1. Finite Element Methods for Stokes Equations

In this section, we study finite element methods for Stokes equations. The discrete inf-sup condition requires careful choices of spaces pair for the velocity and the pressure.

1.1. Babuška theory II. We consider the conforming discretization of the variational problem

(1) \( a(u, v) = \langle f, v \rangle \)

in the finite dimensional subspaces \( U_h \subset U \) and \( V_h \subset V \). Find \( u_h \in U_h \) such that

(2) \( a(u_h, v_h) = \langle f, v_h \rangle, \) for all \( v_h \in V_h \).

The existence and uniqueness of (2) is equivalent to the following discrete inf-sup conditions:

(\(D\)) \( \inf_{u_h} \sup_{v_h} \frac{a(u_h, v_h)}{\|u_h\| \|v_h\|} = \inf_{v_h} \sup_{u_h} \frac{a(u_h, v_h)}{\|u_h\| \|v_h\|} = \alpha_h > 0. \)

With appropriate choice of basis, (2) has a matrix form. To be well defined, first the matrix should be square. Second the matrix should be full rank (non singular). For a square matrix, the two inf-sup conditions are merged into one.

An abstract error analysis can be established. The key property for the conforming Garlerkin orthogonality

\( a(u - u_h, v_h) = 0, \) for all \( v_h \in V_h \).

\textbf{Theorem 1.1.} If the bilinear form \( a(\cdot, \cdot) \) satisfies (C), (E), (U) and (D), there exists a unique solution \( u \in U \) to (1) and a unique solution \( u_h \in U_h \) to (2). Furthermore

\( \|u - u_h\| \leq \frac{\|a\|}{\alpha_h} \inf_{v_h \in U_h} \|u - v_h\|. \)
Then for any $w_h \in U_h$, note that $P_h w_h = w_h$,

$$
\|u - u_h\| = \|(I - P)(u - w_h)\| \leq \|I - P_h\| \|u - w_h\|.
$$

Since $P_h^2 = P_h$, we use the identity in [10]:

$$
\|I - P_h\| = \|P_h\|,
$$

to get the desired result. □

1.2. Brezzi theory II. We consider finite element approximation to the mixed problem: Find $u_h \in \mathbb{V}_h$ and $p_h \in \mathbb{P}_h$ such that

\begin{align*}
(3) \quad & a(u_h, v_h) + b(v_h, p_h) = (f, v_h), \quad \text{for all } v_h \in \mathbb{V}_h, \\
(4) \quad & b(u_h, q_h) = (g, q_h), \quad \text{for all } q_h \in \mathbb{P}_h.
\end{align*}

We shall mainly consider the conforming case $\mathbb{V}_h \subset \mathbb{V}$ and $\mathbb{P}_h \subset \mathbb{P}$. We denote $B_h$, the restriction of $B$ to $\mathbb{V}_h$ and $\mathbb{Z}_h = N(B_h)$. Recall that $\mathbb{Z} = N(B)$. In the application to Stokes equations $B = -\text{div}$, so $\mathbb{Z}$ is called divergence free space and $\mathbb{Z}_h$ is discrete divergence free space.

**Remark 1.2.** In general $\mathbb{Z}_h \not\subset \mathbb{Z}$. Namely a discrete divergence free function may not be divergence free. Just compare $B_h u_h = 0$ in $(\mathbb{P}_h)'$

$$
\langle B_h u_h, q_h \rangle = 0, \quad \text{for all } q_h \in \mathbb{P}_h,
$$

with $B u_h = 0$ in $\mathbb{P}'$

$$
\langle B u_h, q \rangle = 0, \quad \text{for all } q \in \mathbb{P}.
$$

If we identify $\mathbb{P} = \mathbb{P}'$ and $\mathbb{P}_h = (\mathbb{P}_h)'$ using Riesz representation theorem, then $N(B_h) \subset (\mathbb{P}_h)^{\perp}$ which may contains non-trivial elements in $\mathbb{P}$. To enforce $\mathbb{Z}_h \subset \mathbb{Z}$, it suffices to have $B(\mathbb{V}_h) \subset \mathbb{P}_h$.

