theta-\alpha_i,$$

and

$$\Delta_3'=\alpha_3',\ \Delta_4'=\alpha_4'+2\alpha_3'^2,\ \Delta_5'=\alpha_5'-5\alpha_3'^3,$$
$$\Delta_6'=\alpha_6'+14\alpha_3'^4,\ \Theta_i'=\theta'-\alpha_i',\quad \text{for}\ \ i=4,5,6. \tag{3}$$

Observe that $\Delta_i = \alpha_i$ $(i = 4, 5, 6)$ as $\alpha_3 = 0$. After these notations, the algebra $L(\alpha_3, \alpha_4, \ldots, \alpha_n, \theta)$ from $F\text{Leib}_{n+1}$ becomes $L(\Delta_3, \Delta_4, \ldots, \Delta_n, \Theta_n)$.

4.1. *Dimension 5*

According to the above notations, one can rewrite Theorem 3.1 for $F\text{Leib}_5$ as follows.

Theorem 4.1 *Two algebras $L(\Delta)$ and $L(\Delta')$ from $F\text{Leib}_5$, where $\Delta = (\Delta_3, \Delta_4, \Theta_4)$ and $\Delta' = (\Delta_3', \Delta_4', \Theta_4')$, are isomorphic, if and only if there exist complex numbers A and B, such that $A(A+B) \neq 0$ and the following conditions hold*:

$$\begin{aligned}\Delta_3' &= \frac{1}{A}\left(1 + \frac{B}{A}\right)\Delta_3,\\ \Delta_4' &= \frac{1}{A^2}\left(1 + \frac{B}{A}\right)\Delta_4,\\ \Theta_4' &= \frac{1}{A^2}\Theta_4.\end{aligned} \tag{4}$$

In order to describe orbits in $F\text{Leib}_5$ under the action of the adapted base change, we split it into the following subsets:

$$\begin{aligned}U_1 &= \{L(\Delta) \in F\text{Leib}_5\colon \Delta_3 \neq 0, \Delta_4 \neq 0\},\\ U_2 &= \{L(\Delta) \in F\text{Leib}_5\colon \Delta_3 \neq 0, \Delta_4 = 0, \Theta_4 \neq 0\},\\ U_3 &= \{L(\Delta) \in F\text{Leib}_5\colon \Delta_3 \neq 0, \Delta_4 = 0, \Theta_4 = 0\},\\ U_4 &= \{L(\Delta) \in F\text{Leib}_5\colon \Delta_3 = 0, \Delta_4 \neq 0, \Theta_4 \neq 0\},\\ U_5 &= \{L(\Delta) \in F\text{Leib}_5\colon \Delta_3 = 0, \Delta_4 \neq 0, \Theta_4 = 0\},\\ U_6 &= \{L(\Delta) \in F\text{Leib}_5\colon \Delta_3 = 0, \Delta_4 = 0, \Theta_4 \neq 0\},\\ U_7 &= \{L(\Delta) \in F\text{Leib}_5\colon \Delta_3 = 0, \Delta_4 = 0, \Theta_4 = 0\}.\end{aligned}$$

It is obvious that $\{U_i\}$, $i = 1, 2, \ldots, 7$, is a partition of $F\text{Leib}_5$. The following proposition shows that U_1 is a union of infinitely many orbits and these orbits can be parametrized by $\mathbb{C}$.

Proposition 4.1

(i) *Two algebras $L(\Delta)$ and $L(\Delta')$ from U_1 are isomorphic if and only if*

$$\left(\frac{\Delta_3}{\Delta_4}\right)^2 \Theta_4 = \left(\frac{\Delta_3'}{\Delta_4'}\right)^2 \Theta_4'. \tag{5}$$

(ii) *Orbits in U_1 can be parametrized as $L(1, 1, \lambda)$, $\lambda \in \mathbb{C}$.*

The next proposition is a description of subsets U_i, $i = 2, \ldots, 7$.

PROPOSITION 4.2 *The subsets U_2, U_3, U_4, U_5, U_6 and U_7 are single orbits with the representatives $L(1,0,1)$, $L(1,0,0)$, $L(0,1,1)$, $L(0,1,0)$, $L(0,0,1)$ and $L(0,0,0)$, respectively.*

We summarize the above observations under the following classification theorem.

