A review of "Radicals" on NON Associative structures

- First Meeting 199A, September 30, 2016 Meyberg ch 1 (10 pp) Ex 1–6

  www.math.uci.edu/~brusso/ch1marked.pdf

1.5 (page 5) $\boxed{A \text{ is any algebra.}}$ The derived series is $A = A^{(0)} \supset A^{(1)} \supset A^{(2)} \supset \cdots \supseteq A^{(k)} \supseteq \cdots$

defined by Inductively $A^{(0)} = A \qquad A^{(k+1)} = A^{(k)} A^{(k)}$

(= all finite sums of products of elements of $A^{(k)}$)

definition A (non-associative algebra) is solvable if $A^{(n)} = 0$ for some $n \geq 0$

Theorem 2 (p. 5) If A is Noetherian, then A has a unique maximal solvable ideal $\boxed{\mathcal{R}(A)}$ and $\mathcal{R}(A/\mathcal{R}(A)) = 0$. (Noetherian means every non-empty set of ideals has a maximal element)

definition $\boxed{\mathcal{R}(A)}$ is called the solvable radical of A

We say A is $\boxed{\text{semi-simple}}$ if $\mathcal{R}(A) = 0$.

- Third Meeting 199A, October 14, 2016 Solvable radical (informal notes Meyberg pp 5–6)

  www.math.uci.edu/~brusso/solvableradical.pdf

  (details of the proof of Theorem 2 above)

Theorem 1 p. 41 of chpt 5 Meyberg Notes. If A is a Lie algebra (over char 0 field) then A is semisimple $\Leftrightarrow$ its Killing form is non-degenerate.

- Sixth Meeting (199A) November 4, 2016 Meyberg ch 3 pp 21–25
  www.math.uci.edu/~brusso/meyberg21-25marked.pdf

$\mathcal{J}$ is any triple system: $\langle\ ,\ ,\ \rangle : \mathcal{J}\times\mathcal{J}\times\mathcal{J} \to \mathcal{J}$
is tri-linear (that's all!)

Exercise 2 (p. 23) If $\mathcal{J}$ is Noetherian, then there exists a unique maximal solvable ideal $\boxed{\mathrm{Rad}(\mathcal{J})}$ (called the solvable radical of $\mathcal{J}$)

$\boxed{3.2}$ p. 22 definition "derivatives" of a triple system $\mathcal{J}$ are defined recursively $\mathcal{J}^{(0)} = \mathcal{J}$, $\mathcal{J}^{(n+1)} = \langle \mathcal{J}^{(n)}, \mathcal{J}^{(n)}, \mathcal{J}^{(n)} \rangle$
(= all finite sums of triple products of elements of $\mathcal{J}^{(n)}$)
$\mathcal{J}$ is solvable if $\mathcal{J}^{(n)} = (0)$ for some $n \geq 1$
$\mathcal{J}$ is $\boxed{\text{semisimple}}$ if $\mathrm{Rad}(\mathcal{J}) = 0$

- Sixth Meeting (199A), November 4, 2016 Meyberg ch 3 Triple Systems; solvable and nilradicals (informal notes pp 21–25)
  www.math.uci.edu/~brusso/meyberg21-25NOTES.pdf

(See pages 7 8 for an ~~precise~~ outline of the proof of Exercise 2 on p. 23; better yet see Problems 15 and 16* on "Final Exam" problems from 199A
www.math.uci.edu/~brusso/199AExam121116.pdf)

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* In Problem 16 we used the notation $R(\mathcal{J})$ instead of $\mathrm{Rad}(\mathcal{J})$. We also stated $R(\mathcal{J}/R(\mathcal{J})) = 0$.

$\boxed{\text{Theorem 10 (p. 57)}}$ If $\mathcal{J}$ is a semisimple Lie triple system (over F of char 0) then every derivation D of $\mathcal{J}$ is of the form $D = \sum_{i=1}^{n} L(u_i, v_i)$

Proof. Step 1 Define $\delta : \mathcal{L} \to \mathcal{L}$ $\quad \delta(X) = [D, H] \oplus Da$
if $X = H \oplus a$

Exercise 6
Verify that δ is a derivation of $\mathcal{L}$.
i.e. $\delta[X_1, X_2] = [X_1, \delta(X_2)] + [\delta(X_1), X_2]$

step 2 $\mathcal{L}$ is semisimple so $\delta(X) = [U, X] = \mathrm{ad}\,U(X)$ $\quad U = H_1 \oplus a_1 \in \mathcal{L}$
(since $\mathcal{J}$ is semi-simple - this will be discussed next week (March 17))

By Theorem 3 of chapter 5 of Meyberg (p. 42) every derivation is inner (of the form ad U for some $U \in \mathcal{L}$)

Step 3 $\quad Da = 0 \oplus Da = \delta(0 \oplus a) = [H_1 \oplus a_1, 0 \oplus a]$

$= \cancel{0 \oplus (H_1 a + L(a_1, a))} = L(a_1, a) \oplus H_1(a)$

$\therefore \quad 0 = L(a_1, a) \qquad Da = H_1(a) \qquad H_1 \in \mathcal{H}$ as required.

Recall Theorem 1 p. 45 of Meyberg Notes - chapter 6

$\mathcal{L} = \mathcal{H} \oplus \mathcal{J} \qquad \mathcal{H} = \operatorname{span} L(\mathcal{J}, \mathcal{J}) = \{ \sum_{i=1}^{n} L(x_i, y_i) : x_i, y_i \in \mathcal{J}, n \geq 1 \} \subseteq \mathrm{Der}(\mathcal{J})$

$[H_1 \oplus x_1, H_2 \oplus x_2] = ([H_1, H_2] + L(x_1, x_2)) \oplus (H_1 x_2 - H_2 x_1)$