In operator form $B_h = Q_h B I_h$ where $I_h : \mathbb{V}_h \hookrightarrow \mathbb{V}$ and $Q_h : \mathbb{P}' \to (\mathbb{P}_h)'$ is the natural inclusion. So for $u_h \in N(B_h)$, i.e., $B_h u_h = 0$, it is possible that $B u_h \in \ker(Q_h) \cap B(\mathbb{V}_h)$. When $B(\mathbb{V}_h) \subset \mathbb{P}_h$, $Q_h B u_h = B u_h$ and thus $B_h u_h = 0$ implies $B u_h = 0$. □

The discrete inf-sup conditions for the finite element approximation will be
Theorem 1.3. If (A), (B), (C) and (D) hold, then the discrete problem is well-posed and
\[ \|u - u_h\|_V + \|p - p_h\|_p \leq C \inf_{v_h \in V_h, q_h \in P_h} \|u - u_h\|_V + \|p - p_h\|_p. \]

Exercise 1.4. Let \( \mathbb{U} = V \times P \) and rewrite the mixed formulation using one bilinear form defined on \( \mathbb{U} \). Then use Babuska theory to prove the above theorem. Write explicitly how the constant \( C \) depends on the constants in all inf-sup conditions.

1.3. Fortin operator. Verification of discrete inf-sup condition for the bilinear form \( a(\cdot, \cdot) \) is relatively easy. Again the difficult part is the verification of the inf-sup condition for the bilinear form \( b(\cdot, \cdot) \) or simply called \textit{div-stability} for Stokes equations.

Note that the inf-sup condition (B) in the continuous level implies: for any \( q_h \in P_h \), there exists \( v \in V \) such that \( b(v, q_h) \geq \beta \|v\|_V \|q_h\|_P \) and \( \|v\| \leq C \|q_h\| \). For the discrete inf-sup condition, we need a \( v_h \in V_h \) satisfying such property. One approach is to use the so-called Fortin operator [6] to get such a \( v_h \) from \( v \).

Definition 1.5 (Fortin operator). A linear operator \( \Pi_h : V \to V_h \) is called a Fortin operator if
\[ (1) \ b(\Pi_h v, q_h) = b(v, q_h) \text{ for all } q_h \in P_h \]
\[ (2) \ \|\Pi_h v\|_V \leq C \|v\|_V. \]

Theorem 1.6. Assume the continuous inf-sup condition (B) holds and there exists a Fortin operator \( \Pi_h \), then the discrete inf-sup condition (B) holds.

Proof. The inf-sup condition (B) in the continuous level implies: for any \( q_h \in P_h \), there exists \( v \in V \) such that \( b(v, q_h) \geq \beta \|v\|_V \|q_h\|_P \) and \( \|v\| \leq C \|q_h\| \). We choose \( v_h = \Pi_h v \).

By the definition of Fortin operator
\[ b(v_h, q_h) = b(v, q_h) \geq \beta \|v\|_V \|q_h\|_P \geq \beta C \|v_h\|_V \|q_h\|_P. \]
The discrete inf-sup condition then follows. \( \square \)

In the application to Stokes equations, \( P = L^2_0(\Omega) \) endowed with \( L^2 \)-norm \( \| \cdot \| \) and \( V = H^1_0(\Omega) \) with norm \( |v|_1 := \| \nabla v \| \). In the definition of Fortin operator, we require the operator is stable in \( | \cdot |_1 \)-norm and call it the \( H^1 \)-stability of the operator \( \Pi_h \).

When velocity spaces containing linear finite element space, it suffices to construct a Fortin operator stable in a weaker norm. Let us define a mesh dependent norm
\[ \|v\|_h = \|v\| + h|v|_1. \]
For \( v \in V_h \), by the inverse inequality \( \|v\|_h \approx \|v\| \).