THEOREM 4.2 *Let L be an element of $F\mathrm{Leib}_5$. Then, it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

(1) $L(0,0,0) = L_5^s : e_0e_0 = e_2,\ e_ie_0 = e_{i+1},\ 1 \leq i \leq 3.$
(2) $L(0,0,1)$: $L_5^s,\ e_0e_1 = e_4.$
(3) $L(0,1,0)$: $L_5^s,\ e_0e_1 = e_4,\ e_1e_1 = e_4.$
(4) $L(0,1,1)$: $L_5^s,\ e_0e_1 = 2e_4,\ e_1e_1 = e_4.$
(5) $L(1,0,0)$: $L_5^s,\ e_0e_1 = e_3 - 2e_4,\ e_1e_1 = e_3 - 2e_4,\ e_2e_1 = e_4.$
(6) $L(1,0,1)$: $L_5^s,\ e_0e_1 = e_3 - e_4,\ e_1e_1 = e_3 - 2e_4,\ e_2e_1 = e_4.$
(7) $L(1,1,\lambda)$: $L_5^s,\ e_0e_1 = e_3 + (\lambda - 1)e_4,\ e_1e_1 = e_3 - e_4,\ e_2e_1 = e_4, \quad \lambda \in \mathbb{C}.$

4.2. *Dimension 6*

This section concerns $F\mathrm{Leib}_6$. According to the notations (3), we rephrase Theorem 3.1 as follows.

THEOREM 4.3 *Two algebras $L(\Delta)$ and $L(\Delta')$ from $F\mathrm{Leib}_6$, where $\Delta = (\Delta'_3, \Delta'_4, \Delta'_5, \Theta'_5)$ and $\Delta' = (\Delta'_3, \Delta'_4, \Delta'_5, \Theta'_5)$, are isomorphic, if and only if there exist complex numbers A and B, such that $A(A+B) \neq 0$ and the following conditions hold*:

$$\begin{aligned}
\Delta'_3 &= \frac{1}{A}\left(1 + \frac{B}{A}\right)\Delta_3, \\
\Delta'_4 &= \frac{1}{A^2}\left(1 + \frac{B}{A}\right)\Delta_4, \\
\Delta'_5 + 5\Delta'_3\Delta'_4 &= \frac{1}{A^3}\left(1 + \frac{B}{A}\right)(\Delta_5 + 5\Delta_3\Delta_4), \\
\Theta'_5 &= \frac{1}{A^3}\Theta_5.
\end{aligned} \tag{6}$$

To classify $F\mathrm{Leib}_6$, we represent it as a disjoint union of the following subsets:

$$\begin{aligned}
U_1 &= \{L(\Delta) \in F\mathrm{Leib}_6\colon \Delta_3 \neq 0, \Delta_4 \neq 0\}, \\
U_2 &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 \neq 0\}, \\
U_3 &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 = 0, \Delta_4 \neq 0, \Delta_5 \neq 0\}, \\
U_6 &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 = 0\}, \\
U_5 &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 \neq 0\}, \\
U_6 &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 = 0\}, \\
U_7 &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Theta_5 \neq 0\}, \\
U_8 &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Theta_5 = 0\}, \\
U_9 &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 \neq 0\}, \\
U_{10} &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Theta_5 = 0\}, \\
U_{11} &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 \neq 0\}, \\
U_{12} &= \{L(\Delta) \in F\mathrm{Leib}_6 : \Delta_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Theta_5 = 0\}.
\end{aligned}$$

Since each of these sets is a G_{ad}-stable, the isomorphism problem, for each of them, can be attacked separately. Here each of the subsets U_1, U_2 and U_3 turns out to be a union of infinitely many orbits and they can be described as follows.

Proposition 4.3

(i) *Two algebras* $L(\Delta)$ *and* $L(\Delta')$ *from* U_1 *are isomorphic if and only if*

$$\frac{\Delta_3(\Delta_5 + 5\Delta_3\Delta_4)}{\Delta_4^2} = \frac{\Delta_3'(\Delta_5' + 5\Delta_3'\Delta_4')}{\Delta_4'^2}, \tag{7}$$

and

$$\frac{\Delta_3^3\Theta_5}{\Delta_4^3} = \frac{\Delta_3'^3\Theta_5'}{\Delta_4'^3}. \tag{8}$$

(ii) *Orbits in* U_1 *can be parametrized as* $L(1, 1, \lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

Proposition 4.4

(i) *Two algebras* $L(\Delta)$ *and* $L(\Delta')$ *from* U_2 *are isomorphic if and only if*

$$\frac{\Delta_5^3}{\Delta_3^3\Theta_5^2} = \frac{\Delta_5'^3}{\Delta_3'^3\Theta_5'2}. \tag{9}$$

(ii) *Orbits in* U_2 *can be parametrized as* $L(1, 0, \lambda, \lambda)$, $\lambda \in \mathbb{C}^*$.

