MATHEMATICS OF COMPUTATION
Volume 93, Number 345, January 2024, Pages 55–110
https://doi.org/10.1090/mcom/3859
Article electronically published on June 7, 2023

FINITE ELEMENT DE RHAM AND STOKES COMPLEXES IN THREE DIMENSIONS

LONG CHEN AND XUEHAI HUANG

Abstract. Finite element de Rham complexes and finite element Stokes complexes with varying degrees of smoothness in three dimensions are systematically constructed in this paper. Smooth scalar finite elements in three dimensions are derived through a non-overlapping decomposition of the simplicial lattice. $H(\mathrm{div})$-conforming finite elements and $H(\mathrm{curl})$-conforming finite elements with varying degrees of smoothness are devised based on these smooth scalar finite elements. The finite element de Rham complexes with corresponding smoothness and commutative diagrams are induced by these elements. The div stability of the $H(\mathrm{div})$-conforming finite elements is established, and the exactness of these finite element complexes is proven.

1. Introduction

Hilbert complexes are of fundamental importance in theoretical analysis and the design of stable numerical methods for partial differential equations, as demonstrated in works such as [2–4, 14]. Recently, in [17], we constructed two-dimensional finite element complexes with varying degrees of smoothness, including finite element de Rham and Stokes complexes, finite element elasticity complexes, and finite element divdiv complexes. In the present work, we tackle an even more challenging problem, that of constructing finite element de Rham and Stokes complexes with varying degrees of smoothness in three dimensions.

Introduce the following Sobolev spaces on a domain $\Omega \subseteq \mathbb{R}^3$

$$\begin{aligned}
H^1(\Omega) &= \{\phi \in L^2(\Omega) : \operatorname{grad}\phi \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3)\},\\
\boldsymbol{H}(\mathrm{curl},\Omega) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3) : \operatorname{curl}\boldsymbol{u} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3)\},\\
\boldsymbol{H}(\operatorname{grad}\operatorname{curl},\Omega) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3) : \operatorname{curl}\boldsymbol{u} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3)\},\\
\boldsymbol{H}^1(\mathrm{curl},\Omega) &= \{\boldsymbol{u} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3) : \operatorname{curl}\boldsymbol{u} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^3)\},\\
\boldsymbol{H}(\mathrm{div},\Omega) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3) : \operatorname{div}\boldsymbol{u} \in L^2(\Omega)\}.
\end{aligned}$$

The de Rham complex reads as

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\operatorname{grad}} \boldsymbol{H}(\mathrm{curl},\Omega) \xrightarrow{\operatorname{curl}} \boldsymbol{H}(\mathrm{div},\Omega) \xrightarrow{\operatorname{div}} L^2(\Omega) \to 0. \tag{1}$$

Received by the editor June 8, 2022, and, in revised form, January 27, 2023, and April 8, 2023.
2020 *Mathematics Subject Classification.* Primary 65N30, 58J10, 65N12.
The first author was supported by NSF DMS-2012465, DMS-2136075, and DMS-2309785. The second author was supported by the National Natural Science Foundation of China Project 12171300, and the Natural Science Foundation of Shanghai 21ZR1480500.

The second author is the corresponding author.

The Stokes complexes read as

$$\mathbb{R} \hookrightarrow H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\operatorname{grad}\operatorname{curl}, \Omega) \xrightarrow{\text{curl}} \boldsymbol{H}^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} L^2(\Omega) \to 0, \tag{2}$$

$$\mathbb{R} \hookrightarrow H^2(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}^1(\operatorname{curl}, \Omega) \xrightarrow{\text{curl}} \boldsymbol{H}^1(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} L^2(\Omega) \to 0. \tag{3}$$

For simplicity, we assume Ω is homeomorphic to a ball and thus the de Rham complex (1) and Stokes complexes (2)-(3) are exact. That is

$$\mathbb{R} = \ker(\operatorname{grad}), \ker(\operatorname{curl}) = \operatorname{img}(\operatorname{grad}), \ker(\operatorname{div}) = \operatorname{img}(\operatorname{curl}), \operatorname{img}(\operatorname{div}) = L^2(\Omega).$$

The surjectivity of the div operator is also called the div stability.

In a recent work [32], Hu, Lin and Wu have constructed a C^m-conforming finite element on simplexes in arbitrary dimension. We use the simplicial lattice to give a geometric decomposition of such smooth finite elements $\mathbb{V}^{\text{grad}}$ and $\mathbb{V}^{L^2}$ for the scalar function spaces $H^1(\Omega)$ and $L^2(\Omega)$ in Section 3.2, respectively. Following our recent work [17], we introduce the simplicial lattice $\mathbb{T}^3_k = \big\{\alpha = (\alpha_0, \dots, \alpha_3) : \alpha_i \text{ are non-negative integers}, i = 0, \dots, 3, \text{ and } \alpha_0 + \cdots + \alpha_3 = k\big\}$. The Bernstein basis for polynomial space $\mathbb{P}_k(T)$ is $\{\lambda^\alpha := \lambda_0^{\alpha_0} \cdots \lambda_3^{\alpha_3}, \alpha \in \mathbb{T}^3_k\}$, where $\lambda = (\lambda_0, \dots, \lambda_3)$ is the barycentric coordinate.

An integer vector $\boldsymbol{r} = (r^{\text{v}}, r^e, r^f)$ is called a smoothness vector if $r^f \geq -1$, $r^e \geq \max\{2r^f, -1\}$ and $r^{\text{v}} \geq \max\{2r^e, -1\}$. For a smoothness vector $\boldsymbol{r}$, define $\boldsymbol{r} \ominus 1 := \max\{\boldsymbol{r} - 1, -1\}$ and $\boldsymbol{r}_+ := \max\{\boldsymbol{r}, \boldsymbol{0}\}$ applied component-wise. For integer $k \geq \max\{2r^{\text{v}} + 1, 0\}$, we shall derive a geometric decomposition of the polynomial space $\mathbb{P}_k(T)$ based on a partition of the simplicial lattice $\mathbb{T}^3_k$. With such geometric decomposition, we can give a precise characterization of the polynomial bubble space

$$\begin{aligned}\mathbb{B}_k(T; \boldsymbol{r}) := \{u \in \mathbb{P}_k(T) : \;& \nabla^j u \text{ vanishes at all vertices of } T \text{ for } j = 0, \dots, r^{\text{v}},\\ & \nabla^j u \text{ vanishes on all edges of } T \text{ for } j = 0, \dots, r^e,\\ & \text{and } \nabla^j u \text{ vanishes on all faces of } T \text{ for } j = 0, \dots, r^f\},\end{aligned}$$

and of edge bubble $\mathbb{B}_k(e; \boldsymbol{r})$ and face bubble $\mathbb{B}_k(f; \boldsymbol{r})$. Given a triangulation $\mathcal{T}_h$, we then construct C^{r^f}-continuous finite element spaces $\mathbb{V}_k(\boldsymbol{r})$ with $C^{r^{\text{v}}}$-smoothness at vertices, C^{r^e}-smoothness on edges, and C^{r^f}-smoothness on faces. Here C^{-1}-smoothness means discontinuity. Therefore if $r^f = -1$, $\mathbb{V}_k(\boldsymbol{r}) \subset L^2(\Omega)$ and for $r^f \geq 0$, $\mathbb{V}_k(\boldsymbol{r}) \subset H^1(\Omega)$.

Combining with the $t - n$ decomposition on subsimplexes introduced in [16], we can further characterize the polynomial bubble spaces

$$\begin{aligned}\mathbb{B}^{\text{curl}}_k(T, \boldsymbol{r}) &:= \{\boldsymbol{v} \in \mathbb{B}^3_k(T, \boldsymbol{r}) : \boldsymbol{v} \times \boldsymbol{n}|_{\partial T} = \boldsymbol{0}\},\\ \mathbb{B}^{\text{div}}_k(T, \boldsymbol{r}) &:= \{\boldsymbol{v} \in \mathbb{B}^3_k(T, \boldsymbol{r}) : \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial T} = 0\},\end{aligned}$$

and construct $H(\text{div})$-conforming finite element space $\mathbb{V}^{\text{div}}_k(\boldsymbol{r}_2) = \mathbb{V}^3_k(\boldsymbol{r}_2) \cap \boldsymbol{H}(\text{div}, \Omega)$ in Section 3.3. In Section 4, we establish the discrete div stability between finite element spaces

$$\operatorname{div} \mathbb{V}^{\text{div}}_k(\boldsymbol{r}_2) = \mathbb{V}^{L^2}_{k-1}(\boldsymbol{r}_2 \ominus 1). \tag{4}$$

The key is to prove the discrete div stability of bubble spaces in Section 4.2

$$\operatorname{div} \mathbb{B}^{\text{div}}_k(T; \boldsymbol{r}_2) = \mathbb{B}_{k-1}(T; \boldsymbol{r}_2 \ominus 1)/\mathbb{R}.$$

When (4) holds, we call parameters $(\boldsymbol{r}_2, \boldsymbol{r}_2 \ominus 1, k)$ div stable. See (30) for sufficient conditions of parameters for such discrete div stability. In Section 4.4, we further construct $H(\operatorname{div})$-conforming finite element space $\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3)$ with an inequality constraint on the smoothness vectors $\boldsymbol{r}_3 \geq \boldsymbol{r}_2 \ominus 1$, and prove the div stability

$$\operatorname{div} \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) = \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) \quad \boldsymbol{r}_3 \geq \boldsymbol{r}_2 \ominus 1.$$

When $r_2^f \geq 0$, $\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) \subset \boldsymbol{H}^1(\Omega; \mathbb{R}^3)$ and $(\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3), \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3))$ is a divergence free and stable finite element velocity-pressure pair for the Stokes equation.

By the aid of the degrees of freedom (DoFs) of spaces $\mathbb{V}_{k+2}^{\operatorname{grad}}(\boldsymbol{r}_0)$ and $\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3)$, we construct $H(\operatorname{curl})$-conforming finite element space $\mathbb{V}_{k+1}^{\operatorname{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \{\boldsymbol{v} \in \mathbb{V}_{k+1}^{\operatorname{curl}}(\boldsymbol{r}_1) : \operatorname{curl} \boldsymbol{v} \in \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2) \cap \ker(\operatorname{div})\}$ in Section 5.3. When identifying DoFs, we first keep DoFs for $\operatorname{curl} \boldsymbol{v} \in \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2)$, combine DoFs for $\mathbb{V}_{k+1}^3(\boldsymbol{r}_1)$, and eliminate linear dependent ones.

Let $\boldsymbol{r}_0 \geq 0, \boldsymbol{r}_1 = \boldsymbol{r}_0 - 1, \boldsymbol{r}_2 \geq \boldsymbol{r}_1 \ominus 1, \boldsymbol{r}_3 \geq \boldsymbol{r}_2 \ominus 1$ be a sequence of smoothness vectors. Assume $(\boldsymbol{r}_2, \boldsymbol{r}_3, k)$ is div stable, and $k \geq \max\{2r_1^{\mathrm{v}} + 1, 2r_2^{\mathrm{v}} + 1, 2r_3^{\mathrm{v}} + 2, 1\}$, we acquire the finite element de Rham complexes with various smoothness in three dimensions in Section 5.5

$$\mathbb{R} \xrightarrow{\subset} \mathbb{V}_{k+2}^{\operatorname{grad}}(\boldsymbol{r}_0) \xrightarrow{\operatorname{grad}} \mathbb{V}_{k+1}^{\operatorname{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) \xrightarrow{\operatorname{curl}} \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) \xrightarrow{\operatorname{div}} \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) \to 0, \tag{5}$$

and the following commutative diagram in Section 5.6

$$\begin{array}{ccccccccc}
\mathbb{R} & \xrightarrow{\subset} & C^\infty(\Omega) & \xrightarrow{\operatorname{grad}} & C^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{\operatorname{curl}} & C^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{\operatorname{div}} & C^\infty(\Omega) \longrightarrow 0 \\
 & & \big\downarrow I_h^{\operatorname{grad}} & & \big\downarrow I_h^{\operatorname{curl}} & & \big\downarrow I_h^{\operatorname{div}} & & \big\downarrow I_h^{L^2} \\
\mathbb{R} & \xrightarrow{\subset} & \mathbb{V}_{k+2}^{\operatorname{grad}}(\boldsymbol{r}_0) & \xrightarrow{\operatorname{grad}} & \mathbb{V}_{k+1}^{\operatorname{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) & \xrightarrow{\operatorname{curl}} & \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) & \xrightarrow{\operatorname{div}} & \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) \longrightarrow 0,
\end{array}$$

where $I_h^{\operatorname{grad}}, I_h^{\operatorname{curl}}, I_h^{\operatorname{div}}$, and $I_h^{L^2}$ are the canonical interpolation operators using the DoFs. In Section 5.7 we construct the first type finite element de Rham complexes.

When $r_2^f \geq 0$, space $\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) \subset \boldsymbol{H}^1(\Omega; \mathbb{R}^3)$ and $\mathbb{V}_{k+1}^{\operatorname{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) \subset \boldsymbol{H}(\operatorname{grad}\operatorname{curl}, \Omega)$. Therefore (5) becomes a finite element Stokes complex. Existing works on finite element Stokes complexes [41] and finite element de Rham complexes [21] on simplicial meshes are examples of (5). On finite element de Rham and Stokes complexes not covered by (5), we refer to [22, 27, 34] for some discrete Stokes complexes based on split meshes, whose shape functions are piece-wise polynomials. Non-conforming discretization of Stokes complex (2) in [36] and non-conforming discretizations of Stokes complex (3) in [28, 49] are conforming finite element de Rham complexes. More divergence free Stokes finite elements on simplicial meshes can be found in [42] and references therein. Divergence free Stokes finite elements on cubic meshes in arbitrary dimension and macro-elements on general convex quadrilaterals in two dimensions are designed in [43] and [44], respectively. In [7, 8], finite element complexes of arbitrary smoothness on cubic meshes in arbitrary dimension are constructed through the tensor product. Isogeometric divergence free discretizations of Stokes equation can be found in [12, 25].

The rest of this paper is organized as follows. The simplicial lattice and barycentric calculus are introduced in Section 2. In Section 3, the geometric decomposition of C^m-conforming finite elements in three dimensions and $H(\operatorname{div})$-conforming finite elements are studied. In Section 4, the div stability is proved, and smooth $H(\operatorname{div})$-conforming finite elements are constructed. Finite element de Rham complexes with various smoothness are devised in Section 5. Smooth scalar finite elements in arbitrary dimension are constructed in Appendix A.

2. Simplicial lattice and barycentric calculus

Let $T \subset \mathbb{R}^n$ be an n-dimensional simplex with vertices $\mathtt{v}_0, \mathtt{v}_1, \ldots, \mathtt{v}_n$ in general position. That is

$$T = \left\{ \sum_{i=0}^{n} \lambda_i \mathtt{v}_i : 0 \leq \lambda_i \leq 1, \sum_{i=0}^{n} \lambda_i = 1 \right\},$$

where $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ is called the barycentric coordinate, and volume of T is non-zero. We will write $T = \mathrm{Convex}(\mathtt{v}_0, \ldots, \mathtt{v}_n)$, where Convex stands for the convex combination. Some content of this section can be found in the book [37] but with different notation. Here notation on subsimplexes is adapted from [5].

2.1. **The simplicial lattice.** For two non-negative integers $l \leq m$, we will use the multi-index notation $\alpha \in \mathbb{N}^{l:m}$, meaning $\alpha = (\alpha_l, \cdots, \alpha_m)$ with integer $\alpha_i \geqslant 0$. The sum of a multi-index is $|\alpha| := \sum_{i=l}^{m} \alpha_i$ for $\alpha \in \mathbb{N}^{l:m}$. We can also treat α as a row vector with non-negative integer valued coordinates. We use the convention that: a vector $\alpha \geq c$ means $\alpha_i \geq c$ for all components $i = 0, 1, \ldots, n$. We define $\lambda^\alpha := \lambda_0^{\alpha_0} \cdots \lambda_n^{\alpha_n}$ for $\alpha \in \mathbb{N}^{0:n}$.

A simplicial lattice, also known as the principal lattice [45], of degree k and dimension n is a multi-index set of $n+1$ components and with fixed sum k, i.e.,

$$\mathbb{T}^n_k = \left\{ \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{0:n} : |\alpha| = \alpha_0 + \alpha_1 + \cdots + \alpha_n = k \right\}.$$

An element $\alpha \in \mathbb{T}^n_k$ is called a node of the lattice. It holds that

$$|\mathbb{T}^n_k| = \binom{n+k}{k} = \dim \mathbb{P}_k(T),$$

where $\mathbb{P}_k(T)$ denotes the set of real valued polynomials defined on T of degree less than or equal to k. Indeed the Bernstein basis of $\mathbb{P}_k(T)$ is

$$\{\lambda^\alpha := \lambda_0^{\alpha_0} \lambda_1^{\alpha_1} \ldots \lambda_n^{\alpha_n} : \alpha \in \mathbb{T}^n_k\}.$$

For a subset $S \subseteq \mathbb{T}^n_k$, we define

$$\mathbb{P}_k(S) = \mathrm{span}\{\lambda^\alpha, \alpha \in S \subseteq \mathbb{T}^n_k\}.$$

With such one-to-one mapping between the lattice node α and the Bernstein polynomial λ^α, we can study properties of polynomials through the simplicial lattice.