Theorem 1.7. Suppose the velocity space \( V_h \) contains piecewise linear and continuous function space. Suppose there exists an operator \( \Pi_B : H^1_0(\Omega) \to V_h \) such that \( (\text{div} v - \text{div} \Pi_B v, q_h) = 0 \) for all \( q_h \in P_h \) and stable in \( \| \cdot \|_h \) norm which is equivalent to
\[ (5) \quad \|\Pi_B u\| \lesssim \|u\| + h|u|_1, \quad \text{for all } u \in H^1_0(\Omega), \]
then there exists a Fortin operator \( \Pi_h : H^1_0(\Omega) \to V_h \) and stable in \( H^1 \) norm.
Proof. Let $\Pi_1 : H^1_0(\Omega) \to P^1$ be the Scott-Zhang quasi-interpolation [?] which satisfies
\begin{equation}
|\Pi_1 u|_1 + h^{-1}||u - \Pi_1 u|| \lesssim |u|_1, \quad ||\Pi_1 u|| \lesssim ||u||.
\end{equation}
We define the Fortin operator as
$$\Pi_h u = \Pi_1 u + \Pi_B(u - \Pi_1 u).$$
Then $(\text{div} u - \text{div} \Pi_h u, q_h) = 0$ for all $q_h \in P_h$ by definition.

Now we prove the $H^1$-stability. By the triangle inequality, inverse inequality, stability of $\Pi_B$, and the property (6) of $\Pi_1$, we get the desired inequality
$$|\Pi_h u|_1 \leq |\Pi_1 u|_1 + |\Pi_B(u - \Pi_1 u)|_1 \lesssim |\Pi_1 u|_1 + h^{-1}||\Pi_B(u - \Pi_1 u)|| \lesssim |u|_1.$$

The idea is to apply a weaker stable Fortin operator to the high frequency $u - \Pi_1 u$. For high frequency functions, the weaker stability will imply the $H^1$ stability.

1.4. Finite Element Spaces for Stokes Equations. Given a triangulation $T$ of the domain $\Omega$, we shall use the following piecewise polynomial spaces
$$P_k(T) = \{ v \in C(\Omega) : v|_\tau \in P_k, \text{for all } \tau \in T \}, \text{ for } k \geq 1$$
$$P_k^{-1}(T) = \{ v \in L^2(\Omega) : v|_\tau \in P_k, \text{for all } \tau \in T \}, \text{ for } k \geq 0.$$

Here the superscript $^{-1}$ means the space is discontinuous. Finite element spaces will be chosen as $\forall h = (P_k(T))^n \cap H^1_0(\Omega)$ and $P_h = P_k(T) \cap L^2(\Omega)$ or $P_k^{-1}(T) \cap L^2(\Omega)$ for careful chosen integers $k$ and $l$. To simplify the notation, we simply write the space as $(P_k, P_k^{-1})$ or $(P_k, P_l)$. And we use $P_0$ for $P_0^{-1}$ since piecewise constant function is obviously discontinuous.

Here is a list of stable spaces pairs for Stokes equations.
- $(P_2, P_0)$: A simple element. Local mass conservation.
- $(P_1^{0,R}, P_0)$: Non-conforming velocity. Local divergence free.
- $(P_0^{0,0}, P_0)$ and $(P_0^{0,0}, P_0)$: Easy to code.
- $(P_k, P_k^{-1})$: stable if $k \geq 4$ in $\mathbb{R}^2$ and for meshes without singular-vertex. Exact divergence free. Scott-Vogelius element.
- $(P_k, P_k^{-1})$: Taylor-Hood element. Optimal convergent rate. Lowest order: $(P_2, P_1)$.
- $(P_1 + B_3, P_1)$: Mini element. Most economic element.
- $(P_k + B_{k+1}, P_k^{-1})$: stabilization using bubble functions. Lowest order: $(P_2 + B_3, P_1^{-1})$.