Proposition 4.5

(i) *Two algebras* $L(\Delta)$ *and* $L(\Delta')$ *from* U_3 *are isomorphic if and only if*

$$\frac{\Delta_4^3\Theta_5}{\Delta_5^3} = \frac{\Delta_4'^3\Theta_5'}{\Delta_5'^3}. \tag{10}$$

(ii) *Orbits in* U_3 *can be parameterized as* $L(0, 1, 1, \lambda)$, $\lambda \in \mathbb{C}$.

However, each of the sets $U_4, U_5, \dots, U_{12}$ is a single orbit, here is a description of them.

Proposition 4.6 *The subsets* U_4, U_5, U_6, U_7, U_8, U_9, U_{10}, U_{11} *and* U_{12} *are single orbits with the representatives* $L(1, 0, 1, 0)$, $L(1, 0, 0, 1)$, $L(1, 0, 0, 0)$, $L(0, 1, 0, 1)$, $L(0, 1, 0, 0)$, $L(0, 0, 1, 1)$, $L(0, 0, 1, 0)$, $L(0, 0, 0, 1)$ *and* $L(0, 0, 0, 0)$, *respectively*.

The result of all the above observations can be spelled out as follows.

Theorem 4.4 *Let* L *be an element of* $F\mathrm{Leib}_6$. *Then, it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

(1) $L(0, 0, 0, 0) = L_6^s$: $e_0e_0 = e_2,\ e_ie_0 = e_{i+1},\ 1 \le i \le 4$.
(2) $L(0, 0, 0, 1)$: $L_6^s,\ e_0e_1 = e_5$.
(3) $L(0, 0, 1, 0)$: $L_6^s,\ e_0e_1 = e_5,\ e_1e_1 = e_5$.
(4) $L(0, 0, 1, 1)$: $L_6^s,\ e_0e_1 = 2e_5,\ e_1e_1 = e_5$.
(5) $L(0, 1, 0, 0)$: $L_6^s,\ e_0e_1 = e_4,\ e_1e_1 = e_4,\ e_2e_1 = e_5$.
(6) $L(0, 1, 0, 1)$: $L_6^s,\ e_0e_1 = e_4 + e_5,\ e_1e_1 = e_4,\ e_2e_1 = e_5$.

(7) $L(1,0,0,1)$: L_6^s, $e_0e_1 = e_3 - 2e_4 + 6e_5$, $e_1e_1 = e_3 - 2e_4 + 5e_5$, $e_2e_1 = e_4 - 2e_5$, $e_3e_1 = e_5$.
(8) $L(1,0,1,0)$: L_6^s, $e_0e_1 = e_3 - 2e_4 + 6e_5$, $e_1e_1 = e_3 - 2e_4 + 6e_5$, $e_2e_1 = e_4 - 2e_5$, $e_3e_1 = e_5$.
(9) $L(0,1,1,\lambda)$: L_6^s, $e_0e_1 = e_4 + (\lambda+1)e_5$, $e_1e_1 = e_4 + e_5$, $e_2e_1 = e_5$, $\lambda \in \mathbb{C}$.
(10) $L(1,0,\lambda,\lambda)$: L_6^s, $e_0e_1 = e_3 - 2e_4 + (2\lambda+5)e_5$, $e_1e_1 = e_3 - 2e_4 + (\lambda+5)e_5$, $e_2e_1 = e_4 - 2e_5$, $e_3e_1 = e_5$, $\lambda \in \mathbb{C}$.
(11) $L(1,1,\lambda_1,\lambda_2)$: L_6^s, $e_0e_1 = e_3 - e_4 + (\lambda_1 + \lambda_2 + 5)e_5$, $e_1e_1 = e_3 - e_4 + (\lambda_1 + 5)e_5$, $e_2e_1 = e_4 - e_5$, $e_3e_1 = e_5$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

Note 4.1 The orbit U_6 with the representative $L(1,0,0,0)$ can be included in the parametric family of orbits with the representatives $L(1,0,\lambda,\lambda)$ at $\lambda = 0$.

4.3. *Dimension 7*

This section deals with $F\text{Leib}_7$. For $F\text{Leib}_7$, the isomorphism criteria (Theorem 3.1) can be rewritten as follows.

