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Bernstein algebras which are Jordan algebras

By

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0. Introduction. A finite dimensional commutative algebra A over a field K is called *baric* if there exists a nontrivial homomorphism $\omega\colon A \to K$. A baric algebra is called a Bernstein algebra if

(1) $$x^2 x^2 - \omega(x)^2 x^2 = 0$$

for all $x \in A$. (See Wörz-Busekros [6], which will be used as a basic reference, for these definitions.) The origin of Bernstein algebras lies in genetics, see Bernstein [2] and Ljubič [4]. Holgate [3] was the first to translate the problem into the language of nonassociative algebras. A summary of known results can be found in [6], Ch. 9.

For applications, the underlying field K is usually $\mathbb{R}$ or $\mathbb{C}$. In order to avoid major difficulties with Bernstein algebras, one should exclude char $K = 2$. A classification of Bernstein algebras is far beyond reach, and probably the general case will remain untractable.

It has been observed, however (Holgate [3], Wörz-Busekros [7]), that some well-known classes of Bernstein algebras are also Jordan algebras. This includes the simplest type, $x^2 = \omega(x)x$, and also Ljubič's normal Bernstein algebras. Some results about Bernstein algebras which are Jordan algebras can be found in [7]. The purpose of this note is to present a characterization of Bernstein algebras which are Jordan algebras (called Jordan Bernstein algebras from now on) over a field of characteristic not 2 or 3, and list some of their properties.

1. Some identities. In the following let A be a Bernstein algebra over a field K of characteristic not 2 or 3. (The exclusion of characteristic 3 is not always necessary, but it is for the main theorem.) Linearizing (1), we obtain the well-known identities

(2) $$2x^2(xy) - \omega(xy)x^2 - \omega(x^2)xy = 0$$

and

(3) $$4(xy)(xz) + 2x^2(yz) - \omega(yz)x^2 - 2\omega(xy)xz - 2\omega(xz)xy - \omega(x^2)yz = 0.$$

We do not need the full linearization of (1), therefore we have not written it down.

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The property of being a Bernstein algebra is preserved by extension of the underlying field.

Putting $y = x^2$ in (2), we obtain

$$2x^2x^3 = \omega(x)^3x^2 + \omega(x)^2x^3. \tag{4}$$

In fact, $2x^ix^j = \omega(x)^ix^j + \omega(x)^jx^i$ is easily established by induction for $i, j \geqq 2$, but this is more than we need here. (Powers of x are defined as usual: $x^{i+1} = xx^i$.)

2. Characterization of Jordan Bernstein algebras.

Theorem. *Let A be a baric algebra. Then the following statements are equivalent*:

a) *A is a Jordan Bernstein algebra.*
b) *A is a power-associative Bernstein algebra.*
c) *$x^3 - \omega(x)x^2 = 0$ for all $x \in A$.*

Proof. "a) ⇒ b)": Jordan algebras are power-associative; see, for instance, Schafer [5], Ch. 4.

"b) ⇒ c)": From the remark above, it is sufficient to prove this for the case of an infinite underlying field. We have $x^4 = x^2x^2$ from power-associativity, and $x^2x^2 - \omega(x)^2x^2 = 0$ from Bernstein, hence $x^4 - \omega(x)^2x^2 = 0$.

Linearizing this, we find

$$2x(x(xy)) + x(x^2y) + x^3y - 2\omega(x)\omega(y)x^2 - 2\omega(x)^2xy = 0.$$

Put $y = x^2$:

$$2x^5 + x(x^2x^2) + x^3x^2 - 2\omega(x)^3x^2 - 2\omega(x)^2x^3 = 0.$$

Use (1) and (4) to obtain

$$2x^5 - \tfrac{1}{2}\omega(x)^2x^3 - \tfrac{3}{2}\omega(x)^3x^2 = 0.$$

On the other hand, multiplying $x^4 - \omega(x)^2x^2 = 0$ by $2x$ yields

$$2x^5 - 2\omega(x)^2x^3 = 0.$$

Now take the difference:

$$\tfrac{3}{2}(\omega(x)^2x^3 - \omega(x)^3x^2) = 0.$$

Since char $K \neq 3$,

$$\omega(x)^2(x^3 - \omega(x)x^2) = 0.$$

Therefore, the polynomial identity $x^3 - \omega(x)x^2 = 0$ is valid for all x with $\omega(x) \neq 0$. K being infinite, the set of all these x is (Zariski-)dense in A, and therefore the asserted identity is valid in all of A.

