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Bernstein algebras

By

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0. Introduction. It is an old problem posed by S. Bernstein [4] in 1923 to classify all evolutionary operators satisfying the stationarity principle, i.e. all idempotent, quadratic operators acting on the stochastic simplex. Thereby the principle of stationarity in genetics is a generalization of the Mendelian laws (cf. Mendel [12]) and of the theorem of Hardy [6] and Weinberg [14]. Bernstein himself solved the problem in the three-dimensional case and he found two classes of idempotent evolutionary operators in the general case [2], [3], [4].

Nearly half a century later the classification problem has been taken up again by Ju. I. Ljubič [9]. He derived a canonical form for idempotent evolutionary operators. In 1975 P. Holgate [8] gave an algebraic formulation of Bernstein's problem, whereby he introduced the so-called Bernstein algebras. Afterwards Ljubič [10], [11] considered Bernstein algebras, too, especially the normal ones.

A survey on the results about Bernstein algebras up to 1980 is given in Sect. 9 of [15]. Further investigations on Bernstein algebras, e.g. their derivation algebra, have been taken up by M. R. Alcalde, C. Burgueño, A. Labra, A. Micali [1].

In the present paper we show that the well-known decomposition of a Bernstein algebra with respect to an idempotent is nothing else but the Peirce decomposition known for finite-dimensional, power associative algebras with idempotent, especially for Jordan algebras with idempotent. It turns out that all idempotents of a Bernstein algebra are principal and hence primitive. Therefore the Peirce decomposition can be no more decomposed. Furthermore we deduce a necessary and sufficient condition for a Bernstein algebra to be a Jordan algebra.

All trivial Bernstein algebras are Jordan algebras, and beyond this, they are special Jordan algebras. Furthermore all normal Bernstein algebras are Jordan algebras.

1. Preliminaries. In the following let $\mathbb{K}$ be a commutative field of characteristic different from 2. Let A be a commutative, i.g. not associative algebra over $\mathbb{K}$. A is called a *baric* algebra, if there exists a non-trivial algebra homomorphism $\omega: A \to \mathbb{K}$. Then

(1.1) $$N := \operatorname{Ker} \omega$$

is an ideal of A of codimension 1 ($A/N \cong \mathbb{K}$).

A is called a *Bernstein* algebra, cf. Holgate [8], if and only if

(1.2) $$(x^2)^2 = \omega(x)^2 x^2 \quad \text{for all} \quad x \in A.$$

In the following we list several results on Bernstein algebras which are due to Ljubič [10] who formulated them in terms of idempotent operators and to Holgate [8], who gave them in terms of algebras; for a survey compare also Chapter 9 in [15].

For every Bernstein algebra A the non-trivial homomorphism $\omega: A \to \mathbb{K}$ is uniquely determined.

Every Bernstein algebra A possesses at least one idempotent e (i.e. $0 \neq e \in A$, $e^2 = e$).

Every Bernstein algebra A with idempotent e can be decomposed into the internal direct sum of subspaces

$$(1.3) \qquad A = E \oplus U \oplus Z$$

with

$$(1.4) \qquad E := \mathbb{K}e, \quad U := \{ey \mid y \in N\}, \quad Z := \{z \in A \mid ez = 0\}.$$

If $x = \alpha e + u + z \in E \oplus U \oplus Z$ is an arbitrary element of A, then

$$(1.5) \qquad \omega(x) = \alpha.$$

Hence the kernel N of ω splits into

$$(1.6) \qquad N = U \oplus Z.$$

Furthermore the following equations are satisfied for all $u \in U$ and all $z \in Z$:

$$(1.7) \qquad eu = \tfrac{1}{2}u,$$

$$(1.8) \qquad u^3 = 0,$$

$$(1.9) \qquad u(uz) = 0,$$

$$(1.10) \qquad uz^2 = 0,$$

$$(1.11) \qquad u^2(uz) = 0,$$

$$(1.12) \qquad (uz)z^2 = 0,$$

$$(1.13) \qquad (uz)^2 = 0,$$

$$(1.14) \qquad u^2z^2 = 0.$$

The complex products of the subspaces U and Z of A satisfy

$$(1.15) \qquad U^2 \subseteqq Z, \quad UZ \subseteqq U, \quad Z^2 \subseteqq U, \quad UZ^2 = \langle 0 \rangle.$$