THEOREM 4.5 *Two algebras $L(\Delta)$ and $L(\Delta')$ from $F\text{Leib}_7$, where $\Delta = (\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)$ and $\Delta' = (\Delta'_3, \Delta'_4, \Delta'_5, \Delta'_6, \Theta'_6)$, are isomorphic, if and only if there exist complex numbers A and B, such that $A(A+B) \neq 0$ and the following conditions hold*:

$$
\begin{aligned}
\Delta'_3 &= \frac{1}{A}\left(1 + \frac{B}{A}\right)\Delta_3,\\
\Delta'_4 &= \frac{1}{A^2}\left(1 + \frac{B}{A}\right)\Delta_4,\\
\Delta'_5 + 5\Delta'_3\Delta'_4 &= \frac{1}{A^3}\left(1 + \frac{B}{A}\right)(\Delta_5 + 5\Delta_3\Delta_4),\\
\Delta'_6 + 6\Delta'_3\Delta'_5 + 9\Delta_3^{'2}\Delta'_4 + 3\Delta_4^{'2} &= \frac{1}{A^3}\left(1 + \frac{B}{A}\right)(\Delta_6 + 6\Delta_3\Delta_5 + 9\Delta_3^2\Delta_4 + 3\Delta_4^2),\\
\Theta'_6 &= \frac{1}{A^4}\Theta_6.
\end{aligned}
\tag{11}
$$

Under the appropriate constraints to $\Delta_3, \Delta_4, \dots, \Delta_6$ and Θ_6, the set $F\text{Leib}_7$ can be written as a union of the following subsets:

$$
\begin{aligned}
U_1 &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 \neq 0, \Delta_4 \neq 0\},\\
U_2 &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 + 6\Delta_3\Delta_5 \neq 0\},\\
U_3 &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 \neq 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 + 6\Delta_3\Delta_5 = 0\},\\
U_4 &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 \neq 0, \Theta_6 \neq 0\},\\
U_5 &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 = 0, \Delta_4 \neq 0, \Delta_5 \neq 0\},\\
U_6 &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Delta_6 + 3\Delta_4^2 \neq 0,\\
U_7 &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 \neq 0\},\\
U_8 &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 \neq 0, \Theta_6 = 0\},\\
U_9 &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 = 0, \Theta_6 \neq 0\},\\
U_{10} &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 \neq 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 = 0, \Theta_6 = 0\},\\
U_{11} &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Delta_6 + 3\Delta_4^2 = 0, \Theta_6 \neq 0\},\\
U_{12} &= \{L(\Delta) \in F\text{Leib}_7\colon \Delta_3 = 0, \Delta_4 \neq 0, \Delta_5 = 0, \Delta_6 + 3\Delta_4^2 = 0, \Theta_6 = 0\},
\end{aligned}
$$

$U_{13} = \{L(\Delta) \in F\mathrm{Leib}_7\colon \Delta_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 = 0, \Theta_6 \neq 0\},$
$U_{14} = \{L(\Delta) \in F\mathrm{Leib}_7\colon \Delta_3 = 0, \Delta_4 = 0, \Delta_5 \neq 0, \Delta_6 = 0, \Theta_6 = 0\},$
$U_{15} = \{L(\Delta) \in F\mathrm{Leib}_7\colon \Delta_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 \neq 0, \Theta_6 \neq 0\},$
$U_{16} = \{L(\Delta) \in F\mathrm{Leib}_7\colon \Delta_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 \neq 0, \Theta_6 = 0\},$
$U_{17} = \{L(\Delta) \in F\mathrm{Leib}_7\colon \Delta_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 = 0, \Theta_6 \neq 0\},$
$U_{18} = \{L(\Delta) \in F\mathrm{Leib}_7\colon \Delta_3 = 0, \Delta_4 = 0, \Delta_5 = 0, \Delta_6 = 0, \Theta_6 = 0\}.$

It is clear that these subsets are disjoint. The following propositions show that each of the subsets $U_1 - U_7$ is a union of infinitely many orbits. The second part of each proposition gives a parametrization of the orbits.

Proposition 4.7

(i) *Two algebras $L(\Delta)$ and $L(\Delta')$ from U_1 are isomorphic if and only if*

$$\frac{\Delta_3(\Delta_5 + 5\Delta_3\Delta_4)}{\Delta_4^2} = \frac{\Delta_3'(\Delta_5' + 5\Delta_3'\Delta_4')}{\Delta_4'^2}, \tag{12}$$

$$\frac{\Delta_3^2(\Delta_6 + 6\Delta_3\Delta_5 + 9\Delta_3^2\Delta_4 + 3\Delta_4^2)}{\Delta_4^3} = \frac{\Delta_3'^2(\Delta_6' + 6\Delta_3'\Delta_5' + 9\Delta_3'^2\Delta_4' + 3\Delta_4'^2)}{\Delta_4'^3}, \tag{13}$$

$$\left(\frac{\Delta_3}{\Delta_4}\right)^4 \Theta_6 = \left(\frac{\Delta_3'}{\Delta_4'}\right)^4 \Theta_6'. \tag{14}$$