2.2. **Geometric embedding of a simplicial lattice.** We can embed the simplicial lattice into a geometric simplex by using α/k as the barycentric coordinate of node α. Given $\alpha \in \mathbb{T}^n_k$, the barycentric coordinate of α is given by

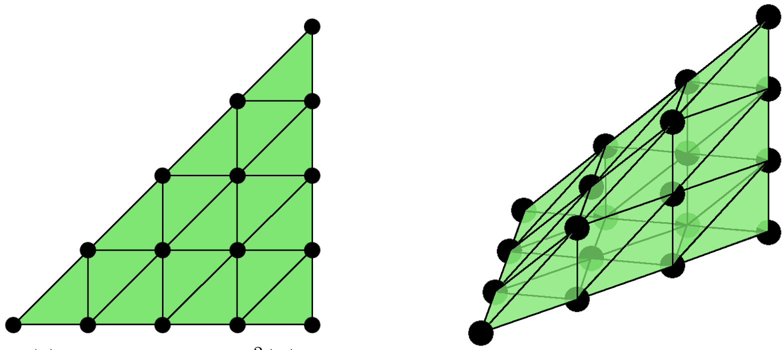
$$\lambda(\alpha) = (\alpha_0, \alpha_1, \ldots, \alpha_n)/k.$$

Let T be a simplex with vertices $\{\mathtt{v}_0, \mathtt{v}_1, \ldots, \mathtt{v}_n\}$. The geometric embedding is

$$x : \mathbb{T}^n_k \to T, \quad x(\alpha) = \sum_{i=0}^{n} \lambda_i(\alpha) \mathtt{v}_i.$$

For a visual representation, please refer to Fig. 1.

A simplicial lattice $\mathbb{T}^n_k$ is, by definition, an algebraic set. Through the geometric embedding $\mathbb{T}^n_k(T)$, we can use operators for the geometric simplex T to study this



(a) Simplicial lattice $\mathbb{T}^2_4(T)$

(b) Simplicial lattice $\mathbb{T}^3_3(T)$

FIGURE 1. Two examples of the simplicial lattices

algebraic set. For example, for a subset $S \subseteq T$, we use $\mathbb{T}^n_k(S) = \{\alpha \in \mathbb{T}^n_k, x(\alpha) \in S\}$ to denote the portion of lattice nodes whose geometric embedding is inside S. The superscript ^{n} will be replaced by the dimension of S when S is a lower dimensional subsimplex.

2.3. **Subsimplicial lattices.** Following [5], we let $\Delta(T)$ denote all the subsimplices of T, while $\Delta_\ell(T)$ denotes the set of subsimplices of dimension ℓ, for $0 \leq \ell \leq n$. The cardinality of $\Delta_\ell(T)$ is $\binom{n+1}{\ell+1}$. Elements of $\Delta_0(T) = \{\mathtt{v}_0, \ldots, \mathtt{v}_n\}$ are $n+1$ vertices of T and $\Delta_n(T) = \{T\}$.

For a subsimplex $f \in \Delta_\ell(T)$ with $\ell = 0, \ldots, n-1$, we will overload the notation f for both the geometric simplex and the algebraic set of indices. Namely $f = \{f(0), \ldots, f(\ell)\} \subseteq \{0, 1, \ldots, n\}$ and

$$f = \mathrm{Convex}(\mathtt{v}_{f(0)}, \ldots, \mathtt{v}_{f(\ell)}) \in \Delta_\ell(T)$$

is the ℓ-dimensional simplex spanned by the vertices $\mathtt{v}_{f(0)}, \ldots, \mathtt{v}_{f(\ell)}$.

If $f \in \Delta_\ell(T)$ with $\ell = 0, \ldots, n-1$, then $f^* \in \Delta_{n-\ell-1}(T)$ denotes the subsimplex of T opposite to f. When treating f as a subset of $\{0, 1, \ldots, n\}$, $f^* \subseteq \{0, 1, \ldots, n\}$ so that $f \cup f^* = \{0, 1, \ldots, n\}$, i.e., f^* is the complement of set f. Geometrically,

$$f^* = \mathrm{Convex}(\mathtt{v}_{f^*(1)}, \ldots, \mathtt{v}_{f^*(n-\ell)}) \in \Delta_{n-\ell-1}(T)$$

is the $(n-\ell-1)$-dimensional simplex spanned by vertices not contained in f.

Given a subsimplex $f \in \Delta_\ell(T)$ with $\ell = 0, \ldots, n-1$, through the geometric embedding $f \hookrightarrow T$, we define the prolongation/extension operator $E : \mathbb{T}^\ell_k \to \mathbb{T}^n_k$ as follows:

$$E(\alpha)_{f(i)} = \alpha_i, i = 0, \ldots, \ell, \qquad \text{and } E(\alpha)_j = 0, j \notin f.$$

For example, assume $f = \{1, 3, 4\}$, then for $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{T}^\ell_k(f)$, the extension $E(\alpha) = (0, \alpha_0, 0, \alpha_1, \alpha_2, \ldots, 0)$. The geometric embedding $x(E(\alpha)) \in f$ justifies the notation $\mathbb{T}^\ell_k(f)$. On the other hand, for $\alpha \in \mathbb{T}^n_k(T)$ and $f \in \Delta_\ell(T)$, the restriction $\alpha_f \in \mathbb{N}^{0:\ell}$ is defined as $(\alpha_f)_i = \alpha_{f(i)}$ for $i = 0, \ldots, \ell$. With a slight abuse of notation, for a node $\alpha_f \in \mathbb{T}^\ell_k(f)$, we still use the same notation $\alpha_f \in \mathbb{T}^n_k(T)$ to denote its extension $E(\alpha_f)$. Then for $\alpha \in \mathbb{T}^n_k(T)$ and $f \in \Delta_\ell(T)$ with $\ell = 0, \ldots,$

$n-1$, we have the following direct decomposition

$$\alpha = E(\alpha_f) + E(\alpha_{f^*}) = \alpha_f + \alpha_{f^*}, \text{ and } |\alpha| = |\alpha_f| + |\alpha_{f^*}|. \tag{6}$$

Based on (6), we can write a Bernstein polynomial as

$$\lambda^\alpha = \lambda_f^{\alpha_f} \lambda_{f^*}^{\alpha_{f^*}},$$

where $\lambda_f = \lambda_{f(0)} \dots \lambda_{f(\ell)} \in \mathbb{P}_{\ell+1}(f)$ is the bubble function on f.

In summary, by treating f as a set of indices, we can apply the operators $\cup, \cap, ^*, \setminus$ on sets. While treating f as a geometric simplex, $\partial f, \mathring{f}$ etc. can be applied.

2.4. **Distance.** Given $f \in \Delta_\ell(T)$, $\ell = 0, 1, \dots, n-1$, we define the distance of a node $\alpha \in \mathbb{T}_k^n$ to f as

$$\operatorname{dist}(\alpha, f) := |\alpha_{f^*}| = \sum_{i \in f^*} \alpha_i.$$

We present a geometric interpretation of $\operatorname{dist}(\alpha, f)$. Set the vertex $\mathtt{v}_{f(0)}$ as the origin and embed the lattice to the scaled reference simplex $k\hat{T} = \operatorname{Convex}\{\mathbf{0}, k\boldsymbol{e}_1, \dots, k\boldsymbol{e}_n\}$, where $\{\boldsymbol{e}_i, i = 1, \dots, n\}$ is the canonical basis of $\mathbb{R}^n$. Then $|\alpha_{f^*}| = s$ becomes the linear equation

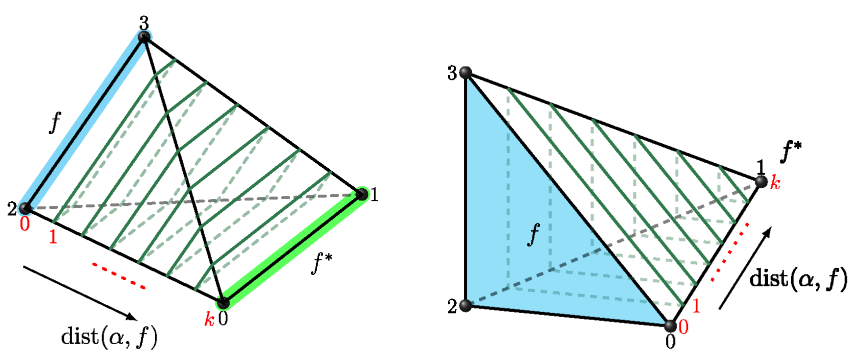
$$x_{f^*(1)} + x_{f^*(2)} + \cdots + x_{f^*(n-\ell)} = s,$$

which defines a hyper-plane in $\mathbb{R}^n$, denoted by $L(f, s)$, with a normal vector $\mathbf{1}_{f^*}$. The simplex f can be thought of as the convex combination of vectors $\{\boldsymbol{e}_{f(0)f(i)}\}_{i=1}^{\ell}$. Obviously $\mathbf{1}_{f^*} \cdot \boldsymbol{e}_{f(0)f(i)} = 0$ as the zero pattern is complementary to each other. So f is parallel to the hyper-plane $L(f, s)$. The distance $\operatorname{dist}(\alpha, f)$ for $\alpha \in L(f, s)$ is the intercept of the hyper-plane $L(f, s)$ and the basis vector $\boldsymbol{e}_{f(0)f(i)}$; see Fig. 2 for an illustration. In particular $f \in L(f, 0)$ and $\lambda_i|_f = 0$ for $i \in f^*$. Indeed $f = \{x \in T : \lambda_i(x) = 0, i \in f^*\}$.

We can extend the definition of distance to two subsimplexes. For $e \in \Delta_\ell(T)$, $f \in \Delta(T)$, define

$$\operatorname{dist}(e, f) := \min_{\alpha \in \mathbb{T}_k^\ell(e)} \operatorname{dist}(\alpha, f).$$

It is easy to verify that: for $e \in \Delta(f^*)$, $\operatorname{dist}(e, f) = k$ and for $e \in \Delta(f)$, i.e., $e \cap f \neq \varnothing$, then $\operatorname{dist}(e, f) = 0$.



(a) Distance to an edge (b) Distance to a face

FIGURE 2. Distance to a subsimplex

We define the lattice tube of f with distance r as

$$D(f,r) = \{\alpha \in \mathbb{T}^n_k, \operatorname{dist}(\alpha, f) \le r\},$$

which contains lattice nodes at most r distance away from f. We overload the notation

$$L(f,s) = \{\alpha \in \mathbb{T}^n_k, \operatorname{dist}(\alpha, f) = s\},$$

which is defined as a plane early but here is a subset of lattice nodes on this plane. Then

$$D(f,r) = \cup_{s=0}^{r} L(f,s), \quad L(f,s) = L(f^*, k-s).$$

By definition $D(f,-1) = \varnothing$, $D(f,0) = L(f,0) = f$, and $L(f,k) = f^*$. We have the following characterization of lattice nodes in $D(f,r)$.

Lemma 2.1. *For lattice node $\alpha \in \mathbb{T}^n_k$, and $f \in \Delta_\ell(T)$, $\ell = 0, 1, \ldots, n-1$,*

$$\alpha \in D(f,r) \iff |\alpha_{f^*}| \le r \iff |\alpha_f| \ge k - r,$$
$$\alpha \notin D(f,r) \iff |\alpha_{f^*}| > r \iff |\alpha_f| \le k - r - 1.$$

Proof. Use the definition of $\operatorname{dist}(\alpha, f)$ and the fact $|\alpha_f| + |\alpha_{f^*}| = k$. □

For each vertex $\mathtt{v}_i \in \Delta_0(T)$,

$$D(\mathtt{v}_i, r) = \{\alpha \in \mathbb{T}^n_k, |\alpha_{i^*}| \le r\},$$

which is isomorphic to a simplicial lattice $\mathbb{T}^n_r$ of degree r; see the green triangle in Fig. 3. For an $(n-1)$-face $f \in \Delta_{n-1}(T)$, $D(f,r)$ is a trapezoid of height r with base f. In general for $f \in \Delta_\ell(T)$, the hyper-plane $L(f,r)$ will cut the simplex T into two parts, and $D(f,r)$ is the part containing f. See Fig. 2 for illustration.

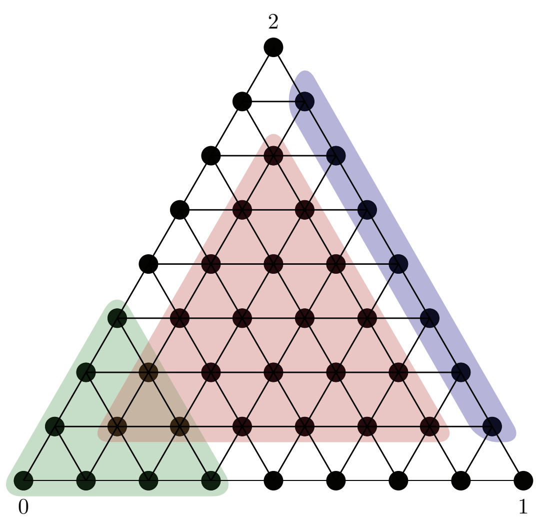


FIGURE 3. A simplicial lattice $\mathbb{T}^2_8(T)$ in two dimensions. The green triangle contains $D(\mathtt{v}_0, 3)$. The purple trapezoid is $D(\mathring{f}, 0)$. The red triangle is $\mathbb{T}^2_5(\mathring{T})$.

For two nodes $\alpha, \beta \in \mathbb{T}^n_k$, define the distance

$$\operatorname{dist}(\alpha, \beta) = \frac{1}{2}\|\alpha - \beta\|_{\ell_1}.$$

Two nodes $\alpha, \beta \in \mathbb{T}^n_k$ are adjacent if $\operatorname{dist}(\alpha, \beta) = 1$. By assigning edges to all adjacent nodes, the simplicial lattice $\mathbb{T}^n_k$ becomes an undirected graph and denoted by $\mathcal{G}(\mathbb{T}^n_k)$. The distance of two nodes in the graph is the length of a minimal path connecting them, where the length of a path is defined as the number of edges in the path. Obviously the graph $\mathcal{G}(\mathbb{T}^n_k)$ is connected. Graph theory can be further applied for the study of the lattice $\mathbb{T}^n_k$ and in turn the polynomial space.

Define $\epsilon_i \in \mathbb{N}^{0:n}$ as $\epsilon_i = (0, \ldots, 1, \ldots, 0)$ and $\epsilon_{ij} = \epsilon_i - \epsilon_j = (0, \ldots, 1, \ldots, -1, \ldots, 0)$.

Lemma 2.2. *For $\alpha, \beta \in \mathbb{T}^n_k$, it holds*

$$\operatorname{dist}(\alpha, \beta) = 1 \iff \beta = \alpha + \epsilon_{ij}, \quad \text{for some } i, j \in [0:n], i \neq j.$$

Here $[0 : n]$ is the set $\{0, 1, \ldots, n\}$.

Proof. Notice for two non-negative integers, if $\alpha_i \neq \beta_i$, then $|\alpha_i - \beta_i| \geq 1$. As $\alpha, \beta \in \mathbb{T}^n_k$, we have $\sum_{i=0}^n (\alpha_i - \beta_i) = 0$. The condition $\operatorname{dist}(\alpha, \beta) = 1$ means $\sum_{i=0}^n |\alpha_i - \beta_i| = 2$. So the only possibility is: $\alpha_i - \beta_i = -1$ and $\alpha_j - \beta_j = 1$ for some $0 \leq i, j \leq n$ and $i \neq j$. □

Note that $\operatorname{dist}(\alpha, \beta)$ is defined for two nodes while $\operatorname{dist}(\alpha, f)$ is between a node and a subsimplex. We show the two distance definitions are consistent.

Lemma 2.3. *For $\alpha \in \mathbb{T}^n_k$ and $f \in \Delta_\ell(T)$, for $\ell = 0, 1, \ldots, n-1$, it holds*

$$\operatorname{dist}(\alpha, f) = \min_{\beta_f \in \mathbb{T}^n_k(f)} \operatorname{dist}(\alpha, \beta_f).$$

Proof. For $\beta_f \in \mathbb{T}^n_k(f)$, since $|\beta_f| = k = |\alpha_f| + |\alpha_{f^*}|$, we have

$$\operatorname{dist}(\alpha, \beta_f) = \frac{1}{2}(\|\alpha_f - \beta_f\|_{\ell_1} + |\alpha_{f^*}|) \geq \frac{1}{2}(|\beta_f| - |\alpha_f| + |\alpha_{f^*}|) = |\alpha_{f^*}|.$$

Hence

$$\operatorname{dist}(\alpha, f) \leq \min_{\beta_f \in \mathbb{T}^n_k(f)} \operatorname{dist}(\alpha, \beta_f).$$

Then the equality holds by choosing $\beta_f = \alpha_f + |\alpha_{f^*}|\epsilon_{f(0)} \in \mathbb{T}^n_k(f)$. □

2.5. **Derivative and distance.** Recall that in [5] a smooth function u is said to vanish to order r on f if $D^\beta u|_f = 0$ for all $\beta \in \mathbb{N}^{1:n}, |\beta| < r$. The following result shows that the vanishing order r of a Bernstein polynomial λ^α on f is exactly the distance $\operatorname{dist}(\alpha, f)$.

