Meyberg chpt 1 p. 5 – 6 10/12/16 (1)

lemma 2 B ideal in A $\Rightarrow$

A is solvable $\Leftrightarrow$ $\underbrace{A/B}_{\text{homo image}}$ & $\underbrace{B}_{\text{subalg}}$ are solvable

$\boxed{\Rightarrow}$ is lemma 1

To prove $\boxed{\Leftarrow}$

claim $(A/B)^{(k)} = A^{(k)}/B$

First is $B \subset A^{(k)}$ $k=0$ ok $k=1$ $B \overset{?}{\subseteq} A^{(1)} = AA$

$$(A/B)^{(0)} = A/B$$

$$(A/B)^{(1)} = A/B \cdot A/B = \text{span}\{(a+B)(a'+B) : a, a' \in A\}$$

$$= \text{span}\{aa' + B \quad \cancel{= \langle AA + B} \subseteq A/B$$

$$A^{(1)}/B = AA/B \qquad \overset{?}{A^{(k)} \subset B}$$

$$A/B = \{a + B : a \in A\}$$

$$(A/B)^{(1)} = A/B \; A/B = \{aa' + B\} = (A^{(1)} + B)/B$$

$$(A/B)^{(k)} = \frac{A^{(k)} + B}{B} \qquad A^{(k)} + B = B$$

$$A^{(k)} \subset B$$

$$A/B \;\; A/B \;=\; (AA + B)/B$$

$$(a+B)(a'+B) = aa' + B = (aa' + 0) + B$$

$$\in (AA + B)/B$$

$\therefore \;\; A/B \cdot A/B \subseteq (AA + B)/B$

Converse consider $aa' + b$

Is $A^{(2)}$ an algebra ?

let $x, y \in A^{(2)}$ $\quad x = \sum a_i b_i \quad a_i, b_i \in A^{(1)}$

say $\quad x = ab \in A^{(1)} A^{(1)} \quad y = cd$

$$a = a_1 a_2 \in AA$$

$$b = b_1 b_2 \in AA$$

$$x = (a_1 a_2)(b_1 b_2) \qquad y = (c_1 c_2)(c_3 c_4)$$

$$xy = \Big((a_1 a_2)(b_1 b_2) \Big)\Big(\overbrace{(c_1 c_2)}^{A^{(1)}} \overbrace{(c_3 c_4)}^{A^{(1)}} \Big)$$

$A^{(0)} = A$, $A^{(1)} = AA$, $A^{(2)} = (A^1)(A^1)$, ... $A^{(k+1)} = (A^{(k)} A^{(k)})$

assume $A^{(k)}$ is an algebra $A^{(k)} A^{(k)} \subset A^{(k)}$

let ~~x, y~~ $\in$ $A^{(k+1)} A^{(k+1)} \overset{?}{\in} A^{(k+1)}$

$x = \sum a_i b_i$ $y = \sum c_j d_j$ $a, b, c, d \in A^{(k)}$

$$xy = \sum_{i,j} (a_i b_i)(c_j d_j)$$

$$A^{(k+1)} A^{(k+1)} = \left(A^{(k)} A^{(k)}\right)\left(A^{(k)} A^{(k)}\right)$$

$$\subseteq A^{(k)} A^{(k)} \cancel{A^{(k)}}$$

$$= A^{(k+1)}$$

So $A^{(k)}$ are subalgebras

$A^{(1)}$ is an ideal

$$\left(A/B\right)^{(k)} = \left(A^{(k)} + B\right)/B$$ as vector paces

$$A^{(k)} B \overset{?}{\subset} B$$

So $A^{(k)} \subset B$

$$(A^{(k)})^{(s)} \subset B^{(s)}$$
$$\shortparallel$$
$$A^{(k+s)} = 0 \quad \text{for some } s.$$

This proves Lemma 2

A is Noetherian if every nonempty set of ideals has a maximal element

~~Theorem 2~~

i.e. if $B_\alpha \subset A$ are ideals

$\exists\, \alpha$ $\quad B_\alpha \subset B_\beta \subset A$

$\Rightarrow B_\alpha = B_\beta$

Th 2 (i) U, V solv ideals in $A \Rightarrow U+V$ solvable ideal

~~$a(u+v)$~~ $A(U+V) \subseteq AU + AV \subset U+V$

$(U+V)(A) \subseteq UA + VA \subset U+V$

so $U+V$ is an ideal

$$\underbrace{(U+V)/V}_{\therefore \text{ solvable}} \simeq \underbrace{U/U\cap V}_{\text{solvable}} \xrightarrow{\text{Lemma 2}} U+V \text{ solvable}$$

$\boxed{\checkmark}$

(ii) A Noetherian $\Rightarrow \exists 1$ maximal solvable ideal $\mathcal{R}(A) \supset$ all solv. ideals (∴ largest solvable)

& $\mathcal{R}(A/\mathcal{R}(A)) = 0$.

let $R(A)$ be a maximal elt in the set (B_α) of all solvable ideals

($\neq \phi$ since (0) is solvable)

(0) is an algebra $(0) = (0)^{(0)}$

$(0)^{(1)} = (0)(0) = (0)$ ... etc.