"c) ⇒ a)": Linearize $x^3 - \omega(x)x^2 = 0$ to obtain

$$2x(xy) + x^2y - \omega(y)x^2 - 2\omega(x)xy = 0.$$

Multiply this by x:

$$2x(x(xy)) + x(x^2 y) - \omega(y)x^3 - 2\omega(x)x(xy) = 0.$$

On the other hand, replace y by xy:

$$2x(x(xy)) + x^2(xy) - \omega(x)\omega(y)x^2 - 2\omega(x)x(xy) = 0.$$

Take the difference:

$$0 = x(x^2 y) - x^2(xy) - \omega(y)(x^3 - \omega(x)x^2) = x(x^2 y) - x^2(xy),$$

and this is the Jordan identity.

Furthermore, from $x^3 - \omega(x)x^2 = 0$, we get

$$0 = x^4 - \omega(x)x^3 = x^4 - \omega(x)^2 x^2 = x^2 x^2 - \omega(x)^2 x^2,$$

since $x^4 = x^2 x^2$ in Jordan algebras, and therefore A is a Bernstein algebra. □

Remark 1. The "Bernstein" part of c) ⇒ a) it not new; see [6], Thm. 9.12.

Remark 2. One may view "b) ⇒ c)" as the crucial part of the proof. The equation $x^4 - \omega(x)^2 x^2 = 0$ implies that A is a train algebra (see [6], p. 34, for the definition), and the argument shows that $\tau^4 - \tau^2$ cannot occur as rank polynomial of a Bernstein train algebra. In fact, the following can be proved: If a Bernstein algebra is a train algebra, its rank polynomial has degree r and the characteristic of the underlying field is zero or greater than r, then the rank polynomial is $\tau^2(\tau - 1)(\tau - \frac{1}{2})^{r-3}$. The proof of this is quite straightforward, but a little tedious.

3. Some properties of Jordan Bernstein algebras. In the following let A be a Bernstein algebra and $c \in A$ an idempotent (idempotents exist: they are exactly the squares of the elements $y \in A$ with $\omega(y) = 1$; cf. [6]). By $L(c)$ we denote, as usual, the left multiplication by c.

$N := \operatorname{Ker} \omega$ is an ideal of A, especially $L(c)$-invariant. For $x = c$, $y \in N$ we obtain from (2) (cf. Holgate [3]):

$$2c(cy) - cy = 0 \quad \text{or} \quad (2L(c)^2 - L(c))y = 0.$$

Thus, the restriction of $L(c)$ to N is annihilated by the polynomial $2\tau^2 - \tau = \tau(2\tau - 1)$.

It is known from linear algebra that this implies the semisimplicity of $L(c)$, restricted to N, and this has the following consequences: First, every $L(c)$-invariant subspace M of N is a direct sum $M = M_0 \oplus M_{1/2}$, where $M_\lambda := \{y \in M : cy = \lambda y\}$.

Second, there exists an $L(c)$-invariant subspace $M' \subset N$ such that $N = M \oplus M'$. (Of course, the same applies for $L(c)$-invariant subspaces of $L(c)$-invariant subspaces.) Especially, $N = N_0 \oplus N_{1/2}$, and with the help of (3) the following relations can be found (Ljubič [4], see also [6]): $N_0 N_0 \subset N_{1/2}$, $N_0 N_{1/2} \subset N_{1/2}$, $N_{1/2} N_{1/2} \subset N_0$.