Lemma 2.4. *Let $f \in \Delta_\ell(T)$ be a subsimplex of T. For $\alpha \in \mathbb{T}^n_k, \beta \in \mathbb{N}^{1:n}$, then*

$$D^\beta \lambda^\alpha|_f = 0, \quad \text{if } \operatorname{dist}(\alpha, f) > |\beta|.$$

Proof. For $\alpha \in \mathbb{T}^n_k$, we write $\lambda^\alpha = \lambda_f^{\alpha_f} \lambda_{f^*}^{\alpha_{f^*}}$. When $|\alpha_{f^*}| > |\beta|$, the derivative $D^\beta \lambda^\alpha$ will contain a factor $\lambda_{f^*}^\gamma$ with $\gamma \in \mathbb{N}^{1:n-\ell}$, and $|\gamma| = |\alpha_{f^*}| - |\beta| > 0$. Therefore $D^\beta \lambda^\alpha|_f = 0$ as $\lambda_i|_f = 0$ for $i \in f^*$. □

Denote by $\boldsymbol{t}_{j,i}$ the edge vector from $\mathtt{v}_j$ to $\mathtt{v}_i$. By computing the constant directional derivative $\boldsymbol{t}_{j,i} \cdot \nabla\lambda_\ell$ by values on the two vertices, we have

$$\boldsymbol{t}_{j,i} \cdot \nabla\lambda_\ell = \delta_{i\ell} - \delta_{j\ell} = \begin{cases} 1, & \text{if } \ell = i, \\ -1, & \text{if } \ell = j, \\ 0, & \text{if } \ell \neq i, j. \end{cases} \tag{7}$$

Lemma 2.5. *For* $\alpha \in \mathbb{T}^n_k$ *and* $0 \leq i \neq j \leq n$,

$$\nabla(\lambda^{\alpha+\epsilon_i}) \cdot \boldsymbol{t}_{j,i} = (\alpha_i + 1)\lambda^\alpha - \alpha_j \lambda^{\alpha+\epsilon_{ij}}. \tag{8}$$

Proof. By direct calculation and formula (7). □

The normalized basis $\lambda^\alpha/\alpha!$ has the constant integral

$$\int_T \frac{1}{\alpha!}\lambda^\alpha \,\mathrm{d}x = \frac{n!}{(k+n)!}|T|, \quad \forall\, \alpha \in \mathbb{T}^n_k. \tag{9}$$

We give an explicit form for functions in $\mathbb{P}_k(T)\cap L^2_0(T) = \{p \in \mathbb{P}_k(T) : \int_T p \,\mathrm{d}x = 0\}$.

Lemma 2.6. *For* $k \geq 0$, *it holds*

$$\mathbb{P}_k(T) \cap L^2_0(T) = \operatorname{span}\{\lambda^\alpha/\alpha! - \lambda^\beta/\beta! : \alpha, \beta \in \mathbb{T}^n_k \text{ and } \operatorname{dist}(\alpha,\beta) = 1\}.$$

Proof. By the integral formula (9), $\lambda^\alpha/\alpha! - \lambda^\beta/\beta! \in \mathbb{P}_k(T) \cap L^2_0(T)$ for $\alpha, \beta \in \mathbb{T}^n_k$.

As the graph $\mathcal{G}(\mathbb{T}^n_k)$ is connected, we can find a spanning tree $\mathcal{T}$ with number of edges equals $|\mathbb{T}^n_k|-1=\dim(\mathbb{P}_k(T)\cap L^2_0(T))$. So $\{\lambda^\alpha/\alpha!-\lambda^\beta/\beta! : [\alpha,\beta] \text{ is an edge in } \mathcal{T}\}$ is a basis of $\mathbb{P}_k(T) \cap L^2_0(T)$. Then the result follows as the edge of $\mathcal{T}$ is a subset of $\mathcal{G}(\mathbb{T}^n_k)$ and $[\alpha, \beta]$ is an edge iff $\operatorname{dist}(\alpha,\beta) = 1$ by Lemma 2.2. □

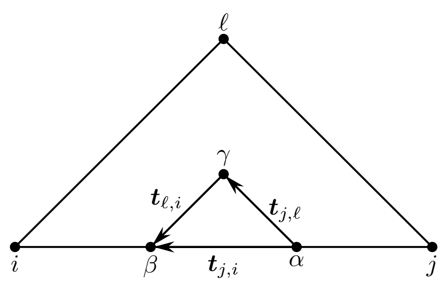


FIGURE 4. Velocity fields $\boldsymbol{u}$ satisfying $\operatorname{div}\boldsymbol{u} = \frac{1}{\alpha!}\lambda^\alpha - \frac{1}{\beta!}\lambda^\beta$ with $\beta = \alpha + \epsilon_{ij}$. One direction is $\boldsymbol{t}_{j,i}$ and another is a detour through $\gamma = \alpha + \epsilon_{\ell j} = \beta + \epsilon_{\ell i}$.

Given a function $p = \frac{1}{\alpha!}\lambda^\alpha - \frac{1}{\beta!}\lambda^\beta$, we can find two vector functions $\boldsymbol{u}$ satisfying $\operatorname{div}\boldsymbol{u} = p$.

Lemma 2.7. *Let* $\alpha, \beta \in \mathbb{T}^n_k$ *and* $\beta = \alpha + \epsilon_{ij}$. *Then*

$$\frac{1}{\beta!\alpha_j}\operatorname{div}(\lambda^{\alpha+\epsilon_i}\boldsymbol{t}_{j,i}) = \frac{1}{\alpha!}\lambda^\alpha - \frac{1}{\beta!}\lambda^\beta. \tag{10}$$

Proof. Direct calculation using $\operatorname{div}(\lambda^{\alpha+\epsilon_i}\boldsymbol{t}_{j,i}) = \nabla\lambda^{\alpha+\epsilon_i} \cdot \boldsymbol{t}_{j,i}$ and formula (8). See Figure 4. □

Lemma 2.8. *Let* $\alpha, \beta \in \mathbb{T}_k^n$ *and* $\beta = \alpha + \epsilon_{ij}$. *Choose an* $\ell \in [0:n], \ell \neq i, j$, *s.t.* $\gamma = \alpha + \epsilon_{\ell j} = \beta + \epsilon_{\ell i} \in \mathbb{T}_k^n$. *Then*

$$\frac{1}{\gamma!\alpha_j}\operatorname{div}(\lambda^{\alpha+\epsilon_\ell}\boldsymbol{t}_{j,\ell}) + \frac{1}{\gamma!\beta_i}\operatorname{div}(\lambda^{\beta+\epsilon_\ell}\boldsymbol{t}_{\ell,i}) = \frac{1}{\alpha!}\lambda^\alpha - \frac{1}{\beta!}\lambda^\beta. \tag{11}$$

Proof. By (10), we have

$$\frac{1}{\gamma!\alpha_j}\operatorname{div}(\lambda^{\alpha+\epsilon_\ell}\boldsymbol{t}_{j,\ell}) = \frac{1}{\alpha!}\lambda^\alpha - \frac{1}{\gamma!}\lambda^\gamma, \quad \frac{1}{\gamma!\beta_i}\operatorname{div}(\lambda^{\beta+\epsilon_\ell}\boldsymbol{t}_{\ell,i}) = \frac{1}{\gamma!}\lambda^\gamma - \frac{1}{\beta!}\lambda^\beta.$$

Then (11) follows. See Figure 4. ☐

3. Smooth finite elements in three dimensions

Previous work has constructed C^m-smooth finite elements in any dimension and a smooth $H(\operatorname{div})$-conforming element in two dimensions (see [32, Section 4.2 and 6.3]). In this section, we introduce a new approach to construct smooth finite elements in three dimensions using the simplicial lattice introduced in the previous section and the $t-n$ decomposition technique outlined in [16].

3.1. A decomposition of the simplicial lattice.

An integer vector $\boldsymbol{r} = (r^{\mathrm{v}}, r^e, r^f)$ is called a smoothness vector if $r^f \geq -1$, $r^e \geq \max\{2r^f, -1\}$ and $r^{\mathrm{v}} \geq \max\{2r^e, -1\}$. It is also denoted as $\boldsymbol{r} = (r^0, r^1, r^2)$, where the superscript $\ell = 0, 1, 2$ represents the dimension of the subsimplex. Sometimes, to simplify notation, we represent the vector $\boldsymbol{r}$ as a column vector.

Lemma 3.1. *For* $\ell = 1, 2$, *if* $r^{\ell-1} \geq 2r^\ell \geq 0$, *the subsets*

$$\left\{D(f, r^\ell)\backslash\left[\cup_{e\in\Delta_{\ell-1}(f)}D(e, r^{\ell-1})\right], f \in \Delta_\ell(T)\right\}$$

are disjoint.

Proof. Consider two different subsimplices $f, \tilde{f} \in \Delta_\ell(T), \ell = 1, 2$. The dimension of their intersection is at most $\ell - 1$. Therefore $f \cap \tilde{f} \subseteq e$ for some $e \in \Delta_{\ell-1}(f)$. For example, two faces will meet on an edge and two edges will share a vertex or with an empty intersection. Then $e^* \subseteq (f \cap \tilde{f})^* = f^* \cup \tilde{f}^*$. For $\alpha \in D(f, r^\ell) \cap D(\tilde{f}, r^\ell)$, we have $|\alpha_{e^*}| \leq |\alpha_{f^*}| + |\alpha_{\tilde{f}^*}| \leq 2r^\ell \leq r^{\ell-1}$. Therefore we have shown the intersection region $D(f, r^\ell) \cap D(\tilde{f}, r^\ell) \subseteq \cup_{e\in\Delta_{\ell-1}(f)}D(e, r^{\ell-1})$ and the result follows. ☐

Next we remove $D(e, r^i)$ from $D(f, r^\ell)$ for all $e \in \Delta_i(T)$ and $i = 0, 1, \ldots, \ell - 1$.

Lemma 3.2. *Given integer* $m \geq 0$, *let non-negative integer array* $\boldsymbol{r} = (r^0, r^1, r^2)$ *satisfy*

$$r^2 = m, \quad r^\ell \geq 2r^{\ell+1} \quad \text{for } \ell = 0, 1.$$

Let $k \geq 2r^0 + 1 \geq 8m + 1$. *For* $\ell = 1, 2$,

$$D(f, r^\ell)\backslash\left[\bigcup_{i=0}^{\ell-1}\bigcup_{e\in\Delta_i(f)}D(e, r^i)\right] = D(f, r^\ell)\backslash\left[\bigcup_{i=0}^{\ell-1}\bigcup_{e\in\Delta_i(T)}D(e, r^i)\right]. \tag{12}$$

Proof. In (12), the relation $\supseteq$ is obvious as $\Delta_i(f) \subseteq \Delta_i(T)$. To prove $\subseteq$, it suffices to show for $\alpha \in D(f, r^\ell)\backslash\left[\bigcup_{i=0}^{\ell-1}\bigcup_{e\in\Delta_i(f)}D(e, r^i)\right]$, it is also not in $D(e, r^i)$ for $e \in \Delta_i(T)$ and $e \notin \Delta_i(f)$.

By definition,

$$|\alpha_{f^*}| \leq r^\ell, \ |\alpha_e| \leq k - r_i - 1 \ \text{ for all } e \in \Delta_i(f), i = 0, \ldots, \ell - 1.$$

For each $e \in \Delta_i(T)$ but $e \notin \Delta_i(f)$, the dimension of the intersection $e \cap f$ is at most $i-1$. It follows from $r^j \geq 2r^{j+1}$ and $k \geq 2r^0+1$ that: when $i > 0$,

$$|\alpha_e| = |\alpha_{e\cap f}| + |\alpha_{e\cap f^*}| \leq k - r^{i-1} - 1 + r^\ell \leq k - r^i - 1,$$

and when $i=0$,

$$|\alpha_e| = |\alpha_{e\cap f^*}| \leq r^\ell \leq k - r^i - 1.$$

So $|\alpha_{e^*}| > r^i$. We conclude that $\alpha \notin D(e, r^i)$ for all $e \in \Delta_i(T)$ and (12) follows. $\square$

We are in the position to present an important partition of the simplicial lattice.

Theorem 3.3. *Given integer $m \geq 0$, let non-negative integer array $\boldsymbol{r} = (r^0, r^1, r^2)$ satisfy*

$$r^2 = m, \quad r^\ell \geq 2r^{\ell+1} \quad \textit{for } \ell = 0, 1.$$

Let $k \geq 2r^0 + 1 \geq 8m+1$. Then we have the following direct decomposition of the simplicial lattice on a tetrahedron T:

$$\mathbb{T}^3_k(T) = \bigoplus_{\ell=0}^3 \bigoplus_{f\in\Delta_\ell(T)} S_\ell(f, \boldsymbol{r}), \tag{13}$$

where

$$S_0(\mathtt{v}, \boldsymbol{r}) = D(\mathtt{v}, r^0),$$

$$S_\ell(f, \boldsymbol{r}) = D(f, r^\ell) \backslash \left[\bigcup_{i=0}^{\ell-1} \bigcup_{e\in\Delta_i(f)} D(e, r^i) \right], \ \ell = 1, 2,$$

$$S_3(T, \boldsymbol{r}) = \mathbb{T}^3_k(T) \backslash \left[\bigcup_{i=0}^{2} \bigcup_{f\in\Delta_i(T)} D(f, r^i) \right].$$

Consequently we have the following geometric decomposition of $\mathbb{P}_k(T)$

$$\mathbb{P}_k(T) = \bigoplus_{\ell=0}^3 \bigoplus_{f\in\Delta_\ell(T)} \mathbb{P}_k(S_\ell(f, \boldsymbol{r})). \tag{14}$$

Proof. First we show that the sets $\{S_\ell(f, \boldsymbol{r}), f \in \Delta_\ell(T), \ell = 0, \ldots, 3\}$ are disjoint. Take two vertices $\mathtt{v}_1, \mathtt{v}_2 \in \Delta_0(T)$. For $\alpha \in D(\mathtt{v}_1, r^0)$, we have $\alpha_{\mathtt{v}_1} \geq k - r^0$. As $\mathtt{v}_1 \in \mathtt{v}_2^*$ and $k \geq 2r^0+1$, $|\alpha_{\mathtt{v}_2^*}| \geq \alpha_{\mathtt{v}_1} \geq k - r^0 \geq r^0 + 1$, i.e., $\alpha \notin D(\mathtt{v}_2, r^0)$. Hence $\{S_0(\mathtt{v}), \mathtt{v} \in \Delta_0(T)\}$ are disjoint and $\bigoplus_{\mathtt{v}\in\Delta_0(T)} S_0(\mathtt{v})$ is a disjoint union. By Lemma 3.1 and (12), we know $\{S_\ell(f, \boldsymbol{r}), f \in \Delta_\ell(T), \ell = 0, \ldots, 3\}$ are disjoint.

Next we inductively prove

$$\bigoplus_{i=0}^{\ell} \bigoplus_{f\in\Delta_i(T)} S_i(f, \boldsymbol{r}) = \bigcup_{i=0}^{\ell} \bigcup_{f\in\Delta_i(T)} D(f, r^i) \quad \text{for } \ell = 0, 1, 2. \tag{15}$$

Obviously (15) holds for $\ell = 0$. Assume (15) holds for $\ell < j$. Then

$$\begin{aligned}
\bigoplus_{i=0}^{j} \bigoplus_{f\in\Delta_i(T)} S_i(f, \boldsymbol{r}) &= \bigoplus_{f\in\Delta_j(T)} S_j(f, \boldsymbol{r}) \oplus \bigcup_{i=0}^{j-1} \bigcup_{e\in\Delta_i(T)} D(e, r^i) \\
&= \bigoplus_{f\in\Delta_j(T)} \left(D(f, r^j) \backslash \left[\bigcup_{i=0}^{j-1} \bigcup_{e\in\Delta_i(T)} D(e, r^i) \right] \right) \oplus \bigcup_{i=0}^{j-1} \bigcup_{e\in\Delta_i(T)} D(e, r^i) \\
&= \bigcup_{i=0}^{j} \bigcup_{f\in\Delta_i(T)} D(f, r^i).
\end{aligned}$$

By induction, (15) holds for $\ell = 0, 1, 2$. Then (13) is true from the definition of $S_3(T, \boldsymbol{r})$ and (15). $\square$

Remark 3.4. The decomposition can be naturally extended to the case $r^\ell = -1$ by treating $D(f, -1) = \varnothing$. For example, when $\boldsymbol{r} = (-1, -1, -1)$, $S_\ell(f, \boldsymbol{r}) = \varnothing$ for $\ell = 0, 1, 2$, and $S_3(T, \boldsymbol{r}) = \mathbb{T}^3_k$.

Remark 3.5. From the implementation point of view, the index set $S_\ell(f, \boldsymbol{r})$ can be found by a logic array and set the entries as true when the distance constraint holds.