(ii) *Orbits in U_1 can be parametrized as $L(1, 1, \lambda_1, \lambda_2, \lambda_3)$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$.*

Proof

(i) $\Rightarrow$: Let $L(\Delta)$ and $L(\Delta')$ be isomorphic. Then, due to Theorem (4.5), there are complex numbers A and B: $A(A+B) \neq 0$, such that the action of the adapted group G_{ad} can be expressed by the system as (11).
Now, if we substitute the value of $\Delta_3', \Delta_4', \Delta_5' + 5\Delta_3'\Delta_4', \Delta_6' + 6\Delta_3'\Delta_5' + 9\Delta_3'^2\Delta_4' + 3\Delta_4'^2$ and Θ_6' to the expressions $\frac{\Delta_3'(\Delta_5'+5\Delta_3'\Delta_4')}{\Delta_4'^2}$, $\frac{\Delta_3'^2(\Delta_6'+6\Delta_3'\Delta_5'+9\Delta_3'^2\Delta_4'+3\Delta_4'^2)}{\Delta_4'^3}$ and $\left(\frac{\Delta_3'}{\Delta_4'}\right)^4 \Theta_6'$ we will get the required equalities.
$\Leftarrow$: Let the equalities (12), (13) and (14) hold.
We put

$$A_0 = \frac{\Delta_4}{\Delta_3}, \quad B_0 = \frac{\Delta_4}{\Delta_3}\left(\frac{\Delta_4}{\Delta_3^2} - 1\right) \tag{15}$$

and

$$A_0' = \frac{\Delta_4'}{\Delta_3'}, \quad B_0' = \frac{\Delta_4'}{\Delta_3'}\left(\frac{\Delta_4'}{\Delta_3'^2} - 1\right). \tag{16}$$

Then, $\Delta_0 = \rho(\frac{1}{A_0}, \frac{B_0}{A_0}, \Delta)$ and $\Delta_0' = \rho(\frac{1}{A_0'}, \frac{B_0'}{A_0'}, \Delta')$ (see the convention in Section 3), where

$$\Delta_0 = L\left(1, 1, \frac{\Delta_3(\Delta_5 + 5\Delta_3\Delta_4)}{\Delta_4^2}, \frac{\Delta_3^2(\Delta_6 + 6\Delta_3\Delta_5 + 9\Delta_3^2\Delta_4 + 3\Delta_4^2)}{\Delta_4^3}, \left(\frac{\Delta_3}{\Delta_4}\right)^4 \Theta_6\right)$$

and

$$\Delta_0' = L\left(1, 1, \frac{\Delta_3'(\Delta_5' + 5\Delta_3'\Delta_4')}{\Delta_4'^2}, \frac{\Delta_3'^2(\Delta_6' + 6\Delta_3'\Delta_5' + 9\Delta_3'^2\Delta_4' + 3\Delta_4'^2)}{\Delta_4'^3}, \left(\frac{\Delta_3'}{\Delta_4'}\right)^4 \Theta_6'\right).$$

Then, the equalities (12), (13) and (14) imply that $\Delta_0 = \Delta'_0$. Now, we make use of the properties $1' - 3'$ of ρ and find the complex numbers A and B such that $A(A+B) \neq 0$:

$$A = \frac{A_0}{A_0'}, \quad B = \frac{B_0 A_0' - B_0' A_0}{A_0'(A_0' + B_0')}. \tag{17}$$

Thus we get

$$A = \frac{\Delta_3'\Delta_4}{\Delta_3\Delta_4'}, \quad B = \frac{\Delta_3'\Delta_4}{\Delta_3\Delta_4'}\left(\frac{\Delta_3'^2\Delta_4}{\Delta_3^2\Delta_4'} - 1\right). \tag{18}$$

For the given A and B, we get the corresponding system of Equation (11):

$$\frac{1}{A}\left(1 + \frac{B}{A}\right)\Delta_3 = \Delta_3',$$

$$\frac{1}{A^2}\left(1 + \frac{B}{A}\right)\Delta_4 = \Delta_4',$$

$$\frac{1}{A^3}\left(1 + \frac{B}{A}\right)(\Delta_5 + 5\Delta_3\Delta_4) = \Delta_5' + 5\Delta_3'\Delta_4',$$

$$\frac{1}{A^4}\left(1 + \frac{B}{A}\right)(\Delta_6 + 6\Delta_3\Delta_5 + 9\Delta_3^2\Delta_4 + 3\Delta_4^2) = \Delta_6' + 6\Delta_3'\Delta_5' + 9\Delta_3'^2\Delta_4' + 3\Delta_4'^2,$$

$$\frac{1}{A^4}\Theta_6 = \Theta_6',$$

meaning that $L(\Delta)$ and $L(\Delta')$ are isomorphic.