Remark. Equations (1.13) and (1.14) have been claimed by Ljubič [10]. In [15] the author only arrived to prove the identity

$$(1.16) \qquad 2(uz)^2 + u^2z^2 = 0.$$

But in 1980 Ljubič communicated to the author the following proof. On the one side $uz \in U$ implies $(uz)^2 \in U^2 \subseteqq Z$, and on the other side $u^2z^2 \in U^2Z^2 \subseteqq U$. Since $U \cap Z = \langle 0 \rangle$ both terms in (1.15) must vanish separately.

The set of idempotents of A is given by

$$(1.17) \qquad \{e + u + u^2 \mid u \in U\}.$$

Although the decomposition (1.3) of a Bernstein algebra depends on the choice of the idempotent $e \in A$, it follows by the above statement, (1.3) and $U^2 \subseteqq Z$, see (1.15), that the dimension of the subspace U of A is an invariant of A.

If A has finite dimension, which is at least 1,

$$\dim A = 1 + n,$$

then one can associate to $A = E \oplus U \oplus Z$ a pair of integers $(r+1, s)$, called the *type* of A, whereby

$$(1.18) \qquad r := \dim U, \quad s := \dim V,$$

hence $r + s = n$.

It has been shown in [16] that for each decomposition $n = r + s$ there exists a Bernstein algebra of type $(r+1, s)$. Thereby the so-called *trivial* Bernstein algebra of type $(r+1, s)$ has been introduced as Bernstein algebra of the corresponding type where $N^2 = \langle 0 \rangle$. Furthermore it is proved that for every decomposition $n = r + s$ there exists, up to isomorphism, exactly one $(n+1)$-dimensional, trivial Bernstein algebra of type $(r+1, s)$. Furthermore A is isomorphic to the external direct sum of the vector spaces $\mathbb{K}$, $\mathbb{K}^r$ and $\mathbb{K}^s$

$$(1.19) \qquad A \cong \mathbb{K} \oplus \mathbb{K}^r \oplus \mathbb{K}^s = \{(\alpha, x, y) \mid \alpha \in \mathbb{K}, x \in \mathbb{K}^r, y \in \mathbb{K}^s\}$$

where the multiplication is given by

$$(1.20) \qquad (\alpha, x_1, y_1)(\beta, x_2, y_2) := (\alpha\beta, \tfrac{1}{2}(\alpha x_2 + \beta x_1), 0),$$
$$\text{for all} \quad \alpha, \beta \in \mathbb{K}, \quad x_i \in \mathbb{K}^r, \quad y_i \in \mathbb{K}^s, \quad i = 1, 2.$$

Furthermore the derivation algebra of a Bernstein algebra of type $(r+1, s)$ has maximal dimension, namely $r + r^2 + s^2$, if and only if it is a trivial one of this type, cf. [16].

2. The Peirce decompostion. For every finite-dimensional, commutative, power-associative algebra A over a field $\mathbb{K}$, char $\mathbb{K} \neq 2$, which possesses an idempotent $e \in A$ (this is equivalent to the fact that A is not a nilalgebra) there exists the Peirce decomposition, cf. Braun and Koecher, [5], § 12, or Schafer, [13], Chapter V.

Under the above assumptions it follows that the linear transformation $L_e: A \to A, x \mapsto e x$, satisfies

$$(2.1) \qquad 2L_e^3 - 3L_e^2 + L_e = 0.$$

Hence the minimal polynomial of L_e is a divisor of

$$(2.2) \qquad p(\lambda) := 2\lambda^3 - 3\lambda^2 + \lambda = (\lambda - 1)(2\lambda - 1)\lambda.$$

Therefore the eigenvalues of L_e are contained in the set $\{1, 1/2, 0\}$.

Define the subspaces

$$A_\lambda := \{x \in A \mid L_e x = \lambda x\}, \quad \lambda = 1, 1/2, 0.$$

Then A splits into the direct sum

$$(2.3) \qquad A = A_1 \oplus A_{1/2} \oplus A_0$$

which is called the *Peirce* decomposition of A with respect to e.