Introduce the polynomial bubble space on T

$$\mathbb{B}_k(T; \boldsymbol{r}) = \mathbb{P}_k(S_3(T, \boldsymbol{r})) = \operatorname{span}\{\lambda^\alpha,\ \alpha \in S_3(T, \boldsymbol{r})\}. \tag{16}$$

By Lemma 2.4, we can also write

$$\begin{aligned}\mathbb{B}_k(T; \boldsymbol{r}) := \{u \in \mathbb{P}_k(T) : \ & \nabla^j u \text{ vanishes at all vertices of } T \text{ for } j = 0, \ldots, r^{\texttt{v}},\\ & \nabla^j u \text{ vanishes on all edges of } T \text{ for } j = 0, \ldots, r^e,\\ & \text{and } \nabla^j u \text{ vanishes on all faces of } T \text{ for } j = 0, \ldots, r^f\}.\end{aligned}$$

Similarly the face bubble space on $f \in \Delta_2(T)$

$$\mathbb{B}_k(f; \begin{pmatrix} r^{\texttt{v}} \\ r^e \end{pmatrix}) = \operatorname{span}\left\{\lambda_f^\alpha,\ \alpha \in \mathbb{T}^2_k(f) \backslash \bigcup_{i=0}^{1} \bigcup_{e \in \Delta_i(f)} D(e, r^i)\right\},$$

and the edge bubble space

$$\mathbb{B}_k(e; r^{\texttt{v}}) = \operatorname{span}\left\{\lambda_e^\alpha = b_e^{r^{\texttt{v}}+1} \lambda_e^{\alpha - r^{\texttt{v}} - 1},\ \alpha \in \mathbb{T}^1_k(e) \backslash \bigcup_{\texttt{v} \in \Delta_0(e)} D(\texttt{v}, r^{\texttt{v}})\right\}.$$

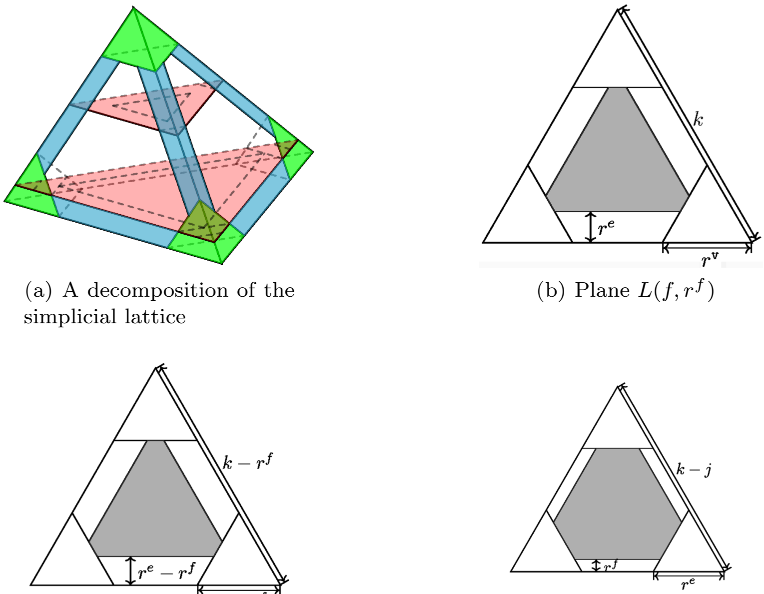
In general, on the lattice plane $L(f, j)$, the face bubble space becomes $\mathbb{B}_{k-j}(f; \begin{pmatrix} r^{\texttt{v}} \\ r^e \end{pmatrix} - j)$ and

$$S_3(T, \boldsymbol{r}) = \bigcup_{j = r^f + 1}^{k - r^{\texttt{v}} - 1} \left(S_2(f, \begin{pmatrix} r^{\texttt{v}} \\ r^e \end{pmatrix} - j) \cap L(f, j) \right).$$

We provide several slides of $L(f, j)$ and corresponding S_ℓ in Fig. 5(b), (c), (d).

3.2. **Smooth scalar finite elements in three dimensions.** In this subsection, we construct finite element spaces with smoothness parameter $\boldsymbol{r} = (r^{\texttt{v}}, r^e, r^f)$, which can be generalized to any dimension. The details of generalization are summarized in Appendix A.

For each edge e, we choose two normal vectors $\boldsymbol{n}_1^e, \boldsymbol{n}_2^e$ and abbreviate as $\boldsymbol{n}_1, \boldsymbol{n}_2$. For each face f, we choose a normal vector $\boldsymbol{n}_f$ and abbreviate as $\boldsymbol{n}$ when f is clear in the context. When in a conforming mesh $\mathcal{T}_h$, $\boldsymbol{n}_1^e, \boldsymbol{n}_2^e$ or $\boldsymbol{n}_f$ depends on e or f, not the element containing it. For face f with normal vector $\boldsymbol{n}$, the tangential part of vector $\boldsymbol{v}$ is denoted by $\Pi_f \boldsymbol{v} := \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{n})\boldsymbol{n}$ and $\Pi_f = I - \boldsymbol{n}\boldsymbol{n}^\intercal$ is the projection matrix. We use the convention: $j = 0, \ldots, -1$ means no such j. Similarly no j exists satisfying $0 \le j \le -1$.



(a) A decomposition of the simplicial lattice

(b) Plane $L(f, r^f)$

(c) Plane $L(f,j)$ for $r^e - r^f \le j \le r^{\mathrm{v}} - r^e$

(d) Plane $L(f,j)$ for $r^{\mathrm{v}} - r^e \le j \le k - r^{\mathrm{v}} - 1$

FIGURE 5. Different planes $L(f,j)$ and $S_2(f, \binom{r^{\mathrm{v}}}{r^e} - j)$ (in gray). The distance to a vertex is decreasing from r^{v} to r^e and the distance to an edge is from r^e to r^f.

Theorem 3.6. *Let $\boldsymbol{r} = (r^{\mathrm{v}}, r^e, r^f)$ with $r^f = m \ge -1$, $r^e \ge \max\{2r^f, -1\}$, $r^{\mathrm{v}} \ge \max\{2r^e, -1\}$, and non-negative integer $k \ge 2r^{\mathrm{v}} + 1$. The shape function space $\mathbb{P}_k(T)$ is determined by the DoFs*

$$\nabla^j u(\mathtt{v}), \quad \mathtt{v} \in \Delta_0(T), j = 0, 1, \ldots, r^{\mathrm{v}}, \tag{17a}$$

$$\int_e \frac{\partial^j u}{\partial n_1^i \partial n_2^{j-i}}\, q \,\mathrm{d}s, \quad e \in \Delta_1(T), q \in \mathbb{P}_{k-2(r^{\mathrm{v}}+1)+j}(e), 0 \le i \le j \le r^e, \tag{17b}$$

$$\int_f \frac{\partial^j u}{\partial n_f^j}\, q \,\mathrm{d}S, \quad f \in \Delta_2(T), q \in \mathbb{B}_{k-j}(f; \begin{pmatrix} r^{\mathrm{v}} \\ r^e \end{pmatrix} - j), 0 \le j \le r^f, \tag{17c}$$

$$\int_T u\, q \,\mathrm{d}x, \quad q \in \mathbb{B}_k(T; \boldsymbol{r}). \tag{17d}$$

With mesh $\mathcal{T}_h$, define the global C^m-continuous finite element space

$$\mathbb{V}_k(\mathcal{T}_h; \boldsymbol{r}) = \{u \in C^m(\Omega) : u|_T \in \mathbb{P}_k(T) \text{ for all } T \in \mathcal{T}_h, \text{ and all the DoFs (17a)-(17d) are single-valued}\}.$$

Proof. Thanks to the geometric decomposition (14), the number of DoFs (17a)-(17d) is same as $\dim \mathbb{P}_k(T)$. Take $u \in \mathbb{P}_k(T)$ and assume all the DoFs (17a)-(17d) vanish. We will prove $u = 0$.

The vanishing DoF (17a) implies $(\nabla^j u)(\mathtt{v}) = 0$ for $\mathtt{v} \in \Delta_0(T)$ and $0 \le j \le r^{\mathrm{v}}$, which combined with the vanishing DoF (17b) yields $(\nabla^j u)|_e = 0$ for $e \in \Delta_1(T)$

and $0 \leq j \leq r^e$. Then $\frac{\partial^j u}{\partial n_f^j}\Big|_f \in \mathbb{B}_{k-j}(f; \begin{pmatrix} r^{\mathrm{v}} \\ r^e \end{pmatrix} - j)$ for $f \in \Delta_2(T)$ and $0 \leq j \leq r^f$. Now the vanishing DoF (17c) indicates $(\nabla^j u)|_f = 0$ for $f \in \Delta_2(T)$ and $0 \leq j \leq r^f$. As a result $u \in \mathbb{B}_k(T; \boldsymbol{r})$. Therefore $u = 0$ follows from the vanishing DoF (17d).

Finally $\mathbb{V}_k(\mathcal{T}_h; \boldsymbol{r}) \subset C^m(\Omega)$ since we derive $(\nabla^j u)|_f = 0$ for $f \in \Delta_2(T)$ and $0 \leq j \leq m$ by using only DoFs (17a)-(17c) on f. ∎

To emphasize $\mathbb{V}_k(\mathcal{T}_h; \boldsymbol{r})$ as the discretization of $H^1(\Omega)$ or $L^2(\Omega)$, we will use notation $\mathbb{V}_k^{\mathrm{grad}}(\mathcal{T}_h; \boldsymbol{r}) \subset H^1(\Omega)$, for $r^f \geq 0$, or $\mathbb{V}_k^{L^2}(\mathcal{T}_h; \boldsymbol{r}) \subset L^2(\Omega)$ for $r^f \geq -1$, respectively.

Remark 3.7. A basis of $\mathbb{P}_k(T)$ dual to DoFs (17a)-(17d) might be computed via a symbolical computation as follows. We compute the distance of each node α to lower dimensional subsimplexes and use logic arrays to find $S_\ell(T, \boldsymbol{r})$. Then we evaluate DoFs for the Bernstein basis $\lambda^\alpha, \alpha \in S_\ell(T, \boldsymbol{r}), \ell = 0, 1, 2, 3$ to get a square matrix Φ. The matrix Φ is non-singular by the unisolvence and it is indeed block lower triangular, cf. the proof of Theorem A.6. The dual basis is then given by Φ^{-1} applied to the Bernstein basis.

Remark 3.8. When $\dim \mathbb{B}_k(T; \boldsymbol{r}) \geq 1$, DoF (17d) can be changed to

$$\int_T u\, q \,\mathrm{d}x, \quad q \in \mathbb{P}_0(T) \oplus (\mathbb{B}_k(T; \boldsymbol{r})/\mathbb{R}),$$

where $\mathbb{B}_k(T; \boldsymbol{r})/\mathbb{R} := \mathbb{B}_k(T; \boldsymbol{r}) \cap L_0^2(T)$. Similarly, when $\dim \mathbb{B}_k(f; \begin{pmatrix} r^{\mathrm{v}} \\ r^e \end{pmatrix}) \geq 1$, the face DoF $\int_f u\, q \,\mathrm{d}S, q \in \mathbb{B}_k(f; \begin{pmatrix} r^{\mathrm{v}} \\ r^e \end{pmatrix})$ can be changed to $\int_f uq \,\mathrm{d}S, q \in \mathbb{P}_0(f) \oplus (\mathbb{B}_k(f; \begin{pmatrix} r^{\mathrm{v}} \\ r^e \end{pmatrix})/\mathbb{R})$.

Lemma 3.9. *Depending on r^{v}, we require*

$$\begin{cases} k \geq 0 & \text{when } r^{\mathrm{v}} = -1, \\ k \geq 3 + r^e & \text{when } r^{\mathrm{v}} = 0, \\ k \geq \max\{2r^{\mathrm{v}} + 1, 3(r^e + 1)\} & \text{when } r^{\mathrm{v}} \geq 1. \end{cases}$$

Then

$$\dim \mathbb{B}_k(f; \begin{pmatrix} r^{\mathrm{v}} \\ r^e \end{pmatrix}) \geq 1.$$

Proof. The case $r^{\mathrm{v}} = -1, k \geq 0$ is trivial. The cases $(r^{\mathrm{v}}, r^e) = (0, -1), k \geq 2$ and $(r^{\mathrm{v}}, r^e) = (0, 0), k \geq 3$ can be verified directly. We only focus on the case $r^{\mathrm{v}} \geq 1$. Recall that $\lfloor x \rfloor$ is the nearest integer less than or equal to x and $\lceil x \rceil$ is the nearest integer greater than or equal to x. Set the node $\alpha = (\lfloor k/3 \rfloor, \lceil k/3 \rceil, k - \lfloor k/3 \rfloor - \lceil k/3 \rceil)$. The third component $\alpha_2 = \lfloor k/3 \rfloor$ if $\mathrm{mod}(k, 3) = 1$ and $\lceil k/3 \rceil$ otherwise. Then

$$\mathrm{dist}(\alpha, e_i) = \alpha_i \geq \lfloor k/3 \rfloor \geq r^e + 1, \quad i = 0, 1, 2,$$

where e_i is the edge opposite to vertex $\mathtt{v}_i$ for $i = 0, 1, 2$. We prove the distance to vertex

$$\mathrm{dist}(\alpha, \mathtt{v}_i) = \alpha_{i-1} + \alpha_{i+1} \geq k - \lceil k/3 \rceil \geq r^{\mathrm{v}} + 1,$$

where the last inequality can be derived from $k \geq 2r^{\mathrm{v}} + 1$ when $r^{\mathrm{v}} \geq 1$. So $\alpha \in S_2(f, \boldsymbol{r})$. ∎

The three-dimensional case is similar.

Lemma 3.10. *Depending on* r^{v}*, we require*

$$\begin{cases} k \geq 0 & \text{when } r^{\mathrm{v}} = -1, \\ k \geq 4 + r^e + r^f & \text{when } r^{\mathrm{v}} = 0, \\ k \geq \max\{2r^{\mathrm{v}} + 1, 4(r^f + 1)\} & \text{when } r^{\mathrm{v}} \geq 1. \end{cases}$$

Then $\dim \mathbb{B}_k(T; \boldsymbol{r}) \geq 1$.

Proof. The case $r^{\mathrm{v}} = -1, k \geq 0$ is trivial. The cases $(r^{\mathrm{v}}, r^e, r^f) = (0, -1, -1), k \geq 2$, $(r^{\mathrm{v}}, r^e, r^f) = (0, 0, -1), k \geq 3$, and $(r^{\mathrm{v}}, r^e, r^f) = (0, 0, 0), k \geq 4$ can be verified directly. So we focus on the case $r^{\mathrm{v}} \geq 1$. The requirement $k \geq 2r^{\mathrm{v}} + 1$ is asymptotically stronger. By enumerating the cases for r^{v} and r^e, we can verify $2\lfloor k/4 \rfloor + (\mathrm{mod}(k, 4) - 2)_+ \geq r^e + 1$, and $3\lfloor k/4 \rfloor + (\mathrm{mod}(k, 4) - 1)_+ \geq r^{\mathrm{v}} + 1$ when $r^{\mathrm{v}} \geq 1$.

We set first $\alpha_i = \lfloor k/4 \rfloor$ for $i = 0, 1, 2, 3$ and distribute the remainder $\mathrm{mod}(k, 4)$ as $\alpha_i \leftarrow \alpha_i + 1$ for $i = 0, \ldots, \mathrm{mod}(k, 4) - 1$. By construction $|\alpha| = k$. The distance to each face f

$$\mathrm{dist}(\alpha, f_{i^*}) = \alpha_i \geq \lfloor k/4 \rfloor \geq r^f + 1, \quad i = 0, 1, 2, 3.$$

The distance to an edge is the sum of two components. As some component can get additional 1, it is bounded by

$$\mathrm{dist}(\alpha, e) \geq 2\lfloor k/4 \rfloor + (\mathrm{mod}(k, 4) - 2)_+ \geq r^e + 1.$$

Similarly the distance to a vertex is bounded by

$$\mathrm{dist}(\alpha, \mathrm{v}) \geq 3\lfloor k/4 \rfloor + (\mathrm{mod}(k, 4) - 1)_+ \geq r^{\mathrm{v}} + 1.$$

So $\alpha \in S_3(T, \boldsymbol{r})$. ∎

Next we count the dimension of the finite element space $\mathbb{V}_k(\mathcal{T}_h; \boldsymbol{r})$. For integers $0 \leq i \leq j \leq n$, recall the combinatorial formula

$$\binom{j}{i} + \binom{j+1}{i} + \cdots + \binom{n}{i} = \binom{n+1}{i+1} - \binom{j}{i+1}, \tag{18}$$

which holds from $\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$. Here we understand $\binom{i}{i+1}$ as 0. For an n-dimensional simplex, the number $\binom{n+1}{i+1}$ of i-dimensional faces equals the sum of the number $\binom{n}{i}$ of i-dimensional faces including v_0 as a vertex and the number $\binom{n}{i+1}$ of i-dimensional faces excluding v_0.