(ii) It is easy to see that, for any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$, there exists an algebra $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)$ from U_1, such that $\lambda_1 = \frac{\Delta_3(\Delta_5 + 5\Delta_3\Delta_4)}{\Delta_4^2}$, $\lambda_2 = \frac{\Delta_3^2(\Delta_6 + 6\Delta_3\Delta_5 + 9\Delta_3^2\Delta_4 + 3\Delta_4^2)}{\Delta_4^3}$ and $\lambda_3 = \left(\frac{\Delta_3}{\Delta_4}\right)^4 \Theta_6$. ■

Proposition 4.8

(i) *Two algebras $L(\Delta)$ and $L(\Delta')$ from U_2 are isomorphic if and only if*

$$\frac{\Delta_5^3}{\Delta_3(\Delta_6 + 6\Delta_3\Delta_5)^2} = \frac{\Delta_5'^3}{\Delta_3'(\Delta_6' + 6\Delta_3'\Delta_5')^2}, \tag{19}$$

$$\left(\frac{\Delta_5}{\Delta_6 + 6\Delta_3\Delta_5}\right)^4 \Theta_6 = \left(\frac{\Delta_5'}{\Delta_6' + 6\Delta_3'\Delta_5'}\right)^4 \Theta_6'. \tag{20}$$

(ii) *Orbits in U_2 can be represented as $L(1, 0, \lambda_1, -5\lambda_1, \lambda_2)$, $\lambda_1 \in \mathbb{C}^*$, $\lambda_2 \in \mathbb{C}$.*

Proposition 4.9

(i) *Two algebras* $L(\Delta)$ *and* $L(\Delta')$ *from* U_3 *are isomorphic if and only if*

$$\left(\frac{\Delta_3}{\Delta_5}\right)^2\Theta_6 = \left(\frac{\Delta'_3}{\Delta'_5}\right)^2\Theta'_6. \tag{21}$$

(ii) *Orbits in* U_3 *can be parametrized as* $L(1,0,1,-6,\lambda)$, $\lambda\in\mathbb{C}$.

Proposition 4.10

(i) *Two algebras* $L(\Delta)$ *and* $L(\Delta')$ *from* U_4 *are isomorphic if and only if*

$$\frac{\Delta_6^4}{\Delta_3^4\Theta_6^3} = \frac{\Delta_6'^4}{\Delta_3'4\Theta_6'^3}. \tag{22}$$

(ii) *Orbits in* U_4 *can be parametrized as* $L(1,0,0,\lambda,\lambda)$, $\lambda\in\mathbb{C}^*$.

Proposition 4.11

(i) *Two algebras* $L(\Delta)$ *and* $L(\Delta')$ *from* U_5 *are isomorphic if and only if*

$$\frac{\Delta_4(\Delta_6+3\Delta_4^2)}{\Delta_5^2} = \frac{\Delta'_4(\Delta'_6+3\Delta_4'^2)}{\Delta_5'^2}, \tag{23}$$

$$\left(\frac{\Delta_4}{\Delta_5}\right)^4\Theta_6 = \left(\frac{\Delta'_4}{\Delta'_5}\right)^4\Theta'_6. \tag{24}$$

(ii) *Orbits in* U_5 *can be parametrized as* $L(0,1,1,\lambda_1,\lambda_2)$, $\lambda_1,\ \lambda_2\in\mathbb{C}$.

Proposition 4.12

(i) *Two algebras* $L(\Delta)$ *and* $L(\Delta')$ *from* U_6 *are isomorphic if and only if*

$$\left(\frac{\Delta_4}{\Delta_6+3\Delta_4^2}\right)^2\Theta_6 = \left(\frac{\Delta'_4}{\Delta'_6+3\Delta_4'^2}\right)^2\Theta'_6. \tag{25}$$

(ii) *Orbits in* U_6 *can be parametrized as* $L(0,1,0,0,\lambda)$, $\lambda\in\mathbb{C}$.