Furthermore the following relations are fulfilled

$$A_1^2 \subseteqq A_1, \quad A_1 A_{1/2} \subseteqq A_{1/2} \oplus A_0, \quad A_{1/2}^2 \subseteqq A_1 \oplus A_0$$
$$A_1 A_0 = \langle 0 \rangle, \quad A_{1/2} A_0 \subseteqq A_1 \oplus A_{1/2}, \quad A_0^2 \subseteqq A_0. \tag{2.4}$$

Since all alternative algebras and all Jordan algebras are power-associative, they possess a Peirce decomposition, provided they are finite-dimensional, commutative and have an idempotent. For Jordan algebras of this kind the complex products of A_1, $A_{1/2}$ and A_0 satisfy in addition to (2.4)

$$A_1 A_{1/2} \subseteqq A_{1/2}, \quad A_{1/2} A_0 \subseteqq A_{1/2}. \tag{2.5}$$

Again let A be a Bernstein algebra with idempotent e and decomposition (1.3)

$$A = E \oplus U \oplus Z.$$

Although A i. g. is not power-associative, as it will be indicated immediately, the algebra A possesses a Peirce decomposition with respect to e which is given by (1.3).

Let $x = \alpha e + u + z$ be an arbitrary element of $A = E \oplus U \oplus Z$. Then in view of (1.4) and (1.7) one has

$$x^2 = \alpha^2 e + (\alpha u + 2uz + z^2) + u^2, \tag{2.6}$$

where the intermediate bracket is an element of U and $u^2 \in Z$. With (1.2) one obtains

$$(x^2)^2 = \alpha^4 e + (\alpha^3 u + 2\alpha^2 uz + \alpha^2 z^2) + \alpha^2 u^2. \tag{2.7}$$

From formula (3.14) with y replaced by x and using (1.8)–(1.10) we obtain

$$\begin{aligned} x^4 = (x^2 x) x = \alpha^4 e + [\alpha^3 u + 2\alpha^2 uz + \tfrac{1}{4}\alpha^2 z^2 + \tfrac{3}{2}\alpha u^2 z + 3\alpha (uz) z \\ + \tfrac{1}{2}\alpha z^3 + (u^2 z) z + 2((uz) z) + 2((uz) u) z + z^4] \\ + [\alpha^2 u^2 + (u^2 z) u + 2((uz) z) z + z^3 u], \end{aligned} \tag{2.8}$$

thereby the first square bracket lies in U while the second square bracket is an element of Z. Comparing (2.7) and (2.8) we see that $z^2 = 0$ for all $z \in Z$ is a necessary condition for A to be power-associative. But there exist Bernstein algebras, where this condition is not fulfilled, cf. [15], p. 224, case (3).

Consider now the linear mapping $L_e: A \to A$, $x \mapsto ex$. Take

$$x = \alpha e + u + z \in E \oplus U \oplus Z.$$

Then from (1.4) and (1.7) we obtain

$$\begin{aligned} L_e x &= \alpha e^2 + eu + ez = \alpha e + \tfrac{1}{2} u \\ L_e^2 x &= \alpha e^2 + \tfrac{1}{2} eu = \alpha e + \tfrac{1}{4} u \\ L_e^3 x &= \alpha e^2 + \tfrac{1}{4} eu = \alpha e + \tfrac{1}{8} u, \end{aligned}$$

hence

$$(2L_e^3 - 3L_e^2 + L_e) x = 0 \quad \text{for all} \quad x \in A,$$

i. e.

$$2L_e^3 - 3L_e^2 + L_e = 0. \tag{2.9}$$

Therefore the eigenvalues of L_e are contained in $\{1, 1/2, 0\}$. From (1.4) and (1.7) it follows that the corresponding subspaces are E, U and Z, respectively. Hence (1.3) is the Peirce decomposition of A with respect to $e \in A$. From (1.4) and (1.15) it follows that the complex products of E, U and Z satisfy

$$\begin{aligned} &E^2 = E, \quad && EU = U, \quad && U^2 \subseteqq Z \\ &EZ = \langle 0 \rangle, \quad && UZ \subseteqq U, \quad && Z^2 \subseteqq U. \end{aligned} \tag{2.10}$$

Hence all relations in (2.4), apart from the last one, and in (2.5) with E, U, Z instead of $A_1, A_{1/2}, A_0$, which are valid for Jordan algebras, are fulfilled for the Peirce decomposition of a Bernstein algebra as well. It follows immediately that $Z^2 = \langle 0 \rangle$ is a necessary condition for a Bernstein algebra to be a Jordan algebra. A necessary and sufficient condition is given in Theorem 3.