Lemma 3.11. *The dimension of* $\mathbb{V}_k(\mathcal{T}_h; \boldsymbol{r})$ *is*

$$\dim \mathbb{V}_k(\mathcal{T}_h; \boldsymbol{r}) = \sum_{i=0}^{3} C_i(k, \boldsymbol{r}) |\Delta_i(\mathcal{T}_h)|,$$

where

$$
\begin{aligned}
C_0(k,\boldsymbol{r}) &= \binom{r^{\mathrm{v}}+3}{3},\\
C_1(k,\boldsymbol{r}) &= (k+r^e-2r^{\mathrm{v}}-1)\binom{r^e+2}{2}-\binom{r^e+2}{3},\\
C_2(k,\boldsymbol{r}) &= \binom{k+3}{3}-3\binom{r^{\mathrm{v}}+3}{3}-3\binom{k-2r^{\mathrm{v}}-1}{3}-\binom{k+2-r^f}{3}\\
&\quad +3\binom{r^{\mathrm{v}}+2-r^f}{3}-3(r^f+1)\binom{k-2r^{\mathrm{v}}+r^e}{2}+3\binom{k-2r^{\mathrm{v}}+r^f}{3},\\
C_3(k,\boldsymbol{r}) &= \binom{k+3}{3}-4C_0(k,\boldsymbol{r})-6C_1(k,\boldsymbol{r})-4C_2(k,\boldsymbol{r}).
\end{aligned}
$$

Proof. By (18) with $i=2$, the number of DoFs on edge $e\in\Delta_1(\mathcal{T}_h)$ is

$$
\begin{aligned}
\sum_{j=0}^{r^e}(j+1)(k-2r^{\mathrm{v}}-1+j) &= (k-2r^{\mathrm{v}}-1)\binom{r^e+2}{2}+2\sum_{j=0}^{r^e}\binom{j+1}{2}\\
&= (k+r^e-2r^{\mathrm{v}}-1)\binom{r^e+2}{2}-\binom{r^e+2}{3}.
\end{aligned}
$$

Applying (18) with $i=2$ again, the number of DoFs on face $f\in\Delta_2(\mathcal{T}_h)$ is

$$
\begin{aligned}
&\sum_{j=0}^{r^f}\dim\mathbb{B}_{k-j}(f;\binom{r^{\mathrm{v}}}{r^e}-j)\\
&=\sum_{j=0}^{r^f}\left[\binom{k+2-j}{2}-3\binom{r^{\mathrm{v}}+2-j}{2}-3\binom{k-2r^{\mathrm{v}}+r^e}{2}+3\binom{k-2r^{\mathrm{v}}-1+j}{2}\right]\\
&=\binom{k+3}{3}-\binom{k+2-r^f}{3}-3\binom{r^{\mathrm{v}}+3}{3}+3\binom{r^{\mathrm{v}}+2-r^f}{3}\\
&\quad -3(r^f+1)\binom{k-2r^{\mathrm{v}}+r^e}{2}+3\binom{k-2r^{\mathrm{v}}+r^f}{3}-3\binom{k-2r^{\mathrm{v}}-1}{3},
\end{aligned}
$$

which ends the proof. ☐

3.3. **Smooth $H(\mathrm{div})$-conforming finite elements.** For a linear space V, denote by $V^3:=V\otimes\mathbb{R}^3$. Let $\mathbb{V}_k^{\mathrm{div}}(\mathcal{T}_h;\boldsymbol{r}):=\mathbb{V}_k^3(\mathcal{T}_h;\boldsymbol{r})\cap\boldsymbol{H}(\mathrm{div},\Omega)$, where $\mathbb{V}_k(\mathcal{T}_h;\boldsymbol{r})$ is the scalar finite element space defined in Theorem 3.6. Define the polynomial bubble space

$$\mathbb{B}_k^{\mathrm{div}}(T;\boldsymbol{r}):=\ker(\mathrm{tr}^{\mathrm{div}})\cap\mathbb{B}_k^3(T;\boldsymbol{r}),$$

where $\mathrm{tr}^{\mathrm{div}}\,\boldsymbol{v}=\boldsymbol{n}\cdot\boldsymbol{v}|_{\partial T}$. When $r^f\geq 0$, $\mathbb{V}_k^{\mathrm{div}}(\mathcal{T}_h;\boldsymbol{r})=\mathbb{V}_k^3(\mathcal{T}_h;\boldsymbol{r})\subset\boldsymbol{H}^1(\Omega;\mathbb{R}^3)$ and $\mathbb{B}_k^{\mathrm{div}}(T;\boldsymbol{r})=\mathbb{B}_k^3(T;\boldsymbol{r})$. When $r^f=-1$, $\mathbb{V}_k^3(\mathcal{T}_h;\boldsymbol{r})$ is discontinuous and to be in $\boldsymbol{H}(\mathrm{div},\Omega)$, the normal direction should be continuous.

A precise characterization of $\mathbb{B}_k^{\mathrm{div}}(T;\boldsymbol{r})$ with $r^f=-1$ is given below, where for an integer $m\geq -1$, $m_+:=\max\{m,0\}$, and the Iverson bracket $[m=-1]=\begin{cases}1 & \text{if } m=-1,\\ 0 & \text{if } m\neq -1.\end{cases}$ For each face f, choose two linearly independent tangent vectors

$\{\boldsymbol{t}_f^1, \boldsymbol{t}_f^2\}$ and for each edge e, choose a tangent vector $\boldsymbol{t}_e$. Define

$$\mathbb{B}_k^{\mathrm{div}}(f;\boldsymbol{r}_+) := \mathbb{B}_k(f;\begin{pmatrix} r^{\mathrm{v}}_+ \\ r^e_+ \end{pmatrix}) \otimes \mathrm{span}\{\boldsymbol{t}_f^1, \boldsymbol{t}_f^2\},$$
$$\mathbb{B}_k^{\mathrm{div}}(e;\boldsymbol{r}_+) := \mathbb{B}_k(e; r^{\mathrm{v}}_+) \otimes \mathrm{span}\{\boldsymbol{t}_e\}.$$

Lemma 3.12. *Consider $\boldsymbol{r} = (r^{\mathrm{v}}, r^e, -1)$ with $r^{\mathrm{v}} \geq \max\{2r^e, -1\}$ and $r^e \geq -1$. We have*

$$\mathbb{B}_k^{\mathrm{div}}(T;\begin{pmatrix} r^{\mathrm{v}} \\ r^e \\ -1 \end{pmatrix}) = \mathbb{B}_k^3(T;\boldsymbol{r}_+) \bigoplus_{f\in\Delta_2(T)} \mathbb{B}_k^{\mathrm{div}}(f;\boldsymbol{r}_+) \bigoplus_{e\in\Delta_1(T)} [r^e = -1] \mathbb{B}_k^{\mathrm{div}}(e;\boldsymbol{r}_+). \tag{19}$$

Proof. For $\boldsymbol{r} = (-1,-1,-1)$, we have proved the desired decomposition in [16]

$$\mathbb{B}_k^{\mathrm{div}}(T;\boldsymbol{-1}) = \mathbb{B}_k^3(T;\boldsymbol{0}) \bigoplus_{f\in\Delta_2(T)} \mathbb{B}_k^{\mathrm{div}}(f;\boldsymbol{0}) \bigoplus_{e\in\Delta_1(T)} \mathbb{B}_k^{\mathrm{div}}(e;0).$$

We then consider the case $r^{\mathrm{v}} \geq 0$. By definition,

$$\mathbb{B}_k^3(T;\begin{pmatrix} r^{\mathrm{v}} \\ r^e \\ -1 \end{pmatrix}) = \mathbb{B}_k^3(T;\begin{pmatrix} r^{\mathrm{v}} \\ r^e_+ \\ 0 \end{pmatrix}) \bigoplus_{f\in\Delta_2(T)} \mathbb{B}_k^3(f;\begin{pmatrix} r^{\mathrm{v}} \\ r^e_+ \end{pmatrix}) \bigoplus_{e\in\Delta_1(T)} [r^e = -1] \mathbb{B}_k^3(e;r^{\mathrm{v}}).$$

We write

$$\mathbb{B}_k^3(f;\begin{pmatrix} r^{\mathrm{v}} \\ r^e_+ \end{pmatrix}) = \mathbb{B}_k(f;\begin{pmatrix} r^{\mathrm{v}} \\ r^e_+ \end{pmatrix}) \otimes \left(\mathrm{span}\{\boldsymbol{t}_f^1, \boldsymbol{t}_f^2\} + \mathrm{span}\{\boldsymbol{n}_f\}\right).$$

The intersection with $\ker(\mathrm{div})$ will keep the tangential components only. Similarly only $\mathbb{B}_k(e;r^{\mathrm{v}}) \otimes \mathrm{span}\{\boldsymbol{t}_e\}$ is left in the $t-n$ decomposition of $\mathbb{B}_k^3(e;r^{\mathrm{v}})$. ☐

Notice that we have the relation

$$\mathbb{B}_k^3(T;\begin{pmatrix} r^{\mathrm{v}}_+ \\ r^e_+ \\ 0 \end{pmatrix}) \subset \mathbb{B}_k^{\mathrm{div}}(T;\begin{pmatrix} r^{\mathrm{v}} \\ r^e \\ -1 \end{pmatrix}) \subset \mathbb{B}_k^3(T;\begin{pmatrix} r^{\mathrm{v}} \\ r^e \\ -1 \end{pmatrix})$$

and

$$\mathbb{B}_k^{\mathrm{div}}(T;\begin{pmatrix} r^{\mathrm{v}} \\ r^e \\ -1 \end{pmatrix}) \subseteq \mathbb{B}_k^{\mathrm{div}}(T;\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}) = (\ker(\mathrm{tr}^{\mathrm{div}}) \cap \mathbb{P}_k^3(T)).$$

Theorem 3.13. *Let $\boldsymbol{r} = (r^{\mathrm{v}}, r^e, r^f)$ with $r^f = -1$, $r^e \geq -1$, $r^{\mathrm{v}} \geq \max\{2r^e, -1\}$, and non-negative integer $k \geq 2r^{\mathrm{v}}_+ + 1$. The shape function space $\mathbb{P}_k^3(T)$ is determined by the DoFs*

$$\nabla^j \boldsymbol{v}(\mathrm{v}), \quad \mathrm{v} \in \Delta_0(T), j = 0,1,\ldots,r^{\mathrm{v}}, \tag{20a}$$

$$\int_e \frac{\partial^j \boldsymbol{v}}{\partial n_1^i \partial n_2^{j-i}} \cdot \boldsymbol{q}\, \mathrm{d}s, \quad e \in \Delta_1(T), \boldsymbol{q} \in \mathbb{P}^3_{k-2(r^{\mathrm{v}}+1)+j}(e), 0 \leq i \leq j \leq r^e, \tag{20b}$$

$$\int_f \boldsymbol{v}\cdot\boldsymbol{n}\, q\, \mathrm{d}S, \quad q \in \mathbb{B}_k(f;\begin{pmatrix} r^{\mathrm{v}} \\ r^e \end{pmatrix}), f \in \Delta_2(T), \tag{20c}$$

$$\int_T \boldsymbol{v}\cdot\boldsymbol{q}\, \mathrm{d}x, \quad \boldsymbol{q} \in \mathbb{B}_k^{\mathrm{div}}(T;\boldsymbol{r}). \tag{20d}$$

With mesh $\mathcal{T}_h$, define the global $H(\operatorname{div})$-conforming finite element space

$$\mathbb{V}_k^{\operatorname{div}}(\mathcal{T}_h;\boldsymbol{r}) = \{\boldsymbol{v}\in \boldsymbol{H}(\operatorname{div},\Omega) : \boldsymbol{v}|_T\in \mathbb{P}_k^3(T) \text{ for all } T\in\mathcal{T}_h, \\ \text{and all the DoFs (20a)-(20d) are single-valued}\}.$$

Proof. The unisolvence of DoFs (20a)-(20d) for $\mathbb{P}_k^3(T)$ follows from Theorem 3.6 and decomposition (19). More precisely, let us first consider the case $r^{\mathrm{v}}\geq 0, r^e\geq 0, r^f=-1$. Then using the DoFs for $\boldsymbol{r}_+=(r^{\mathrm{v}},r^e,0)$, we know $\mathbb{P}_k^3(T)$ is determined by (20a)-(20b) and

$$\int_f \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}S,\quad q\in\mathbb{B}_k^3(f;\begin{pmatrix} r^{\mathrm{v}}\\ r^e\end{pmatrix}), f\in\Delta_2(T), \tag{21}$$

$$\int_T \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}x,\quad \boldsymbol{q}\in\mathbb{B}_k^3(T;\boldsymbol{r}_+). \tag{22}$$

On each face, we use the decomposition $\mathbb{R}^3=\operatorname{span}\{\boldsymbol{t}_f^1,\boldsymbol{t}_f^2\}\oplus\operatorname{span}\{\boldsymbol{n}_f\}$ and move the tangential components into the bubble space $\mathbb{B}_k^{\operatorname{div}}(T;\boldsymbol{r})$. Therefore only the normal component (20c) is left.

When $r^{\mathrm{v}}\geq 0, r^e=-1, r^f=-1$, we consider the DoFs for $\boldsymbol{r}_+=(r^{\mathrm{v}},0,0)$. That is vertex DoF (20a), volume DoF (22), and the edge and face DoFs

$$\int_e \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}s,\quad \boldsymbol{q}\in\mathbb{B}_k^3(e;r^{\mathrm{v}}), e\in\Delta_1(T), \tag{23}$$

$$\int_f \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}S,\quad \boldsymbol{q}\in\mathbb{B}_k^3(f;\begin{pmatrix} r^{\mathrm{v}}\\ r^e_+\end{pmatrix}), f\in\Delta_2(T). \tag{24}$$

As before on each face, we move the tangential components into the bubble space $\mathbb{B}_k^{\operatorname{div}}(T;\boldsymbol{r})$ and keep only normal component with the test function $q\in\mathbb{B}_k(f;\begin{pmatrix} r^{\mathrm{v}}\\ r^e_+\end{pmatrix})$. On each edge e, we use the decomposition $\mathbb{R}^3=\operatorname{span}\{\boldsymbol{n}_{f_1},\boldsymbol{n}_{f_2}\}\oplus\operatorname{span}\{\boldsymbol{t}_e\}$ where f_1,f_2 are two faces containing e. Again the tangential component $\mathbb{B}_k(e;r^{\mathrm{v}})\otimes\operatorname{span}\{\boldsymbol{t}_e\}$ is moved into the bubble space $\mathbb{B}_k^{\operatorname{div}}(T;\boldsymbol{r})$. The normal components will be redistributed to the two faces containing e so that $\mathbb{B}_k(f;\begin{pmatrix} r^{\mathrm{v}}\\ r^e_+\end{pmatrix})\bigoplus_{e\in\Delta_1(f)}\mathbb{B}_k(e;r^{\mathrm{v}})=\mathbb{B}_k(f;\begin{pmatrix} r^{\mathrm{v}}\\ r^e\end{pmatrix})$ for $r^e=-1$, which leads to (20c). When $r^{\mathrm{v}}=-1$, we can redistribute 3 components of a vector into 3 faces containing that vertex so that (20c) still holds. We refer to [16, Fig. 3] for an illustration. ☐

Example 3.14 ($H(\operatorname{div})$-conforming element)**.** We recover the following known $H(\operatorname{div})$-conforming finite elements:

(i) When $k\geq 1$, $\boldsymbol{r}=-1$, it is the second family of Nédélec face element ($\mathrm{ND}_k^{(2)}$) [10,40] which is Brezzi-Douglas-Marini (BDM_k) [11] in two dimensions.

(ii) When $k\geq 2$, $\boldsymbol{r}=(0,-1,-1)$, it is Stenberg's element [48].

Remark 3.15. For space $\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r})$ with $r^{\mathrm{v}}=0$, we can derive the explicit basis from the Bernstein basis by using the geometric decomposition of the vector Lagrange element and $t-n$ decomposition on subsimplexes, cf. [16]. For general $\boldsymbol{r}$, the explicit basis corresponding to the interior DoF (20d) follows from the geometric decomposition of the bubble functions, cf. (19). Basis for the boundary DoFs

(20a)-(20c) can be modified from that of smooth scalar finite elements using the $t-n$ decomposition. See Remark 3.7.

Corollary 3.16. *Let $\boldsymbol{r} = (r^{\mathrm{v}}, r^e, r^f)$ with $r^f = -1$, $r^e \geq -1$, $r^{\mathrm{v}} \geq \max\{2r^e, -1\}$, and non-negative integer $k \geq 2r^{\mathrm{v}}_+ + 1$. We have the dimension formula*

$$\begin{aligned}\dim \mathbb{V}_k^{\mathrm{div}}(\mathcal{T}_h; \boldsymbol{r}) = \dim \mathbb{V}_k^3(\mathcal{T}_h; \boldsymbol{r}_+) - 3[r^{\mathrm{v}} = -1]|\Delta_0(\mathcal{T}_h)| \\ -3[r^e = -1](k - 2r^{\mathrm{v}}_+ - 1)|\Delta_1(\mathcal{T}_h)| \\ + \left(-2C_2(k, \boldsymbol{r}_+) + 3[r^e = -1](k - 2r^{\mathrm{v}}_+ - 1) + 3[r^{\mathrm{v}} = -1]\right)|\Delta_2(\mathcal{T}_h)| \\ + \left(8C_2(k, \boldsymbol{r}_+) + 6[r^e = -1](k - 2r^{\mathrm{v}}_+ - 1)\right)|\Delta_3(\mathcal{T}_h)|,\end{aligned}$$

where the constant $C_2(k, \boldsymbol{r}_+)$ is defined in Lemma 3.11.