Before we discuss these pairs in detail, we emphasis several considerations:
- Since the inf-sup condition for Stokes equations holds in continuous level, for a fixed pressure space, the velocity space can be enlarged to get discrete inf-sup condition. The enlargement can be done by increasing the polynomial order or refining the mesh.
- We use Fortin operator approach to verify the div stability. This approach is relatively simple but has its own limitation. There are other methods to verify the inf-sup condition for Stokes equations: Verfurth [9], Boland and Nicolaides [2], and Stenberg [8].
- The equation $\text{div} u_h = 0$ holds in a weak topology and in general $\text{div} u_h \neq 0$ point-wise. To enforce $\text{div} u_h = 0$ point-wise, it is better to use $(P_k, P_k^{-1})$ since $\text{div} P_k \subset P_k^{-1}$. 
• Due to the coupling of \( u_h \) and \( p_h \), it is efficient to equilibrate the rates of convergence. Note the error measured in \( H^1 \) norm is usually one order lower than in \( L^2 \) norm. To balance the approximation order, it is better to use \((P_k, P_k^{-1})\) or \((P_k, P_{k-1})\).

• The trade-off between the increased accuracy of high-order elements and the increased complexity of those elements should be taken into account. Piecewise linear or constant function spaces will be much simpler to programming in practice.

1.4.1. \((P_1, p_0)\). The simplest and straightforward pair is \((P_1, P_0)\), i.e. using piecewise linear and continuous space for velocity and piecewise constant space for pressure. The continuity of the velocity space is due to the requirement \( \mathbb{V}_h \subset H^1_0(\Omega) \). Recall that a piecewise smooth function to be in \( H^1(\Omega) \) is equivalent to be globally continuous. The space for pressure is not necessary continuous since only \( L^2 \) integrable is required.

Unfortunately this simple pair is not suitable for the Stokes equations. The velocity space is not big enough to provide meaningful approximation. The discrete inf-sup condition cannot be true. The rectangular matrix representation \( B \) of the divergence operator is of dimension \( NT \times 2N \), where \( N \) is the number of interior nodes and \( NT \) is the number of triangles. Counting the angles nodal-wise and element-wise, we obtain the inequality \( 2\pi N < \pi NT \). Note that the inf-sup condition for \( B \) is equivalent to asking \( B \) is onto. So \( \text{rank}(B) = NT \). But it is impossible since \( 2N < NT \).

In other words, the gradient operator \( B^t \) contains kernel more than a global constant function. For the stable pair, \( B^t p = 0 \) implies \( p = \text{constant} \). For \((P_1, P_0)\) pair, there exists non-constant pressure \( p \) s.t. \( B^t p = 0 \) which is called spurious pressure modes. One way to stabilize the \((P_1, P_0)\) pair is to remove the spurious pressure modes. But this process is highly mesh dependent.

1.4.2. \((P_2, P_0)\). We enlarge the space of velocity to quadratic polynomials to get a stable pair.

We prove the discrete inf-sup condition by constructing a Fortin operator. By Theorem 1.7, we need only a \( L^2 \)-stable Fortin operator. Apply the integration by parts element by element, we obtain

\[
\sum_{\tau \in T} \int_{\tau} \text{div}(v - \Pi_h v) q_h = \sum_{\tau \in T} \int_{\partial\tau} (v - \Pi_h v) \cdot n \, q_h.
\]

Since \( q_h \) is piecewise constant, it is sufficient to construct a stable operator \( \Pi_h v \)

\[
(7) \quad \int_e v \, ds = \int_e \Pi_h v \, ds \quad \text{for all edges } e \text{ of } \mathcal{T}_h,
\]

and that \( \|\Pi_h v\| \leq \|v\|_h \).

Let us write \( P_2 = P_1 \oplus B_E \), where \( B_E \) is the quadratic bubble functions associated to edges. Then (7) is indeed define a function in \( B_E \). More specifically, let \( e \) be an edge with vertices \( v_i, v_j \). Denoted by \( b_e = 6\phi_i\phi_j/|e| \) where \( \phi_i \) is standard hat basis for \( P_1 \). By Simpson rule, the integral \( \int_e b_e = 1 \). Then the operator \( \Pi^B_h v := \sum_{e \in E} \left( \int_e v \, ds \right) b_e \)
satisfies (7). Now we check the stability. For bubble function spaces, since \( b_e \) are finite overlapping,

\[
\|\Pi^B_{e} v\|^2 \lesssim \sum_{e \in E} \left( \int_{e} v \, dt \right)^2 \|b_e\|^2 \lesssim \sum_{T} \left( \int_{T} |w|^2 + h^2 |\nabla w|^2 \, dx \right) = \|v\|^2 + h^2 \|\nabla v\|^2.
\]