If one has a Peirce decomposition of an algebra A with respect to an idempotent e, one can ask, whether this decomposition can be refined, whereby the idempotent e is decomposed into the sum of pairwise orthogonal, primitive idempotents. But it turns out that the Peirce decomposition (1.3) of a Bernstein algebra can be no more refined, cf. the subsequent theorem. Let me recall two definitions, cf. Schafer [13], p. 39:

An idempotent e of an arbitrary algebra A is called *primitive* if there do not exist two orthogonal idempotents $e_1, e_2 \in A$ ($e_1 e_2 = e_2 e_1 = 0$), such that $e = e_1 + e_2$. An idempotent $e \in A$ is called *principal*, if there do not exist idempotents which are orthogonal to e.

Theorem 1. *In every Bernstein algebra all idempotents are principal and primitive.*

P r o o f. Let e be an idempotent of a Bernstein algebra A, and let $A = E \oplus U \oplus Z$ be the corresponding Peirce decomposition. Then the set of idempotents of A is given by

$$\{e + u + u^2 \mid u \in U\}.$$

Two arbitrary idempotents of A have by (1.17) the form

$$e_i = e + u_i + u_i^2 \text{ for some } \quad u_i \in U, \quad i = 1, 2.$$

From (1.4) and (1.7) it follows that their product reduces to

$$e_1 e_2 = e + [\tfrac{1}{2}(u_1 + u_2) + u_1^2 u_2 + u_1 u_2^2 + u_1^2 u_2^2] + u_1 u_2,$$

which is different from zero in every case. Hence every idempotent is principal. Therefore every idempotent is primitive, too. □

In arbitrary, finite-dimensional algebra every idempotent can be decomposed into the sum of pairwise orthogonal, primitive idempotents, cf. [15], p. 35. The above theorem implies that in every Bernstein algebra A every idempotent $e \in A$ can be no more decomposed, and hence there is no refinement of the Peirce decomposition.

3. Jordan algebras. For the proof of the main Theorem 3 we need parts of the following lemma.

Lemma 2. *Let $A = E \oplus U \oplus Z$ be a Bernstein algebra over a field* $\mathbb{K}$, char $\mathbb{K} \neq 2$. *Then the following equations are satisfied for all $u, u_i \in U$ and for all $z, z_i \in Z$, $i = 1, 2$:*

$$u_1^2 u_2 + 2(u_1 u_2) u_1 = 0, \tag{3.1}$$

$$u_1 (u_2 z) + u_2 (u_1 z) = 0, \tag{3.2}$$

$$u(z_1 z_2) = 0, \tag{3.3}$$

$$u_1^2 (u_2 z) + 2(u_1 u_2)(u_1 z) = 0, \tag{3.4}$$

$$(u z_2) z_1^2 + 2(u z_1)(z_1 z_2) = 0, \tag{3.5}$$

$$(u_1 z)(u_2 z) = 0, \tag{3.6}$$

$$(u z_1)(u z_2) = 0, \tag{3.7}$$

$$(u_1 u_2) z^2 = 0, \tag{3.8}$$

$$u^2 (z_1 z_2) = 0. \tag{3.9}$$

Proof. These identities are obtained from (1.8)–(1.14) by polarization, i.e. in the present case we replace u or z, if it appears at least quadratically, by $\alpha u_1 + \beta u_2$ or $\alpha z_1 + \beta z_2$, respectively, and then compare terms with equal powers in α and β, whereby we use (1.8)–(1.14). For example from (1.8) we obtain

$$(\alpha u_1 + \beta u_2)^3 = 0,$$

hence, using (1.8) for u_1 and u_2 instead of u, we obtain (3.1). Similarly $(1.8 + k)$ implies $(3.1 + k)$ for $k = 1, \ldots, 4$. Thereby (3.3) is equivalent to $UZ^2 = \langle 0 \rangle$ in (1.15). Furthermore (1.13) implies (3.6) and (3.7), and (1.14) implies (3.8) and (3.9). □

Before stating the main Theorem 3 let me recall the following definition: A commutative algebra A over a field $\mathbb{K}$ is called a (commutative) *Jordan* algebra, if the identity

$$x^2 (y x) = (x^2 y) x, \quad x, y \in A, \tag{3.10}$$

is satisfied, cf. [13] Chapter IV. In the following let $\mathbb{K}$ be a field of characteristic different from 2.