Proof. When counting the dimension, we compare $\mathbb{V}_k^{\mathrm{div}}(\mathcal{T}_h; \boldsymbol{r})$ with the continuous element $\mathbb{V}_k^3(\mathcal{T}_h; \boldsymbol{r}_+)$. As $r^f = -1$, the two tangential components of the face DoFs are considered as interior DoFs and thus subtracted from coefficients of $\Delta_2(\mathcal{T}_h)$. The cumulation of 4 faces tangential bubbles contributes to the increase $8C_2(k, \boldsymbol{r}_+)$ in the coefficient of $\Delta_3(\mathcal{T}_h)$. Similarly when $r^e = -1$, we add total 6 tangential edge bubbles to the interior and redistribute the two normal components of edge bubbles to each face. When $r^{\mathrm{v}} = -1$, the three components of the vector function at vertices are redistributed to three faces containing that vertex. Therefore facewisely we add $3(k - 2r^{\mathrm{v}}_+ - 1)$ edge DoFs and 3 vertices DoFs. When $r^e = -1$, all 3 components of edge DoFs of a vector are removed and when $r^{\mathrm{v}} = -1$, all 3 components of a vector are removed. ☐

4. Div stability between finite elements spaces

In this section, for two smoothness vectors $(\boldsymbol{r}_2, \boldsymbol{r}_3)$ with relation $\boldsymbol{r}_3 \geq \max\{\boldsymbol{r}_2 - 1, -1\}$, we aim to prove the so-called *discrete div stability*, i.e. the div operator is surjective

$$\operatorname{div} \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) = \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3). \tag{25}$$

4.1. **Overview.** To simplify the notation, introduce $r \ominus n := \max\{r - n, -1\}$ so that the result will stagnate at -1 when $r - n \leq -1$. Additional conditions on $(\boldsymbol{r}_2, \boldsymbol{r}_3, k)$ are needed to establish the div stability (25). For example, $\boldsymbol{r}_2 = 0, \boldsymbol{r}_3 = -1$ is the notorious Stokes finite element pair for which the div stability is hard to verify and may require conditions on the triangulation [47, 54]. While $\boldsymbol{r}_2 = -1, \boldsymbol{r}_3 = -1$ corresponds to the div stability for the $\mathrm{ND}_k^{(2)}/\mathrm{BDM}_k$ element which is relatively easy. In all cases, the degree of polynomial k should be large enough. We shall call $(\boldsymbol{r}_2, \boldsymbol{r}_3, k)$ div stable if (25) holds and summarize several examples in Table 1.

When the space $\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$ contains piecewise constant function, to have the discrete div stability, we require $\dim \mathbb{B}_k(f; \begin{pmatrix} r_2^{\mathrm{v}} \\ r_2^e \end{pmatrix}) \geq 1$ which is ensured when $k \geq \max\{2r_2^{\mathrm{v}} + 1, r_2^{\mathrm{v}} + 2, 3(r_2^e + 1)\}$ by Lemma 3.9.

4.2. **Div stability of bubble spaces.** The essential difficulty is the div stability of bubble spaces

$$\operatorname{div} \mathbb{B}_k^{\mathrm{div}}(T; \boldsymbol{r}_2) = \mathbb{B}_{k-1}(T; \boldsymbol{r}_3)/\mathbb{R},$$

where $\mathbb{B}_{k-1}(T; \boldsymbol{r}_3)/\mathbb{R} = \mathbb{B}_{k-1}(T; \boldsymbol{r}_3) \cap L_0^2(T)$ and $\boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$. Let us refine the notation $S_\ell(f, \boldsymbol{r})$ to $S_\ell(f, \boldsymbol{r}, k)$ to include the degree of polynomial. When

TABLE 1. Examples of discrete div stability for $\boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$

$(r_2^{\mathrm{v}}, r_2^e, r_2^f)$	$(r_3^{\mathrm{v}}, r_3^e, r_3^f)$	$k \geq$	Results	Constraint
$(r_2^{\mathrm{v}}, -1, -1)$	$(r_3^{\mathrm{v}}, -1, -1)$	$\max\{2r_2^{\mathrm{v}}+1, r_2^{\mathrm{v}}+2\}$	Lemma 4.1	
$(r_2^{\mathrm{v}}, 0, -1)$	$(r_3^{\mathrm{v}}, -1, -1)$	$2r_2^{\mathrm{v}}+1$	Lemma 4.2	$r_2^{\mathrm{v}} \geq 1$
$(2,1,0)$	$(1,0,-1)$	6	Lemma 4.3	$r_2^e \geq 2r_2^f + 1$
$(2,1,-1)$	$(1,0,-1)$	6	Corollary 4.4	$r_2^e \geq 1, r_2^f = -1$
$(0,0,-1)$	$(-1,-1,-1)$		Not valid	

$\dim \mathbb{B}_{k-1}(T;\boldsymbol{r}_3) = 1$, we have $\operatorname{div} \mathbb{B}_k^{\operatorname{div}}(T;\boldsymbol{r}_2) = \mathbb{B}_{k-1}(T;\boldsymbol{r}_3) \cap L_0^2(T)$ as $\mathbb{B}_{k-1}(T;\boldsymbol{r}_3) \cap L_0^2(T) = \{0\}$. So we only consider the case $\dim \mathbb{B}_{k-1}(T;\boldsymbol{r}_3) > 1$, i.e., $|S_3(T,\boldsymbol{r}_3,k-1)| \geq 2$.

Similar to Lemma 2.6, we have

$$\mathbb{B}_{k-1}(T;\boldsymbol{r}_3) \cap L_0^2(T) = \operatorname{span}\{\lambda^\alpha/\alpha! - \lambda^\beta/\beta! : \alpha,\beta \in S_3(T,\boldsymbol{r}_3,k-1), \operatorname{dist}(\alpha,\beta)=1\},$$

as the subgraph $\mathcal{G}(S_3(T,\boldsymbol{r}_3,k-1))$ is connected. It suffices to prove that: given

$$p(\alpha,\beta) = \lambda^\alpha/\alpha! - \lambda^\beta/\beta!, \quad \alpha,\beta \in S_3(T,\boldsymbol{r}_3,k-1), \operatorname{dist}(\alpha,\beta)=1,$$

we can find a function

$$\boldsymbol{u} \in \mathbb{B}_k^{\operatorname{div}}(T;\boldsymbol{r}_2) \quad \text{s.t. } \operatorname{div} \boldsymbol{u} = p.$$

In the proofs of results in this subsection, by Lemma 2.2, without loss of generality, we assume

$$\beta = \alpha + \epsilon_{01} = (\alpha_0+1, \alpha_1-1, \alpha_2, \alpha_3). \tag{26}$$

Recall that $\epsilon_i \in \mathbb{N}^{0:n}, \epsilon_i = (0,\ldots,1,\ldots,0)$, $\epsilon_{ij} = \epsilon_i - \epsilon_j = (0,\ldots,1,\ldots,-1,\ldots,0)$, and $\boldsymbol{t}_{i,j}$ is the edge vector from v_i to v_j.

We start from a simple case $r_2^e = -1, r_2^f = -1$ as tangential components on edges and faces are included in the div bubble space; see (19).

Lemma 4.1. *Assume*

$$r_2^{\mathrm{v}} \geq -1, \quad r_2^e = -1, \quad r_2^f = -1, \quad \boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1, \quad k \geq \max\{2r_2^{\mathrm{v}}+1, r_2^{\mathrm{v}}+2\}.$$

It holds that

$$\operatorname{div} \mathbb{B}_k^{\operatorname{div}}(T;\boldsymbol{r}_2) = \mathbb{B}_{k-1}(T;\boldsymbol{r}_3)/\mathbb{R}.$$

Proof. With $k \geq \max\{2r_2^{\mathrm{v}}+1, r_2^{\mathrm{v}}+2\}$, by Lemma 3.10, we can show $\dim \mathbb{B}_k^{\operatorname{div}}(T;\boldsymbol{r}_2) \geq 1$ and $\dim \mathbb{B}_{k-1}(T;\boldsymbol{r}_3) \geq 1$. Taking $\boldsymbol{u} = \lambda^{\alpha+\epsilon_0}\boldsymbol{t}_{1,0}/(\beta!\alpha_1)$, by (10) we have $\operatorname{div} \boldsymbol{u} = p$. Notice that the edge div bubble function $\lambda_0\lambda_1\boldsymbol{t}_{1,0} \in \boldsymbol{H}_0(\operatorname{div},T)$. By writing $\lambda^{\alpha+\epsilon_0}\boldsymbol{t}_{1,0} = (\lambda_0^{\alpha_0}\lambda_1^{\beta_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3})\lambda_0\lambda_1\boldsymbol{t}_{1,0}$, we conclude $\lambda^{\alpha+\epsilon_0}\boldsymbol{t}_{1,0} \in \boldsymbol{H}_0(\operatorname{div},T)$.

We then verify $\alpha + \epsilon_0 \in S_3(T,\boldsymbol{r}_2,k)$ by considering the distance to vertices as follows

$$\operatorname{dist}(\alpha+\epsilon_0, \mathrm{v}_i) = \operatorname{dist}(\alpha,\mathrm{v}_i) + 1 > r_3^{\mathrm{v}} + 1 > r_2^{\mathrm{v}} \quad \text{for } i = 1,2,3,$$
$$\operatorname{dist}(\alpha+\epsilon_0, \mathrm{v}_0) = \operatorname{dist}(\alpha,\mathrm{v}_0) = \operatorname{dist}(\beta,\mathrm{v}_0) + 1 > r_3^{\mathrm{v}} + 1 > r_2^{\mathrm{v}}.$$

We refer to Fig. 6 (Case (2)) for illustration of changing $\alpha \in S_3(T,\boldsymbol{r}_3,k-1)$ to $\alpha + \epsilon_0 \in S_3(T,\boldsymbol{r}_2,k)$. ∎

Next we set $r_2^e = 0$. The tangential component of edge bubbles will be excluded from $\mathbb{B}_k^{\mathrm{div}}(T; \boldsymbol{r}_2)$. The nodes in $S_3(T, \boldsymbol{r}_2, k)$ should be away from edges which in turn requires condition $r_2^{\mathrm{v}} \geq 1$ stronger than the standard one $r_2^{\mathrm{v}} \geq 2r_2^e \geq 0$.

Lemma 4.2. *Assume*

$$r_2^{\mathrm{v}} \geq 1, \quad r_2^e = 0, \quad r_2^f = -1, \quad \boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1, \quad k \geq 2r_2^{\mathrm{v}} + 1.$$

It holds that

$$\operatorname{div} \mathbb{B}_k^{\mathrm{div}}(T; \boldsymbol{r}_2) = \mathbb{B}_{k-1}(T; \boldsymbol{r}_3)/\mathbb{R}.$$

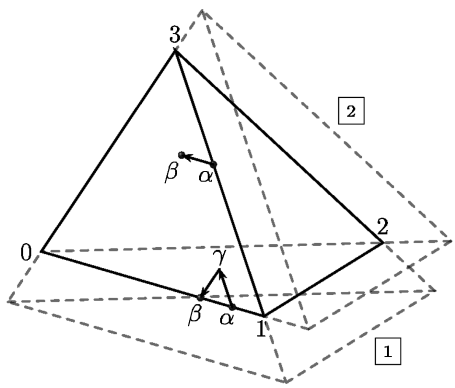


FIGURE 6. Two cases when $r_2^{\mathrm{v}} \geq 1, r_2^e = 0, r_2^f = -1$: Case (1) $\alpha_2 = \alpha_3 = 0$; Case (2) $\alpha_2 + \alpha_3 \geq 1$. The dash line represents different extension of the simplicial lattice so that lattice nodes α, β are away from the edges in the extended lattice.

Proof. By Lemma 3.10 and the setting of parameters, $\dim \mathbb{B}_{k-1}(T; \boldsymbol{r}_3) \geq 1$.

Case 1. Consider case $\alpha_2 = \alpha_3 = 0$ and consequently $\alpha_0 + \alpha_1 = k - 1$. In this case, $\alpha, \beta \in f_{01}$, and $\operatorname{dist}(\alpha, \mathrm{v}_i) > r_3^{\mathrm{v}}$; See Fig. 6.

By Lemma 2.8, we can choose

$$\boldsymbol{u} = \frac{1}{\gamma! \alpha_1} \lambda^{\alpha + \epsilon_3} \boldsymbol{t}_{1,3} + \frac{1}{\gamma! \beta_0} \lambda^{\beta + \epsilon_3} \boldsymbol{t}_{3,0}, \quad \gamma = \alpha + \epsilon_{31}, \tag{27}$$

and verify $\lambda^{\alpha+\epsilon_3} \boldsymbol{t}_{1,3}, \lambda^{\beta+\epsilon_3} \boldsymbol{t}_{3,0} \in \mathbb{B}_k^{\mathrm{div}}(T; \boldsymbol{r}_2)$. We focus on $\lambda^{\alpha+\epsilon_3} \boldsymbol{t}_{1,3}$ as $\lambda^{\beta+\epsilon_3} \boldsymbol{t}_{3,0}$ is symbolically identical. Write $\lambda^{\alpha+\epsilon_3} \boldsymbol{t}_{1,3} = (\lambda_0^{\alpha_0} \lambda_1^{\beta_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3}) \lambda_1 \lambda_3 \boldsymbol{t}_{1,3}$. As $\lambda_1 \lambda_3 \boldsymbol{t}_{1,3} \in \boldsymbol{H}_0(\operatorname{div}, T)$, we conclude $\lambda^{\alpha+\epsilon_3} \boldsymbol{t}_{1,3} \in \boldsymbol{H}_0(\operatorname{div}, T)$. Next we verify $\alpha + \epsilon_3 \in S_3(T, \boldsymbol{r}_2, k)$.

Distance to vertices. For vertices on the plane f_{012}, the distance is increased by 1 as $\alpha_3 \to \alpha_3 + 1$. Then

$$\operatorname{dist}(\alpha + \epsilon_3, \mathrm{v}_i) = \operatorname{dist}(\alpha, \mathrm{v}_i) + 1 > r_3^{\mathrm{v}} + 1 = r_2^{\mathrm{v}} \quad \text{for } i = 0, 1, 2.$$

The distance to v_3 is $k - 1$ which is far larger than r_2^{v}:

$$\operatorname{dist}(\alpha + \epsilon_3, \mathrm{v}_3) = \alpha_0 + \alpha_1 + \alpha_2 = k - 1 > r_2^{\mathrm{v}}.$$

Distance to edges. Similarly when computing the distance to edges not containing v_3, $\alpha_3 \to \alpha_3 + 1$ will increase the distance by 1:

$$\operatorname{dist}(\alpha + \epsilon_3, f_{ij}) = \operatorname{dist}(\alpha, f_{ij}) + 1 \geq 1 > r_2^e = 0 \quad \text{for } i, j \neq 3.$$

When computing the distance to f_{03}, we use the distance to vertices on face f_{012}:

$$\operatorname{dist}(\alpha, \mathrm{v}_0) = \alpha_1 + \alpha_2 + \alpha_3 = \alpha_1 > r_3^{\mathrm{v}} \geq 0.$$

Then

$$\operatorname{dist}(\alpha + \epsilon_3, f_{03}) = \alpha_1 + \alpha_2 = \alpha_1 > 0 = r_2^e.$$

The bound $\operatorname{dist}(\alpha + \epsilon_3, f_{13}) > 0$ is similar. The distance to edge f_{23} is far away as

$$\operatorname{dist}(\alpha + \epsilon_3, f_{23}) = \alpha_0 + \alpha_1 = k - 1 \geq 2r_2^{\mathrm{v}} > 0.$$

Case 2. Consider case $\alpha_2 + \alpha_3 \geq 1$. Namely $\operatorname{dist}(\alpha, f_{01}) \geq 1$. Setting $\boldsymbol{u} = \lambda^{\alpha+\epsilon_0}\boldsymbol{t}_{1,0}/(\beta!\alpha_1)$ and by Lemma 2.7, we have $\operatorname{div} \boldsymbol{u} = p$. Again we have $\lambda^{\alpha+\epsilon_0}\boldsymbol{t}_{1,0} \in \boldsymbol{H}_0(\operatorname{div}, T)$. We only need to show $\alpha + \epsilon_0 \in S_2(T, \boldsymbol{r}_2, k)$. The simplicial lattice containing $\alpha + \epsilon_0$ is extended in α_0 direction; see Fig. 6.

Distance to vertices. This case has been proved in Lemma 4.1. That is

$$\begin{aligned}\operatorname{dist}(\alpha + \epsilon_0, \mathrm{v}_i) &= \operatorname{dist}(\alpha, \mathrm{v}_i) + 1 > r_3^{\mathrm{v}} + 1 = r_2^{\mathrm{v}} \quad \text{for } i = 1, 2, 3,\\ \operatorname{dist}(\alpha + \epsilon_0, \mathrm{v}_0) &= \operatorname{dist}(\alpha, \mathrm{v}_0) = \operatorname{dist}(\beta, \mathrm{v}_0) + 1 > r_3^{\mathrm{v}} + 1 = r_2^{\mathrm{v}}.\end{aligned}$$

Distance to edges. We have

$$\begin{aligned}\operatorname{dist}(\alpha + \epsilon_0, f_{ij}) &= \operatorname{dist}(\alpha, f_{ij}) + 1 > r_3^e + 1 \geq 1 > r_2^e = 0 \quad \text{for } 1 \leq i, j \leq 3,\\ \operatorname{dist}(\alpha + \epsilon_0, f_{01}) &= \alpha_2 + \alpha_3 \geq 1 > r_2^e,\\ \operatorname{dist}(\alpha + \epsilon_0, f_{0i}) &= \operatorname{dist}(\alpha, f_{0i}) = \operatorname{dist}(\beta, f_{0i}) + 1 \geq 1 > r_2^e \quad \text{for } i = 2, 3.\end{aligned}$$

□

We then move to the most difficult case: the velocity is continuous and the pressure is discontinuous. Supersmoothness on vertices and edges is added to ensure the discrete div stability. In the following, $\boldsymbol{r}_2 \geq (2, 1, 0)$ and $k \geq 5$.