It has been noted by Wörz-Busekros [7] that this decomposition of N (and the corresponding decomposition of A) resembles the Peirce decomposition in some well-known classes of algebras, and that one may well adopt the name for the case of Bernstein algebras. The following observation may be of interest for itself.

Lemma. *Let U and V be $L(c)$-invariant subspaces of N. Then UV is $L(c)$-invariant.*

Proof. Let $x = c$, $y \in U$ and $z \in V$ in (3). Then $4(cy)(cz) + 2c(yz) - yz = 0$. Since the first and last term are in UV, so is the second. □

Defining recursively $N^1 := N$, $N^{k+1} := N N^k$ $(k \geqq 1)$, we obtain a chain of ideals of N. Following [6], p. 22f., N is called nilpotent if $N^r = 0$ for some r. (By [5], Thm. 2.4, this is equivalent to the definition of nilpotency given there.)

Using the lemma, we see by induction that N^k is an ideal of A for each k. Now recall the definitions or a genetic algebra ([6], p. 40) and a special train algebra ([6], p. 55) and the fact that N is nilpotent if A is genetic ([6], Lemma 3.19), and put pieces together to obtain

Proposition 1. *Let A be a Bernstein algebra. If N is nilpotent then A is a special train algebra. Especially, A is genetic if and only if A is a special train algebra.* □

Let us return to Jordan Bernstein algebras. From the theorem, we find $x^3 = 0$ for all $x \in N$, and N is a nil Jordan algebra, which implies the nilpotency of N by [5], Thm. 4.3. (One should note here that Abraham [1] has provided a proof that any algebra satisfying $x^3 = 0$ for all x is nilpotent; cf. [6], Thm. 3.33. Since every nilalgebra of index 3 is Jordan – imitate the "c) ⇒ a)"-part of the proof of the theorem – this fact may also be seen as a consequence of [5], Thm. 4.3.) From Proposition 1 we find that every Jordan Bernstein algebra is genetic. Thus, for $\dim A = m + 1$, we have a chain of ideals of A:

$$N = N_1 \supset N_2 \supset \cdots \supset N_m \supset \{0\}$$

such that $\dim N_i = m + 1 - i$ and $N_i N_j \subset N_{k+1}$, where $k := \max\{i, j\}$, for all i and j. (cf. [6], Thm. 3.18. No field extension is necessary here, since the eigenvalues of $L(c)$ lie in K.)

Using the remarks above on $L(c)$-invariant subspaces of N, we obtain

Proposition 2. *Let A be a Jordan Bernstein algebra of dimension $m + 1$. Then there exists a basis $v_1, \ldots, v_m$ of N such that v_i is an eigenvector of $L(c)$ for $1 \leqq i \leqq m$ and N_i is spanned by $v_i, \ldots, v_m$ $(1 \leqq i \leqq m)$.*

Proof. N_2 is an $L(c)$-invariant subspace of $N = N_1$, hence there exists an $L(c)$-invariant complementary subspace W_1 to N_2 in N_1. W_1 being one-dimensional, it is spanned by an eigenvector v_1 of $L(c)$. The rest is induction. □

Besides this, we have to take into account the composition rules for the eigenspaces $N_0, N_{1/2}$ of $L(c)$: $N_0 N_0 = \{0\}$, $N_0 N_{1/2} \subset N_{1/2}$, $N_{1/2} N_{1/2} \subset N_0$ (The additional restriction stems from the "usual" Peirce-decomposition in Jordan algebras; see [7].) The characterization of Jordan Bernstein algebras we have obtained is, of course, not perfect. It would be desirable to have a decomposition into "building blocks" which are put together in a prescribed manner. But note that the bare existence of such a description is everything else but guaranteed.

The results presented here should at least make the construction of Jordan Bernstein algebras a manageable task: Start with a basis of eigenvectors in N (the eigenvalues preassigned), take into account the composition rules for the eigenspaces and note that the only thing to be checked besides this is the identity $x^3 = 0$ in N.

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