Theorem 3. *Let $A = E \oplus U \oplus Z$ be a Bernstein algebra over $\mathbb{K}$. Then A is a Jordan algebra if and only if*

$$Z^2 = \langle 0 \rangle \tag{3.11}$$

and the following equations are fulfilled for all $u, u_i \in U$ and $z, z_i \in Z$, $i = 1, 2$:

$$(u z_1) z_2 + (u z_2) z_1 = 0, \tag{3.12}$$

$$(u_1^2 u_2) z + 2((u_1 z) u_2) u_1 = 0, \tag{3.13}$$

$$((u z_1) z_2) z_1 = 0, \tag{3.14}$$

$$(u_1^2 u_2) u_1 = 0, \tag{3.15}$$

$$((u z_1) z_2) u = 0. \tag{3.16}$$

In view of $U^2 \subseteq Z$ and $UZ \subseteq U$, cf. (1.15), we obtain the following.

Corollary 4. *Let* $A = E \oplus U \oplus Z$ *be a Bernstein algebra. If* $Z^2 = \langle 0 \rangle$ *and* $(UZ)Z = \langle 0 \rangle$, *then* A *is a Jordan algebra.*

Proof of Theorem 3. Let $x = \alpha e + u_1 + z_1$ and $y = \beta e + u_2 + z_2$ be arbitrary elements of a Bernstein algebra $A = E \oplus U \oplus Z$. We compute the products $xy = yx$, x^2, $x^2(yx)$ and $(x^2 y)x$, whereby we use the relations $e^2 = e$, $e u_i = \frac{1}{2} u_i$ (1.7) and $e z_i = 0$ (1.4), $i = 1, 2$, and we separate the E-, U-, and Z-components; in every case the first square bracket lies in U, while the second square bracket is an element of Z.

$$xy = \alpha\beta e + [\tfrac{1}{2}\alpha u_2 + \tfrac{1}{2}\beta u_1 + u_1 z_2 + u_2 z_1 + z_1 z_2] + [u_1 u_2],$$

$$x^2 = \alpha^2 e + [\alpha u_1 + 2 u_1 z_1 + z_1^2] + [u_1^2],$$

$$\begin{aligned} x^2(yx) = \alpha^3 \beta e &+ [\tfrac{1}{4}\alpha^3 u_2 + \tfrac{3}{4}\alpha^2 \beta u_1 + \tfrac{1}{2}\alpha^2 (u_1 z_2 + u_2 z_1 + z_1 z_2) \\ &\quad + \tfrac{1}{2}\alpha\beta(2u_1 z_1 + z_1^2) + \tfrac{1}{2}\alpha(2u_1(u_1 u_2) + u_1^2 u_2) + \tfrac{1}{2}\beta u_1^3 \\ &\quad + 2(u_1 z_1)(u_1 u_2) + u_1^2(u_2 z_1) + z_1^2(u_1 u_2) \\ &\quad + u_1^2(u_1 z_2) + u_1^2(z_1 z_2) + u_1^2(u_1 u_2)] \\ &+ [\tfrac{1}{2}\alpha^2 u_1 u_2 + \tfrac{1}{2}\alpha\beta u_1^2 + \alpha(u_1(u_1 z_2) + u_1(u_2 z_1) + (u_1 z_1) u_2 \\ &\quad + u_1(z_1 z_2) + \tfrac{1}{2} z_1^2 u_2) + \tfrac{1}{2}\beta(2(u_1 z_1) u_1 + z_1^2 u_1) \\ &\quad + 2(u_1 z_1)(u_1 z_2) + 2(u_1 z_1)(u_2 z_1) + (u_1 z_1)(z_1 z_2) \\ &\quad + z_1^2(u_1 z_2) + z_1^2(u_2 z_1) + z_1^2(z_1 z_2)], \end{aligned}$$