Proposition 4.13

(i) *Two algebras* $L(\Delta)$ *and* $L(\Delta')$ *from* U_7 *are isomorphic if and only if*

$$\left(\frac{\Delta_5}{\Delta_6}\right)^4\Theta_6 = \left(\frac{\Delta'_5}{\Delta'_6}\right)^4\Theta'_6. \tag{26}$$

(ii) *Orbit in* U_7 *can be parametrized as* $L(0,0,1,1,\lambda)$, $\lambda\in\mathbb{C}$.

Proposition 4.14 *The subsets* U_8, U_9, U_{10}, U_{11}, U_{12}, U_{13}, U_{14}, U_{15}, U_{16}, U_{17} *and* U_{18} *are single orbits with the representatives* $L(1,0,0,1,0)$, $L(1,0,0,0,1)$, $L(1,0,0,0,0)$, $L(0,1,0,-3,1)$, $L(0,1,0,-3,0)$, $L(0,0,1,0,1)$, $L(0,0,1,0,0)$, $L(0,0,0,1,1)$, $L(0,0,0,1,0)$, $L(0,0,0,0,1)$ *and* $L(0,0,0,0,0)$, *respectively*.

Proof The appropriate base changes leading to the representatives are indicated below:

U_8: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_8$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(1, 0, 0, 1, 0),$$

where $A = \sqrt[3]{\frac{\Delta_6}{14\Delta_3}}$ and $B = \sqrt[3]{\frac{\Delta_6}{14\Delta_3}}(\sqrt[3]{\frac{\Delta_6}{14\Delta_3}}\frac{1}{\Delta_3} - 1)$.
U_9: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_9$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(1, 0, 0, 0, 1),$$

where $A = \sqrt[4]{\frac{\Theta_6}{14}}$ and $B = \sqrt[4]{\frac{\Theta_6}{14}}(\sqrt[4]{\frac{\Theta_6}{14}}\frac{1}{\Delta_3} - 1)$.
U_{10}: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_{10}$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(1, 0, 0, 0, 0),$$

where A is any nonzero complex number and $B = A(\frac{A}{\Delta_3} - 1)$.
U_{11}: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_{11}$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(0, 1, 0, -3, 1),$$

where $A = \sqrt[4]{\frac{\Theta_6}{3}}$ and $B = \sqrt[4]{\frac{\Theta_6}{3}}(\sqrt{\frac{\Theta_6}{3}}\frac{1}{\Delta_4} - 1)$.
U_{12}: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_{12}$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(0, 1, 0, -3, 0),$$

where A is any nonzero complex number and $B = A(\frac{A^2}{\Delta_4} - 1)$.
U_{13}: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_{13}$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(0, 0, 1, 0, 1),$$

where $A = \sqrt[4]{\Theta_6}$ and $B = \sqrt[4]{\Theta_6}(\frac{\sqrt[4]{\Theta_6^3}}{\Delta_5} - 1)$.
U_{14}: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_{14}$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(0, 0, 1, 0, 0),$$

where A is any nonzero complex number and $B = A(\frac{A^3}{\Delta_5} - 1)$.
U_{15}: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_{15}$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(0, 0, 0, 1, 1),$$

where $A = \sqrt[4]{-\Theta_6}$ and $B = \sqrt[4]{-\Theta_6}(\frac{-\Theta_6}{\Delta_6} - 1)$.

U_{16}: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_{16}$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(0,0,0,1,0),$$

where A is any nonzero complex number and $B = A(\frac{A^4}{\Delta_6} - 1)$.
U_{17}: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_{17}$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(0,0,0,0,1),$$

where $A = \sqrt[4]{\Theta_6}$ and B is any complex number except for $-\sqrt[4]{\Theta_6}$.
U_{18}: For $L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6) \in U_{18}$,

$$\rho\left(\frac{1}{A}, \frac{B}{A} : L(\Delta_3, \Delta_4, \Delta_5, \Delta_6, \Theta_6)\right) = L(0,0,0,0,0),$$

where A is any nonzero complex number and B is any complex number except for $-A$. ■

Theorem 4.6 *Let L be an element of $F\mathrm{Leib}_7$. Then, it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras*:

(1) $L(0,0,0,0,0) = L_7^s : e_0e_0 = e_2,\ e_ie_0 = e_{i+1},\ 1 \le i \le 5.$
(2) $L(0,0,0,0,1)$: $L_7^s,\ e_0e_1 = e_6.$
(3) $L(0,0,0,1,0)$: $L_7^s,\ e_0e_1 = e_6,\ e_1e_1 = e_6.$
(4) $L(0,0,0,1,1)$: $L_7^s,\ e_0e_1 = 2e_6,\ e_1e_1 = e_6.$
(5) $L(0,0,1,0,0)$: $L_7^s,\ e_0e_1 = e_5,\ e_1e_1 = e_5,\ e_2e_1 = e_6.$
(6) $L(0,0,1,0,1)$: $L_7^s,\ e_0e_1 = e_5 + e_6,\ e_1e_1 = e_5,\ e_2e_1 = e_6.$
(7) $L(0,1,0,-3,1)$: $L_7^s,\ e_0e_1 = e_4 - 2e_6,\ e_1e_1 = e_4 - 3e_6,\ e_2e_1 = e_5,\ e_3e_1 = e_6.$
(8) $L(0,1,0,-3,0)$: $L_7^s,\ e_0e_1 = e_4 - 3e_6,\ e_1e_1 = e_4 - 3e_6,\ e_2e_1 = e_5,\ e_3e_1 = e_6.$
(9) $L(1,0,0,1,0)$: $L_7^s,\ e_0e_1 = e_3 - 2e_6,\ e_1e_1 = e_3 - 13e_6,\ e_2e_1 = e_4,\ e_3e_1 = e_5, e_4e_1 = e_6.$
(10) $L(0,0,1,1,\lambda)$: $L_7^s,\ e_0e_1 = e_5 + (\lambda+1)e_6,\ e_1e_1 = e_5 + e_6,\ e_2e_1 = e_6,\quad \lambda \in \mathbb{C}.$
(11) $L(0,1,0,0,\lambda)$: $L_7^s,\ e_0e_1 = e_4 + \lambda e_6,\ e_1e_1 = e_4,\ e_2e_1 = e_5,\ e_3e_1 = e_6,\quad \lambda \in \mathbb{C}.$
(12) $L(0,1,1,\lambda_1,\lambda_2)$: $L_7^s,\ e_0e_1 = e_4 + e_5 + (\lambda_1+\lambda_2)e_6,\ e_1e_1 = e_4 + e_5 + \lambda_1 e_6,\ e_2e_1 = e_5 + e_6,\ e_3e_1 = e_6,\ \lambda_1,\ \lambda_2 \in \mathbb{C}.$
(13) $L(1,0,0,\lambda,\lambda)$: $L_7^s,\ e_0e_1 = e_3 + (2\lambda - 14)e_6,\ e_1e_1 = e_3 + (\lambda - 14)e_6,\ e_2e_1 = e_4,\ e_3e_1 = e_5,\ e_4e_1 = e_6,\ \lambda \in \mathbb{C}.$
(14) $L(1,0,1,-6,\lambda)$: $L_7^s,\ e_0e_1 = e_3 + 6e_5 + (\lambda - 20)e_6,\ e_1e_1 = e_3 + 5e_5 - 20e_6,\ e_2e_1 = e_4 + 6e_6,\ e_3e_1 = e_5,\ e_4e_1 = e_6,\ \lambda \in \mathbb{C}.$
(15) $L(1,0,\lambda_1,-5\lambda_1,\lambda_2)$: $L_7^s,\ e_0e_1 = e_3 + (\lambda_1+5)e_5 + (\lambda_2 - 5\lambda_1 - 14)e_6,\ e_1e_1 = e_3 + (\lambda_1+5)e_5 - (5\lambda_1 + 14)e_6,\ e_2e_1 = e_4 + (\lambda_1+5)e_6,\ e_3e_1 = e_5,\ e_4e_1 = e_6,\ \lambda_1,\ \lambda_2 \in \mathbb{C}.$
(16) $L(1,1,\lambda_1,\lambda_2,\lambda_3)$: $L_7^s,\ e_0e_1 = e_3 - e_4 + (\lambda_1+5)e_5 + (\lambda_3 + \lambda_2 - 14)e_6,\ e_1e_1 = e_3 - e_4 + (\lambda_1+5)e_5 + (\lambda_2 - 14)e_6,\ e_2e_1 = e_4 - e_5 + (\lambda_1+5)e_6,\ e_3e_1 = e_5 - e_4,\ e_4e_1 = e_6,\ \lambda_1,\ \lambda_2,\ \lambda_3 \in \mathbb{C}.$

Note 4.2 The orbits U_9 and U_{10} with the representatives $L(1,0,0,0,1)$ and $L(1,0,0,0,0)$ can be included in the parametric family of orbits with representatives $L(1,0,\lambda_1,\lambda_1,\lambda_2)$ and $L(1,0,0,\lambda,\lambda)$ with the values of parameters $\lambda_1 = 0, \lambda_2 = 1$ and $\lambda = 0$, respectively.

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