In the second step, we have used Cauchy-Schwarz inequality and the scaled trace theorem for integral on edges: for any function \( g \in H^1(T) \)

\[
\|g\|_{e}^2 \leq C \left( h_{T}^{-1} \|g\|_T^2 + h_{T} \|\nabla g\|_T^2 \right).
\]

The drawback of this stable pair is that:

- \( Z^h \not\subset Z \) since \( \text{div} \mathcal{P}_2 \subset \mathcal{P}_1^{-1} \). The velocity approximation \( u_h \) is thus not point-wise divergence free. Nevertheless the mass conservation holds in each element.
- the approximation is only first order since \( \|p - p_h\| \leq Ch \) although the velocity space could provide one order higher approximation.

1.4.3. (\( \mathcal{P}_k, \mathcal{P}_{k-1}^{-1} \)). Scott and Vogelius [7] showed that the inf-sup condition holds for (\( \mathcal{P}_k, \mathcal{P}_{k-1}^{-1} \)) pairs in 2D if \( k \geq 4 \) provided the meshes are singular-vertex free. An internal vertex in 2D is said to be singular if edges meeting at the point fall into two straight lines. Note that one can perturb the singular vertex to easily get singular-vertex free triangulations.

The relation \( \text{div} \mathcal{P}_k \subset \mathcal{P}_{k-1}^{-1} \) implies that the pointwise divergence free for the approximated velocity \( u_h \) which is a desirable property (since the conservation of mass everywhere.) The convergent rate is optimal

\[
\|u - u_h\|_1 + \|p - p_h\| \lesssim h^k,
\]

provided the solution \( (u, p) \) are smooth enough, say \( u \in H^{k+1}(\Omega), p \in H^k(\Omega) \) which is not likely to hold in practice.

The drawback is the complication of programming. There are a lot of unknowns for high order polynomials for vector functions and for discontinuous polynomials. For example, locally for one triangle, the lowest order element (\( \mathcal{P}_4, \mathcal{P}_3^{-1} \)) contains 30 d.o.f for velocity and 10 for pressure. Globally the dimension of the velocity space is \( 2(N+3NE+3NT) \approx 32N \) and the dimension of the pressure space is \( 10NT \approx 20N \). The stability of this type of pair in 3D is not clear and partial results can be found in [11].

1.4.4. (\( \mathcal{P}_k, \mathcal{P}_{k-1} \)). If we use continuous space for the pressure, then the degree of freedom for pressure can be saved a lot. For example, the dimension of \( \mathcal{P}_1^{-1} \) is \( 3NT \) which is almost 6 times larger than \( N \), the dimension of \( \mathcal{P}_1 \).

Going from a discontinuous space to a continuous one, the dimension of pressure space is reduced. Then it is optimistic that the velocity space becomes big enough to have the div-stability. Indeed one can show the pair (\( \mathcal{P}_k, \mathcal{P}_{k-1} \)) for \( k \geq 2 \) satisfy the div-stability. This is known as Taylor-Hood (or Hood-Taylor) elements. The proof of the div-stability is delicate. We shall skip it here and refer to, for example, [3, 4], sketch a proof on \( P_2 - P_1 \) using edge element.

For this stable pair, we still maintain the optimal convergent order; see (9). The pair is stable for \( k \geq 2 \). The simplest case \( k = 2 \) (not \( k = 1 \) since \( \mathcal{P}_1, P_0 \) is unstable), \( \mathcal{P}_2, \mathcal{P}_1 \) is very popular. It uses less degree of freedom than the stable pair \( \mathcal{P}_2, \mathcal{P}_0 \) but provide one order higher approximation.