$$\begin{aligned} (x^2 y)x = \alpha^3 \beta e &+ [\tfrac{1}{4}\alpha^3 u_2 + \tfrac{3}{4}\alpha^2 \beta u_1 + \tfrac{1}{2}\alpha^2 (u_1 z_2 + u_2 z_1) \\ &\quad + \tfrac{1}{2}\alpha\beta(4u_1 z_1 + z_1^2) + \tfrac{1}{2}\alpha(u_1^2 u_2 + 2(u_1 u_2) u_1 + u_1^2 z_2 \\ &\quad + 2(u_1 z_1) z_2 + z_1^2 z_2 + 2(u_1 z_2) z_1 + 2(u_1 u_2) z_1) \\ &\quad + \tfrac{1}{2}\beta(2(u_1 z_1) z_1 + z_1^3) + (u_1^2 u_2) z_1 + (u_1^2 z_2) z_1 \\ &\quad + 2((u_1 z_1) z_2) z_1 + (z_1^2 z_2) z_1 + 2((u_1 z_1) u_2) u_1 \\ &\quad + 2((u_1 z_1) u_2) z_1] \\ &+ [\tfrac{1}{2}\alpha^2 u_1 u_2 + \tfrac{1}{2}\alpha\beta u_1^2 + \alpha(u_1 z_2) u_1 \\ &\quad + \tfrac{1}{2}\beta(2(u_1 z_1) u_1 + z_1^2 u_1) + (u_1^2 u_2) u_1 + (u_1^2 z_2) u_1 \\ &\quad + 2((u_1 z_1) z_2) u_1 + (z_1^2 z_2) u_1]. \end{aligned}$$

Now we use further identities which are satisfied in every Bernstein algebra, namely (1.7)–(1.11), (3.1)–(3.4) and (3.6)–(3.9), thereby the expressions for $x^2(yx)$ and $(x^2 y)x$ reduce to:

$$\begin{aligned} x^2(yx) = \alpha^3 \beta e &+ [\tfrac{1}{4}\alpha^3 u_2 + \tfrac{3}{4}\alpha^2 \beta u_1 + \tfrac{1}{2}\alpha^2 (u_1 z_2 + u_2 z_1 + z_1 z_2) \\ &\quad + \tfrac{1}{2}\alpha\beta(2u_1 z_1 + z_1^2) + u_1^2(u_1 u_2)] + [\tfrac{1}{2}\alpha^2 u_1 u_2 + \tfrac{1}{2}\alpha\beta u_1^2] \end{aligned} \tag{3.17}$$

and

$$(3.18)\qquad \begin{aligned}(x^2 y)x = \alpha^3 \beta e + [&\tfrac{1}{4}\alpha^3 u_2 + \tfrac{3}{4}\alpha^2 \beta u_1 + \tfrac{1}{2}\alpha^2 (u_1 z_2 + u_2 z_1)\\ &+ \tfrac{1}{4}\alpha\beta(4u_1 z_1 + z_1^2) + \tfrac{1}{2}\alpha(u_1^2 z_2 + 2(u_1 z_1) z_2 + z_1^2 z_2\\ &+ 2(u_1 z_2) z_1 + 2(u_1 u_2) z_1) + \tfrac{1}{2}\beta(2(u_1 z_1) z_1 + z_1^3)\\ &+ (u_1^2 u_2) z_1 + (u_1^2 z_2) z_1 + 2((u_1 z_1) z_2) z_1\\ &+ (z_1^2 z_2) z_1 + 2((u_1 z_1) u_2) u_1]\\ + [&\tfrac{1}{2}\alpha^2 u_1 u_2 + \tfrac{1}{2}\alpha\beta u_1^2 + (u_1^2 u_2) u_1 + (u_1^2 z_2) u_1\\ &+ 2((u_1 z_1) z_2) u_1 + (z_1^2 z_2) u_1].\end{aligned}$$

A necessary condition that the identity (3.10), valid in Jordan algebras, is fulfilled in the Bernstein algebra A is not only that the first summands of $x^2(yx)$ and $(x^2 y)x$ as well as their first and second square brackets coincide, but also that summands with equal powers in α and β in the corresponding square brackets are equal. Comparing the factors of α^2 in the first square brackets it follows immediately that

$$(3.19)\qquad z_1 z_2 = 0, \quad \text{i.e.} \quad Z^2 = \langle 0 \rangle,$$

is a necessary condition for A to be a Jordan algebra. This fact is already known from the properties of the Peirce decomposition of a Bernstein algebra and of a Jordan algebra, cf. (2.10) and (2.5).