Lemma 4.3. *Assume*

$$r_2^{\mathrm{v}} \geq 2r_2^e \geq 2, \quad r_2^e \geq 2r_2^f + 1 \geq 1, \quad r_2^f \geq 0, \quad \boldsymbol{r}_3 = \boldsymbol{r}_2 - 1, \quad k \geq 2r_2^{\mathrm{v}} + 1. \tag{28}$$

It holds that

$$\operatorname{div} \mathbb{B}_k^3(T; \boldsymbol{r}_2) = \mathbb{B}_{k-1}(T; \boldsymbol{r}_3)/\mathbb{R}.$$

Proof. By Lemma 3.10 and the setting of parameters, $\dim \mathbb{B}_{k-1}(T; \boldsymbol{r}_3) \geq 1$. Without loss of generality, assume $\beta = \alpha + \epsilon_{01} = (\alpha_0 + 1, \alpha_1 - 1, \alpha_2, \alpha_3) \in S_3(T, \boldsymbol{r}_3, k - 1)$ in the lattice for the pressure. We shall sort the nodes by the distance to the edge f_{01}, i.e., the plane $L(f_{01}, s)$ from $s = r_2^e$ to $k - 1 - r_2^e$; see Fig. 7.

Case 1. We first consider the case: $\alpha_3 = r_2^f$ or $\alpha_2 = r_2^f$. Without loss of generality, we discuss $\alpha_3 = r_2^f$ in detail. To push the node into $S_3(T, \boldsymbol{r}_2, k)$, we need to increase the distance to the face $f_3 = f_{012}$ by one, i.e., lift the nodes one level higher in α_3 direction by changing α to $\alpha + \epsilon_3$; see Fig. 7.

We choose $\boldsymbol{u}$ by (27). It remains to verify $\boldsymbol{u} \in \mathbb{B}_k^3(T; \boldsymbol{r}_2)$, i.e., $\alpha + \epsilon_3, \beta + \epsilon_3 \in S_3(T, \boldsymbol{r}_2, k) \subset \mathbb{T}_k^n$. We focus on $\alpha + \epsilon_3$ as $\beta + \epsilon_3$ is symbolically identical.

Distance to vertices. For $\alpha \in S_3(T, \boldsymbol{r}_3, k-1) \subset \mathbb{T}^n_{k-1}$, $i = 0,1,2$,

$$\operatorname{dist}(\alpha, \mathrm{v}_i) > r_3^{\mathrm{v}} \iff \alpha_i \leq k-1-(r_3^{\mathrm{v}}+1) = k-1-r_2^{\mathrm{v}} \iff \operatorname{dist}(\alpha+\epsilon_3, \mathrm{v}_i) > r_2^{\mathrm{v}}.$$

So only v_3 is left. As we assume $\alpha_3 = r_2^f$,

$$\operatorname{dist}(\alpha+\epsilon_3, \mathrm{v}_3) = k - \operatorname{dist}(\alpha+\epsilon_3, f_3) = k-(\alpha_3+1) = k - r_2^f - 1 \geq 2r_2^{\mathrm{v}} - r_2^f > r_2^{\mathrm{v}}.$$

Distance to edges. For edges on the face f_{012}, w.l.o.g. take edge f_{01}, as no α_3 is involved, we have the equivalence of the bound

$$\operatorname{dist}(\alpha, f_{01}) > r_3^e \iff \alpha_0+\alpha_1 \leq k-1-(r_3^e+1) \iff \operatorname{dist}(\alpha+\epsilon_3, f_{01}) > r_2^e.$$

To estimate the distance to other edges, w.l.o.g. consider the edge f_{03}, we use the bound of the distance to vertices. From

$$\operatorname{dist}(\alpha, \mathrm{v}_0) > r_3^{\mathrm{v}} \iff \alpha_1+\alpha_2+\alpha_3 > r_3^{\mathrm{v}},$$

the fact $\alpha_3 = r_2^f$, and the bound (28) on $\boldsymbol{r}_2$, we conclude

$$\operatorname{dist}(\alpha+\epsilon_3, f_{03}) = \alpha_1+\alpha_2 > r_3^{\mathrm{v}} - r_2^f \geq r_2^e.$$

Distance to faces. Obviously the distance to $f_3 = f_{012}$ is increased by 1, i.e.

$$\operatorname{dist}(\alpha+\epsilon_3, f_3) = \alpha_3+1 = r_2^f+1 > r_2^f.$$

But other $\alpha_i, i \neq 3$, remains unchanged. We will use the bound of the distance to edges. Again w.l.o.g. consider face $f_2 = f_{013}$. From

$$\operatorname{dist}(\alpha, f_{01}) = \alpha_2+\alpha_3 > r_3^e = r_2^e - 1,$$

the fact $\alpha_3 = r_2^f$, and the bound (28) on $\boldsymbol{r}_2$, we conclude

$$\operatorname{dist}(\alpha+\epsilon_3, f_2) = \alpha_2 > r_2^e - r_2^f - 1 \geq r_2^f.$$

The last inequality is the motivation to have the stronger constraint $r_2^e \geq 2r_2^f + 1$ in (28).

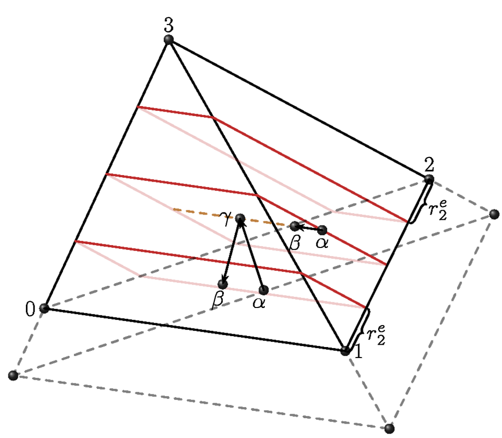


FIGURE 7. Different location of α, β

In summary, for vertices and edges on the face f_{012}, the upper bound on the sum of indices automatically holds as no α_3 is involved. When estimating the distance to edges and faces containing $\texttt{v}_3$, we use the fact $\alpha \in L(f_3, r_2^f)$ is in the face bubble $S_2(f_3, \begin{pmatrix} r_3^{\texttt{v}} \\ r_3^e \end{pmatrix} - r_2^f, k-1-r_2^f)$; see Fig. 5(c).

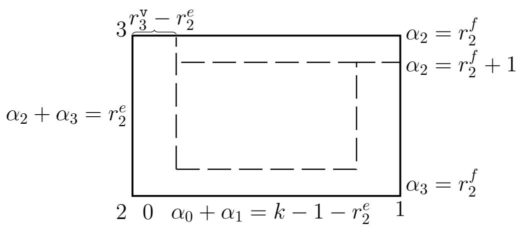


FIGURE 8. On the cut plane $\alpha_2 + \alpha_3 = r_2^e$

Case 2. Consider case $\alpha_2 + \alpha_3 = r_2^e$ and consequently $\alpha_0 + \alpha_1 = k - 1 - r_2^e$. As we have considered the case $\alpha_2 = r_2^f$ or $\alpha_3 = r_2^f$ in Case 1, we can further assume $\alpha_2, \alpha_3 \geq r_2^f + 1 > r_2^f$. Notice that now $r_2^e = \alpha_2 + \alpha_3 \geq 2r_2^f + 2$ which means if $r_2^e = 2r_2^f + 1$ only, either α_2 or $\alpha_3 = r_2^f$ which is covered by Case 1.

We still choose $\boldsymbol{u}$ by (27) and verify $\alpha + \epsilon_3, \beta + \epsilon_3 \in S_3(T, \boldsymbol{r}_2, k)$.

Distance to vertices. Again we only need to consider the distance to $\texttt{v}_3$ which is minimized when $\alpha_2 = r_2^f + 1 > r_2^f$; see Fig. 8. Algebraically, we have

$$\operatorname{dist}(\alpha + \epsilon_3, \texttt{v}_3) = \alpha_0 + \alpha_1 + \alpha_2 > \alpha_0 + \alpha_1 + r_2^f = k - 1 - r_2^e + r_2^f \geq r_2^{\texttt{v}}.$$

Distance to edges. The distance to edges f_{01} and f_{23} is easy to bound as $\alpha_2 + \alpha_3 = r_2^e$:

$$\operatorname{dist}(\alpha + \epsilon_3, f_{01}) = \alpha_2 + \alpha_3 + 1 = r_2^e + 1 > r_2^e,$$
$$\operatorname{dist}(\alpha + \epsilon_3, f_{23}) = \alpha_0 + \alpha_1 = k - 1 - r_2^e > r_2^e.$$

Without loss of generality consider $\operatorname{dist}(\alpha + \epsilon_3, f_{13}) = \alpha_0 + \alpha_2$. From the distance to the vertex and the fact $\alpha_2 + \alpha_3 = r_2^e$, we have the lower bound

$$\operatorname{dist}(\alpha, \texttt{v}_1) = \alpha_0 + \alpha_2 + \alpha_3 > r_3^{\texttt{v}} \quad \Longrightarrow \quad \alpha_0 > r_3^{\texttt{v}} - r_2^e. \tag{29}$$

Together with $\alpha_2 \geq r_2^f + 1$, we have

$$\operatorname{dist}(\alpha + \epsilon_3, f_{13}) = \alpha_0 + \alpha_2 > r_3^{\texttt{v}} - r_2^e + r_2^f + 1 \geq r_2^e.$$

Distance to faces. As we assume $\alpha_2, \alpha_3 > r_2^f$, we have

$$\operatorname{dist}(\alpha + \epsilon_3, f_i) > r_2^f, \quad i = 2, 3.$$

The lower bound (29) implies

$$\operatorname{dist}(\alpha + \epsilon_3, f_0) = \alpha_0 > r_3^{\texttt{v}} - r_2^e \geq r_2^f.$$

Similar to (29), $\operatorname{dist}(\alpha, \texttt{v}_0) > r_3^{\texttt{v}}$ will imply the same lower bound on α_1 and $\operatorname{dist}(\alpha + \epsilon_3, f_1) > r_2^f$.

In summary, when considering the set $L(f_{01}, r_2^e) \cap S_3(T, \boldsymbol{r}_3, k-1)$, the index is well separated from the boundary. See the dash line in Fig. 8.

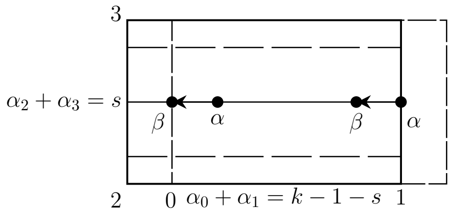


FIGURE 9. On the cut plane $\alpha_2+\alpha_3 = s \geq r_2^e+1$

Case 3. Now only the case $\{L(f_{01}, s), r_2^e+1 \leq s \leq k-1-r_2^e \cap S_3(T, \boldsymbol{r}_3, k-1)\}$ is left which implies $\alpha_2+\alpha_3 \geq r_2^e+1$. See the middle cut plane in Fig. 7.

After Case 1, we can assume $\alpha_2, \alpha_3 \geq r_2^f+1 > r_2^f$. The node α can be on the plane $L(f_{123}, r_2^f)$ and thus lift in α_3 direction will not push it into the interior. We choose to extend in α_0 direction and set $\boldsymbol{u} = \lambda^{\alpha+\epsilon_0}\boldsymbol{t}_{1,0}/(\beta!\alpha_1)$. By Lemma 2.7, $\operatorname{div}\boldsymbol{u} = p$. We verify $\alpha+\epsilon_0 \in S_3(T, \boldsymbol{r}_2, k)$.

Distance to vertices. The trouble case is $\operatorname{dist}(\alpha+\epsilon_0, \mathtt{v}_0)$ as α_0+1 is closer to $\mathtt{v}_0$. We will use the fact that β is closer to $\mathtt{v}_0$, i.e., $\beta_0 = \alpha_0+1$, and

$$\operatorname{dist}(\beta, \mathtt{v}_0) > r_3^{\mathtt{v}} \Longrightarrow k-1-\beta_0 > r_3^{\mathtt{v}},$$

to conclude the desired bound

$$\operatorname{dist}(\alpha+\epsilon_0, \mathtt{v}_0) = \alpha_1+\alpha_2+\alpha_3 = k-\beta_0 > r_2^{\mathtt{v}}.$$

Distance to edges. For edges f_{ij} not containing $\mathtt{v}_0$, no change on α_i, α_j, and thus

$$\operatorname{dist}(\alpha, f_{ij}) > r_3^e \Longrightarrow \alpha_i+\alpha_j \leq k-1-r_3^e-1 = k-r_2^e-1 \Longrightarrow \operatorname{dist}(\alpha+\epsilon_0, f_{ij}) > r_2^e.$$

Consider edge f_{01}. By $\operatorname{dist}(\alpha, f_{01}) = \alpha_2+\alpha_3 \geq r_2^e+1$, we have

$$\operatorname{dist}(\alpha+\epsilon_0, f_{01}) = \alpha_2+\alpha_3 \geq r_2^e+1 > r_2^e.$$

Consider edge f_{03}. We use bound for β

$$\operatorname{dist}(\beta, f_{03}) > r_3^e \Longrightarrow \beta_0+\beta_3 = \alpha_0+1+\alpha_3 \leq k-1-r_3^e-1 = k-1-r_2^e,$$

to conclude $\operatorname{dist}(\alpha+\epsilon_0, f_{03}) > r_2^e$. The $\operatorname{dist}(\alpha+\epsilon_0, f_{02})$ is similar.

Distance to faces. Again the distance to f_0 is easy as

$$\operatorname{dist}(\alpha+\epsilon_0, f_0) = \alpha_0+1 > r_3^f+1 = r_2^f.$$

The distance to faces f_2 and f_3 is from the assumption $\alpha_2, \alpha_3 \geq r_2^f+1 > r_2^f$. So we only need to check $\operatorname{dist}(\alpha+\epsilon_0, f_1) = \alpha_1$. Again we compare with β. By $\operatorname{dist}(\beta, f_1) = \beta_1 > r_3^f$, we have

$$\operatorname{dist}(\alpha+\epsilon_0, f_1) = \alpha_1 = \beta_1+1 > r_3^f+1 = r_2^f.$$

In summary, when $\alpha_2+\alpha_3 \geq r_2^e+1$, we can choose a simple velocity field from α to β and use the distance bound of β to derive the desired distance bound of α; see Fig. 9. $\square$

When changing $r_2^f = 0$ to $r_2^f = -1$, we add more tangential div bubble functions into the bubble space $\mathbb{B}_k^{\operatorname{div}}(T; \boldsymbol{r}_2)$ and thus the following div stability result is trivial.

Corollary 4.4. *Assume*

$$r_2^{\texttt{v}} \geq 2r_2^e \geq 2, \quad r_2^e \geq 1, \quad r_2^f = -1, \quad \boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1, \quad k \geq 2r_2^{\texttt{v}} + 1.$$

It holds that

$$\operatorname{div} \mathbb{B}_k^{\operatorname{div}}(T; \boldsymbol{r}_2) = \mathbb{B}_{k-1}(T; \boldsymbol{r}_3)/\mathbb{R}.$$

Proof. Noting that $\mathbb{B}_k^3(T; (\boldsymbol{r}_2)_+) \subset \mathbb{B}_k^{\operatorname{div}}(T; \boldsymbol{r}_2)$, we have

$$\operatorname{div} \mathbb{B}_k^3(T; (\boldsymbol{r}_2)_+) \subseteq \operatorname{div} \mathbb{B}_k^{\operatorname{div}}(T; \boldsymbol{r}_2) \subseteq \mathbb{B}_{k-1}(T; \boldsymbol{r}_3)/\mathbb{R}.$$

By Lemma 4.3, $\operatorname{div} \mathbb{B}_k^3(T; (\boldsymbol{r}_2)_+) = \mathbb{B}_{k-1}(T; \boldsymbol{r}_3)/\mathbb{R}$, which ends the proof. ☐

We integrate results in Lemmas 4.1-4.2, Lemma 4.3 and Corollary 4.4 into Theorem 4.5.

Theorem 4.5. *Assume* $k \geq \max\{2r_2^{\texttt{v}} + 1, r_2^{\texttt{v}} + 2\}$,

$$\begin{cases} r_2^f \geq 0, & r_2^e \geq 2r_2^f + 1 \geq 1, \quad r_2^{\texttt{v}} \geq 2r_2^e \geq 2, \\ r_2^f = -1, & \begin{cases} r_2^e \geq 1, & r_2^{\texttt{v}} \geq 2r_2^e \geq 2, \\ r_2^e \in \{0, -1\}, & r_2^{\texttt{v}} \geq 2r_2^e + 1, \end{cases} \end{cases} \tag{30}$$

and $\boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$. *It holds that*

$$\operatorname{div} \mathbb{B}_k^{\operatorname{div}}(T; \boldsymbol{r}_2) = \mathbb{B}_{k-1}(T; \boldsymbol{r}_3)/\mathbb{R}.$$

4.3. **Div stability with equality constraint.** For simplicity, hereafter we will omit the triangulation $\mathcal{T}_h$ in the notation of global finite element spaces. For example, $\mathbb{V}_k^{\operatorname{div}}(\mathcal{T}_h; \boldsymbol{r}_2)$ will be abbreviated as $\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2)$.