Let $\mathcal{R}'$ be any solvable ideal

$\mathcal{R}(A) \subseteq \mathcal{R}(A) + \mathcal{R}'$ is solvable $\Rightarrow R(A) = R(A) + \mathcal{R}'$

$\Rightarrow \mathcal{R}' \subset \mathcal{R}(A)$

If $\mathcal{R}'$ is a maximal solvable ideal

then $\mathcal{R}' = \mathcal{R}(A)$ so $R(A)$ is unique

let's now show $\mathcal{R}(A/\mathcal{R}(A)) = (0)$

We have proved that $R(A/R(A))$ is the unique maximal solvable ideal contany all solvable ideals of $A/R(A)$

let $\overline{U} = U/R(A)$ be a solvable ideal in $A/R(A)$

(Exercise: the ideals of $A/R(A)$ are of the form $U/R(A)$ where $R(A) \subset U \subset A$ U is an ideal in A. $R(A)$ is an ideal in U)

claim U is solvable in A

$$(U + R(A))^{(1)} = U^{(1)} + R(A)$$

$$(U + R(A))^{(k)} = U^{(n)} + R(A)$$

$$\left(U/R(A)\right)^{(1)} = U/R(A) \cdot U/R(A)$$

$$= (U + R(A))(U + R(A))$$

$$\subseteq U^{(1)} + R(A)$$

actually $= U^{(1)} + R(A)$

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$$\left(U/_{R(A)}\right)^{(k)} = U^k + R(A) = \cancel{U^{(k)}/_{R(A)}}$$

$$\cancel{= \{ y + R(A) }$$

$$= \frac{U^{(k)} + R(A)}{R(A)}$$

$$U^{(k)} + R(A) = R(A)$$

$$\Rightarrow U^{(k)} \subset R(A)$$

$$\left.\begin{array}{l} U/_{R(A)} \text{ is solvable} \\ R(A) \text{ is solvable} \end{array}\right\} \Rightarrow U \text{ solvable}$$

Proof of the exercise on page 6

Let $R(A)$ be ideal in U $\quad U/_{R(A)}$

U ideal in A $\quad A/U$

$R(A) \subset U \subset A$

$U/_{R(A)} \subset A/_{R(A)}$ so this direction is trivial

$(u + R(A))(a + R(A)) = (ua + R(A)) \in U + R(A)$

let $\mathcal{I}$ be an ideal in $A/R(A)$

let $U = \{a \in A : a + R(A) \in \mathcal{I}\}$

i.e. $\mathcal{I} = U/R(A)$ as vector spaces

claim 1 U is an ideal in A

let $b \in U$, $a \in A$

$b + R(A) \in \mathcal{I}$

$(b + R(A))(a + R(A)) \in \mathcal{I}$

$ba + R(A) \in \mathcal{I}$

$\Rightarrow ba \in U$

Similarly $ab \in U$

U is obviously a subsp. so an ideal

claim 2 $R(A)$ is an ideal in U

clearly $R(A) \subset U$

Let $a \in U$ $\quad b \in R(A)$ $\quad$ To Prove ~~$ab \in$~~

$ab \in R(A)$

$\cancel{a + R(A) \in \mathcal{I}}$ $\quad ab \in U$

$$a + R(A) \in \mathcal{I}$$

$$\underbrace{ab + R(A)}_{} \in \mathcal{I}$$

$$= (a + R(A))(b + R(A))$$

$$= 0$$

$$\Rightarrow ab \in R(A)$$

~~Claim 3~~ $\cancel{\mathcal{I} = U/R(A)}$

$\cancel{U/R(A) = \{a + R(A) : a \in U\} \subseteq \mathcal{I}}$ ~~by def of~~ $\cancel{U}$

?

=

start over

~~Let $V = \{a \in A :$~~

From top of p. 8

Let $\mathfrak{d}$ be an ideal in $A/R(A)$

$$\mathfrak{d} = \{ x + R(A) : x \in U \}$$

i.e. $U = \{ x \in A : x + R(A) \in \mathfrak{d} \}$

$$\therefore \quad \mathfrak{d} = U/R(A)$$