Under the condition (3.19) and in view of $U^2 \subseteqq Z$ (1.15) the equations (3.17) and (3.18) reduce to

$$(3.20)\qquad \begin{aligned}x^2(yx) = \alpha^3 \beta e &+ [\tfrac{1}{4}\alpha^3 u_2 + \tfrac{3}{4}\alpha^2 \beta u_1 + \tfrac{1}{2}\alpha^2(u_1 z_2 + u_2 z_1) + \alpha\beta u_1 z_1]\\ &+ [\tfrac{1}{2}\alpha^2 u_1 u_2 + \tfrac{1}{2}\alpha\beta u_1^2],\end{aligned}$$

and

$$(3.21)\qquad \begin{aligned}(x^2 y)x = \alpha^3 \beta e + [&\tfrac{1}{4}\alpha^3 u_2 + \tfrac{3}{4}\alpha^2 \beta u_1 + \tfrac{1}{2}\alpha^2(u_1 z_2 + u_2 z_1) + \alpha\beta u_1 z_1\\ &+ \alpha((u_1 z_1) z_2 + (u_1 z_2) z_1) + \beta(u_1 z_1) z_1\\ &+ (u_1^2 u_2) z_1 + 2((u_1 z_1) z_2) z_1 + 2((u_1 z_1) u_2) u_1]\\ + [&\tfrac{1}{2}\alpha^2 u_1 u_2 + \tfrac{1}{2}\alpha\beta u_1^2 + (u_1^2 u_2) u_1 + 2((u_1 z_1) z_2) u_1].\end{aligned}$$

If we compare formulas (3.20) and (3.21) we see that in every Bernstein algebra $A = E \oplus U \oplus V$ with $Z^2 = \langle 0 \rangle$ a necessary and sufficient condition to be a Jordan algebra is that the following equations are fulfilled for all $u_i \in U$ and $z_i \in Z$, $i = 1, 2$.

$$(3.22)\qquad (u_1 z_1) z_2 + (u_1 z_2) z_1 = 0,$$

$$(3.23)\qquad (u_1 z_1) z_1 = 0,$$

$$(3.24)\qquad (u_1^2 u_2) z_1 + 2((u_1 z_1) z_2) z_1 + 2((u_1 z_1) u_2) u_1 = 0,$$

$$(3.25)\qquad (u_1^2 u_2) u_1 + 2((u_1 z_1) z_2) u_1 = 0.$$

Equation (3.22) coincides with (3.12) of the above theorem with u replaced by u_1. Furthermore equations (3.22) and (3.23) are equivalent, more precisely (3.23) follows from (3.22) by setting $z_1 = z_2$ and, conversely, (3.22) follows from (3.23) by the process of polarization, i.e. in the present case replacing z_1 by $z_1 + z_2$ and using (3.23). Consider now (3.24) and set $z_2 = 0$, then it follows that

$$(3.26) \qquad (u_1^2 u_2) z_1 + 2((u_1 z_1) u_2) u_1 = 0,$$

setting $u_2 = 0$, we obtain

$$(3.27) \qquad ((u_1 z_1) z_2) z_1 = 0.$$

Conversely, if (3.26) and (3.27) are satisfied, the equation (3.24) follows. From equation (3.25) we obtain for $z_2 = 0$

$$(3.28) \qquad (u_1^2 u_2) u_1 = 0,$$

and for $u_2 = 0$

$$(3.29) \qquad ((u_1 z_1) z_2) u_1 = 0.$$

Conversely (3.28) and (3.29) imply (3.25). Hence under the condition $Z^2 = \langle 0 \rangle$ (3.17) the Bernstein algebra A is a Jordan algebra if and only if (3.12)–(3.16) are satisfied. □

Lemma 5. *Let $A = E \oplus U \oplus Z$ be a trivial Bernstein algebra of type $(r + 1, s)$. Then*

(1) *A is a Jordan algebra,*
(2) *A is a special Jordan algebra.*

Proof. I. Since in every trivial Bernstein algebra A we have $(U \oplus Z)^2 = \langle 0 \rangle$, it follows that $Z^2 = \langle 0 \rangle$ and $UZ = \langle 0 \rangle$, hence equations (3.12)–(3.16) are fulfilled.