The drawback of Hood-Taylor elements is: First it is still not point-wise divergence free. Second since continuous pressure space is used, there is no element-wise mass
conservation. A simple fix is adding the piecewise constant into the pressure space, i.e., \((P_k, P_{k-1} + P_0)\). The div stability of the modified Hood-Taylor elements can be found in xxx.

1.4.5. \((P_1 \bigoplus B_T, P_1)\). We can further reduce the degree of freedom of velocity space to get a stable pair. One well known element is the so-called mini-element developed by Arnold, Brezzi, and Fortin [1].

The idea is to add bubble functions to the velocity space \(B_T = \bigoplus \mathcal{B}_\tau,\ B_\tau = \text{span}\{\lambda_1, \lambda_2, \lambda_3\}\), to stabilize the unstable pair \((P_1, P_1)\).

To construct a Fortin operator \(\Pi_h\), we apply the integration by parts element by element to obtain

\[
\sum_{\tau \in T} \int_{\tau} \text{div}(v - \Pi_h v) q_h = \sum_{\tau \in T} \int_{\partial \tau} (v - \Pi_h v) \cdot n q_h - \sum_{\tau \in T} \int_{\tau} (v - \Pi_h v) \cdot \nabla q_h \]
\[
= - \sum_{\tau \in T} \int_{\tau} (v - \Pi_h v) \cdot \nabla q_h.
\]

Since \(\nabla q_h\) is constant, it suffices to get a stable operator such that \(\int_{\tau} v \, dx = \int_{\tau} \Pi_h v \, dx\) for all \(\tau \in T\). The bubble functions for each element is introduced for this purpose. Let us define \(\Pi_B v \in B_T\) by

\[
\int_{\tau} \Pi_B v \, dx = \int_{\tau} v \, dx, \quad \text{for all } \tau \in T.
\]

It is trivial to show \(\|\Pi_B\|\) is stable in \(L^2\) norm and then the desirable Fortin operator can be constructed using Theorem 1.7.

1.4.6. \((P_1^{CR}, P_0)\). An easy fix of the div-stability is through the sacrifices of conformity of the velocity space. From the proof of the stability of \((P_2, P_0)\) (see (7)), the degree of freedom on edges is important. We then introduce the following piecewise linear finite element space

\[
P_1^{CR} = \{ v \in L^2(\Omega), v|_\tau \in P_1(\tau), \int_e v \text{ is continuous for all } e \}.
\]

The superscript \(^{CR}\) is named after Crouzeix and Raviart who introduced this space in [5].

To impose the boundary condition, one can require \(\int_e v = 0\) for \(e \in \partial \Omega\). That is the boundary condition is not imposed pointwise but in a weak sense. One can easily show functions in \(P_1^{CR}\) is continuous at middle points of edges but not on vertices and thus \(P_1^{CR} \not\subset H^1(\Omega)\).

Follow the proof of the stability of \((P_2, P_0)\) (see (7)), one can also easily proof the stability of \((P_1^{CR}, P_0^{-1})\). This is the simplest stable element for Stokes equations. sketch a proof here.

The sacrifice is that \(P_1^{CR} \not\subset H^1(\Omega)\). One needs to show the violation is get controlled by estimating the consistency error carefully.
1.4.7. \((P_{1,h/2}, P_{0,h})\) and \((P_{1,h/2}, P_{1,h})\). Another way to enrich the velocity space is through the mesh refinement. We denoted by \(T_{h/2}\) a fine triangulation obtained by regular uniform refinement of \(T_h\), i.e., each triangle in \(T_h\) is divided into 4 similar triangles by connecting middle points of edges. \(P_{1,h/2}\) is piecewise linear and continuous finite element space on \(T_{h/2}\). Comparing with \(P_{1,h}\), new degree of freedoms are created on edges. Then \(P_{1,h/2}\) can be used to replace \(P_2\) in the stable pair \((P_2, P_0)\) and \((P_2, P_1)\). The benefit of replacing a better approximation space by a less accurate one is the simplify of programming.

REFERENCES