II. From [16] we know that A is isomorphic to $B := \mathbb{K} \oplus \mathbb{K}^r \oplus \mathbb{K}^s$, with multiplication given by (1.20). Consider now the algebra $C := \mathbb{K} \oplus \mathbb{K}^r \oplus \mathbb{K}^s$, whose multiplication is given by

$$(3.30) \qquad (\alpha, x_1, y_1) \cdot (\beta, x_2, y_2) := (\alpha\beta, \alpha x_2, 0).$$

This algebra is associative, as one checks easily. Then the multiplication in the associated Jordan algebra C^+ reads

$$\begin{aligned}(\alpha, x_1, y_1)(\beta, x_2, y_2) &= \tfrac{1}{2}[(\alpha, x_1, y_1) \cdot (\beta, x_2, y_2) + (\beta, x_2, y_2) \cdot (\alpha, x_1, y_1)] \\ &= \tfrac{1}{2}[(\alpha\beta, \alpha x_2, 0) + (\alpha\beta, \beta x_1, 0)] \\ &= (\alpha\beta, \tfrac{1}{2}(\alpha x_2 + \beta x_1), 0).\end{aligned}$$

Hence A is a isomorphic to C^+, which is a special Jordan algebra. □

Remark. The above result is a generalization of Holgate's result, cf. [7], who proved that all gametic algebras for simple Mendelian inheritance are special Jordan algebras. Thereby the gametic algebra for simple Mendelian inheritance with $n + 1$ alleles is, what we call, a trivial Bernstein algebra of type $(n + 1, 0)$, cf. [16].

4. Normal Bernstein algebras. Ljubič [10] has introduced the class of normal Bernstein algebras. He called a Bernstein algebra $A = E \oplus U \oplus Z$ over a field $\mathbb{K}$ of type $(r+1, s)$ with weight homomorphism $\omega\colon A \to \mathbb{K}$ normal, if the vector space of all linear forms $\varphi\colon a \to \mathbb{K}$, satisfying

$$\varphi(x, y) = \tfrac{1}{2}(\omega(x)\varphi(y) + \omega(y)\varphi(x)),$$

is of dimension $r+1$. Ljubič himself gave several equivalent characterizations of these algebras, one of which we shall use as definition here (without proving its equivalence to the original definition). Hereby a Bernstein algebra A with weight homomorphism $\omega\colon A \to \mathbb{K}$ is called *normal* if the following identity is valid

$$(4.1) \qquad x^2 y = \omega(x)\, x y \quad \text{for all} \quad x, y \in A.$$

It turns out that the existence of a non-trivial homomorphism $\omega\colon A \to \mathbb{K}$ together with the identity (4.1) already implies that A is a Bernstein algebra, i.e. (1.2) is fulfilled. This can be seen by replacing y by x^2, then using (4.1) twice, from where

$$(x^2)^2 = x^2 x^2 = (\omega(x)\,x)(\omega(x)\,x) = \omega(x)^2 x^2.$$

Therefore it would be more appropriate to use the following

Definition. Let A be an algebra over $\mathbb{K}$ with weight homomorphism $\omega\colon A \to \mathbb{K}$. Then A is called a *normal* algebra, if the identity (4.1) is satisfied in A.

The above calculations imply the following

Lemma 6. *Every normal algebra is a Bernstein algebra, i.e.* (1.2) *is satisfied.*

Beyond this, we have the following.

Theorem 7. *Every normal algebra A is a Jordan algebra.*

Proof. Consider (4.1) and replace y by yx, $x, y \in A$, then on the one side

$$x^2(yx) = (\omega(x)\,x)(yx) = \omega(x)\,x(yx)$$

and on the other side

$$(x^2 y)\,x = (\omega(x)\,xy) = \omega(x)\,x(yx),$$

hence

$$x^2(yx) = (x^2 y)\,x,$$

i.e. A is a Jordan algebra. □

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