Theorem 4.6. *Let* $\boldsymbol{r}_2$ *satisfy* (30) *and* $\boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$. *Assume* k *is a large enough integer satisfying* $k \geq \max\{2r_2^{\texttt{v}} + 1, r_2^{\texttt{v}} + 2, 3(r_2^e + 1)\}$. *It holds that*

$$\operatorname{div} \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2) = \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3). \tag{31}$$

Proof. By Lemma 3.9, the condition on k ensures that $\dim \mathbb{B}_k(f; \begin{pmatrix} r_2^{\texttt{v}} \\ r_2^e \end{pmatrix}) \geq 1$ for all $f \in \Delta_2(\mathcal{T}_h)$. The condition $k \geq r_2^{\texttt{v}} + 2$ is considered for case $\boldsymbol{r}_2 = (0, -1, -1)$. It is obvious that $\operatorname{div} \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2) \subseteq \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$ as $\boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$. For $p \in \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) \subset H^{r_3^f+1}(\Omega)$, we are going to construct $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2)$ s.t. $\operatorname{div} \boldsymbol{v} = p$. To motivate the construction, consider the 1-D case. Given values $p^{(j)}(x)$, $j = 0, \ldots, m$ with some non-negative integer m, to construct u satisfying $u' = p$, we can simply set $u^{(j+1)}(x) = p^{(j)}(x)$, $j = 0, \ldots, m$ and $u(x) = 0$.

For vector function $\boldsymbol{v}$, on each lower subsimplex, we will choose a different frame and pick up one direction to assign the derivative relation.

Case 0. For $\texttt{v} \in \Delta_0(\mathcal{T}_h)$, we use the default Cartesian coordinate and write $\boldsymbol{v} = (v_1, v_2, v_3)$. When $r_3^{\texttt{v}} \geq 0$, set

$$\nabla^j (\partial_1 v_1)(\texttt{v}) = \nabla^j p(\texttt{v}), \quad j = 0, \ldots, r_3^{\texttt{v}}, \tag{32}$$

and all other DoFs are zero. Then

$$\nabla^j (\operatorname{div} \boldsymbol{v})(\texttt{v}) = \nabla^j (\partial_1 v_1)(\texttt{v}) = \nabla^j p(\texttt{v}), \quad j = 0, \ldots, r_3^{\texttt{v}}. \tag{33}$$

Case 1. For $e \in \Delta_1(\mathcal{T}_h)$, we use the frame $(\boldsymbol{t}, \boldsymbol{n}_1, \boldsymbol{n}_2)$, where $\boldsymbol{t}$ is a tangential vector of e and $\boldsymbol{n}_1, \boldsymbol{n}_2$ are two linearly independent normal vectors of e. Set edge DoFs of $\boldsymbol{v}\cdot\boldsymbol{t}$ and $\boldsymbol{v}\cdot\boldsymbol{n}_2$ to zero. Together with DoFs (32) on vertices, $\boldsymbol{v}\cdot\boldsymbol{t}|_e$ is determined. Then set the DoF for $\boldsymbol{v}\cdot\boldsymbol{n}_1$ by

$$\int_e \boldsymbol{v}\cdot\boldsymbol{n}_1 \; q \,\mathrm{d}s = 0, \quad q \in \mathbb{P}_{k-2(r_2^{\mathrm{v}}+1)}(e), \quad \text{if } \; r_2^e \geq 0,$$

$$\int_e \frac{\partial^{j+1}(\boldsymbol{v}\cdot\boldsymbol{n}_1)}{\partial n_1 \partial n_1^i \partial n_2^{j-i}} \; q\,\mathrm{d}s = \int_e \frac{\partial^j p}{\partial n_1^i \partial n_2^{j-i}} \; q\,\mathrm{d}s - \int_e \frac{\partial^{j+1}(\boldsymbol{v}\cdot\boldsymbol{t})}{\partial t\, \partial n_1^i \partial n_2^{j-i}} \; q\,\mathrm{d}s,$$
$$q \in \mathbb{P}_{k-2(r_2^{\mathrm{v}}+1)+j+1}(e), 0 \leq i \leq j \leq r_3^e \; \text{ if } \; r_3^e \geq 0.$$

Then by $\operatorname{div}\boldsymbol{v} = \partial_t(\boldsymbol{v}\cdot\boldsymbol{t}) + \partial_{n_1}(\boldsymbol{v}\cdot\boldsymbol{n}_1) + \partial_{n_2}(\boldsymbol{v}\cdot\boldsymbol{n}_2)$ and the vanishing edge DoFs for $\boldsymbol{v}\cdot\boldsymbol{n}_2$, we have

$$\int_e \frac{\partial^j(\operatorname{div}\boldsymbol{v} - p)}{\partial n_1^i \partial n_2^{j-i}} \; q\,\mathrm{d}s = 0, \quad q \in \mathbb{P}_{k-1-2(r_3^{\mathrm{v}}+1)+j}(e), 0 \leq i \leq j \leq r_3^e. \tag{34}$$

Case 2. For $f \in \Delta_2(\mathcal{T}_h)$, we choose two tangential vectors $\boldsymbol{t}_1, \boldsymbol{t}_2$ and a normal vector $\boldsymbol{n}_f$ as the local frame. Set the face DoFs for $\boldsymbol{v}\cdot\boldsymbol{t}_i, i = 1, 2$ as zero when $r_2^f \geq 0$. Together with edge and vertices DoFs in Cases 0 and 1, the tangential component $\Pi_f \partial_n^j \boldsymbol{v}$, for $j = 0, \ldots, r_2^f$, is determined and thus $\operatorname{div}_f(\partial_n^j \boldsymbol{v})|_f = \partial_n^j(\operatorname{div}_f \boldsymbol{v})|_f$ is well-defined. When $r_3^f \geq 0$, we set

$$\int_f \partial_n^j \partial_n(\boldsymbol{v}\cdot\boldsymbol{n}) \; q\,\mathrm{d}S = \int_f \partial_n^j(p - \operatorname{div}_f \boldsymbol{v}) \; q\,\mathrm{d}S,$$
$$q \in \mathbb{B}_{k-1-j}(f; \begin{pmatrix} r_3^{\mathrm{v}} \\ r_3^e \end{pmatrix} - j), j = 0, \ldots, r_3^f.$$

Then by $\operatorname{div}\boldsymbol{v} = \partial_n(\boldsymbol{v}\cdot\boldsymbol{n}) + \operatorname{div}_f \boldsymbol{v}$, we have

$$\int_f \partial_n^j(\operatorname{div}\boldsymbol{v} - p) \; q\,\mathrm{d}S = 0, \quad q \in \mathbb{B}_{k-1-j}(f; \begin{pmatrix} r_3^{\mathrm{v}} \\ r_3^e \end{pmatrix} - j), j = 0, \ldots, r_3^f. \tag{35}$$

Notice that DoFs $\int_f \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S$ remain open as we assume $\dim \mathbb{B}_k(f; \begin{pmatrix} r_2^{\mathrm{v}} \\ r_2^e \end{pmatrix}) \geq 1$. Recall that there exists a $\widetilde{\boldsymbol{v}} \in \boldsymbol{H}^{r_3^f+2}(\Omega; \mathbb{R}^3)$ [24] such that $\operatorname{div}\widetilde{\boldsymbol{v}} = p$. Then set

$$\int_f (\boldsymbol{v}\cdot\boldsymbol{n}) \; q\,\mathrm{d}S = \int_f (\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n}) \; q\,\mathrm{d}S, \quad q \in \mathbb{P}_0(f) \oplus (\mathbb{B}_k(f; \begin{pmatrix} r_2^{\mathrm{v}} \\ r_2^e \end{pmatrix})/\mathbb{R}). \tag{36}$$

By $\operatorname{div}\widetilde{\boldsymbol{v}} = p$ and (36),

$$\int_T (\operatorname{div}\boldsymbol{v} - p)\,\mathrm{d}x = \int_T \operatorname{div}(\boldsymbol{v} - \widetilde{\boldsymbol{v}})\,\mathrm{d}x = 0. \tag{37}$$

Case 3. For $T \in \mathcal{T}_h$, we split the DoFs as

$$\int_T \operatorname{div}\boldsymbol{v}\, q\,\mathrm{d}x = \int_T p\, q\,\mathrm{d}x, \quad q \in \mathbb{B}_{k-1}(\boldsymbol{r}_3)/\mathbb{R},$$
$$\int_T \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}x = 0, \quad \boldsymbol{q} \in \mathbb{B}_k^{\operatorname{div}}(\boldsymbol{r}_2) \cap \ker(\operatorname{div}).$$

By Theorem 4.5, the mapping div is surjective between bubble spaces. Together with (37),

$$\int_T (\operatorname{div} \boldsymbol{v} - p)\, q \,\mathrm{d}x = 0, \quad q \in \mathbb{B}_{k-1}(\boldsymbol{r}_3). \tag{38}$$

Finally combining (33)-(35) and (38) and the unisolvence for $\mathbb{V}^{L^2}_{k-1}(\boldsymbol{r}_3)$, we conclude $\operatorname{div} \boldsymbol{v} = p$. ☐

Example 4.7 (Neilan's Stokes element)**.** We choose $\boldsymbol{r}_2 = (2, 1, 0), \boldsymbol{r}_3 = (1, 0, -1)$ and polynomial degree $k \geq 6$, to get a stable Stokes-pair,

$$(\mathbb{V}^{\operatorname{div}}_k(\begin{pmatrix}2\\1\\0\end{pmatrix}), \mathbb{V}^{L^2}_{k-1}(\begin{pmatrix}1\\0\\-1\end{pmatrix})).$$

This is the Stokes element constructed by Neilan in [41]. The pressure element is discontinuous on faces but continuous on edges and differentiable at vertices. Notice that the lower bound on k is increased from $2r_2^{\mathrm{v}} + 1 = 5$ to $6 = 3(r_2^e + 1)$ to include a face bubble DoF so that $\operatorname{div} \boldsymbol{u}$ will contain piecewise constant.

Example 4.8 (Stenberg's $H(\operatorname{div})$-conforming element)**.** We choose $\boldsymbol{r}_2 = (0, -1, -1)$, $\boldsymbol{r}_3 = (-1, -1, -1)$ and polynomial degree $k \geq 2$, to get a stable pair for mixed Poisson problem,

$$(\mathbb{V}^{\operatorname{div}}_k(\begin{pmatrix}0\\-1\\-1\end{pmatrix}), \mathbb{V}^{L^2}_{k-1}(\begin{pmatrix}-1\\-1\\-1\end{pmatrix})).$$

The $H(\operatorname{div})$-conforming element $\mathbb{V}^{\operatorname{div}}_k(\begin{pmatrix}0\\-1\\-1\end{pmatrix})$ is the so-called Stenberg's element [48]. The lower bound $k \geq 2 = r_2^{\mathrm{v}} + 2$ is to include a face bubble DoF so that $\operatorname{div} \boldsymbol{u}$ will contain piecewise constant. For $\mathrm{ND}_k^{(2)}/\mathrm{BDM}_k$ element, i.e. $\boldsymbol{r}_2 = \boldsymbol{r}_3 = -1$, $k \geq 1$ is enough to ensure the div stability.

4.4. **Div stability with inequality constraints.** We consider more general cases with an inequality constraint on the smoothness vectors $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$:

$$\boldsymbol{r}_2 \geq -1, \text{ and satisfies } (30), \qquad \boldsymbol{r}_3 \geq \boldsymbol{r}_2 \ominus 1.$$

To define the finite element spaces and have the div stability, we further require

$$k \geq \max\{2r_2^{\mathrm{v}} + 1, r_2^{\mathrm{v}} + 2, 3(r_2^e + 1)\}, \quad k - 1 \geq 2r_3^{\mathrm{v}} + 1.$$

As $\boldsymbol{r}_3 \geq \boldsymbol{r}_2 \ominus 1$, we have the relation $\mathbb{V}^{L^2}_{k-1}(\boldsymbol{r}_3) \subseteq \mathbb{V}^{L^2}_{k-1}(\boldsymbol{r}_2 \ominus 1)$. By the div stability (31) established for the larger space $\mathbb{V}^{L^2}_{k-1}(\boldsymbol{r}_2 \ominus 1)$, we can define a subspace

$$\mathbb{V}^{\operatorname{div}}_k(\boldsymbol{r}_2, \boldsymbol{r}_3) \subseteq \mathbb{V}^{\operatorname{div}}_k(\boldsymbol{r}_2), \quad s.t.\ \operatorname{div} \mathbb{V}^{\operatorname{div}}_k(\boldsymbol{r}_2, \boldsymbol{r}_3) = \mathbb{V}^{L^2}_{k-1}(\boldsymbol{r}_3).$$

Such subspace $\mathbb{V}^{\operatorname{div}}_k(\boldsymbol{r}_2, \boldsymbol{r}_3)$ always exists. The difficulty is to give a finite element definition in terms of local DoFs.

We use $\mathrm{ND}_k^{(2)}/\mathrm{BDM}_k$ element $\boldsymbol{r}_2 = \boldsymbol{r}_3 = -1$ as an example to explain the change of DoFs. As $\boldsymbol{r}_2 = -1$, no vertex and edge DoFs exist. Due to the div stability of the bubble space, we can write DoFs as

$$\int_f \boldsymbol{v}\cdot\boldsymbol{n}\, q\,\mathrm{d}S,\quad q\in\mathbb{P}_k(f)=\mathbb{B}_k(f;\begin{pmatrix} r_2^{\mathrm{v}}\\ r_2^e\end{pmatrix}), f\in\Delta_2(T), \tag{39a}$$

$$\int_T \operatorname{div}\boldsymbol{v}\, q\,\mathrm{d}x,\quad \boldsymbol{q}\in\mathbb{P}_{k-1}(T)/\mathbb{R}=\mathbb{B}_{k-1}(T;\boldsymbol{r}_2)/\mathbb{R}, \tag{39b}$$

$$\int_T \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}x,\quad \boldsymbol{q}\in\mathbb{B}_k^{\operatorname{div}}(T;\boldsymbol{r}_2)\cap\ker(\operatorname{div}). \tag{39c}$$

The range of div operator is the discontinuous $\mathbb{P}_{k-1}$ element. Now choose $\boldsymbol{r}_3 \geq \boldsymbol{r}_2 \ominus 1$, we increase smoothness of $\operatorname{div}\boldsymbol{v}$ on vertices, edges, and faces by adding DoFs (40a), (40b), and (40c), and in turn shrink the interior moments (39b) to (40d):

$$\nabla^j \operatorname{div}\boldsymbol{v}(\mathrm{v}),\quad j=0,\ldots,r_3^{\mathrm{v}}, \tag{40a}$$

$$\int_e \frac{\partial^j(\operatorname{div}\boldsymbol{v})}{\partial n_1^i\partial n_2^{j-i}}\, q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-1-2(r_3^{\mathrm{v}}+1)+j}(e), 0\leq i\leq j\leq r_3^e, \tag{40b}$$

$$\int_f \partial_n^j(\operatorname{div}\boldsymbol{v})\ q\,\mathrm{d}S,\quad q\in\mathbb{B}_{k-1-j}(f;\begin{pmatrix} r_3^{\mathrm{v}}\\ r_3^e\end{pmatrix}-j), 0\leq j\leq r_3^f, \tag{40c}$$

$$\int_T \operatorname{div}\boldsymbol{v}\, q\,\mathrm{d}x,\quad q\in\mathbb{B}_{k-1}(T;\boldsymbol{r}_3)/\mathbb{R}, \tag{40d}$$

$$\int_f \boldsymbol{v}\cdot\boldsymbol{n}\, q\,\mathrm{d}S,\quad q\in\mathbb{P}_k(f)=\mathbb{B}_k(f;\begin{pmatrix} r_2^{\mathrm{v}}\\ r_2^e\end{pmatrix}), f\in\Delta_2(T), \tag{40e}$$

$$\int_T \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}x,\quad \boldsymbol{q}\in\mathbb{B}_k^{\operatorname{div}}(T;\boldsymbol{r}_2)\cap\ker(\operatorname{div}). \tag{40f}$$

DoFs (40a)-(40d) on $\operatorname{div}\boldsymbol{v}$ will determine $\operatorname{div}\boldsymbol{v}$ up to a constant and the sum of number of these DoFs is always $\dim\mathbb{P}_{k-1}(T)-1$. The rest DoFs (40e) and (40f) are independent of $\boldsymbol{r}_3$. Hence the total number of DoFs remains unchanged. The unisolvence is also easy as the modified DoFs are to determine $\operatorname{div}\boldsymbol{v}$.

We then explain the general case. For an edge e, we use the frame $(\boldsymbol{t},\boldsymbol{n}_1,\boldsymbol{n}_2)$, where $\boldsymbol{t}$ is a tangential vector of e and $\boldsymbol{n}_1,\boldsymbol{n}_2$ are two linearly independent normal vectors of e. For a face f, we choose two tangential vectors $\boldsymbol{t}_1,\boldsymbol{t}_2$ and a normal vector $\boldsymbol{n}$ as the local frame. We first add DoFs on $\operatorname{div}\boldsymbol{v}\in\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$ to the original DoFs (20a)-(20d), thus in the sequel, $\operatorname{div}\boldsymbol{v}$ is considered as determined, and then remove redundant DoFs. For example, on a face f, we write $\operatorname{div}\boldsymbol{v}$ in the local frame $(\boldsymbol{t}_1,\boldsymbol{t}_2,\boldsymbol{n})$

$$\operatorname{div}\boldsymbol{v}=\partial_{t_1}(\boldsymbol{v}\cdot\boldsymbol{t}_1)+\partial_{t_2}(\boldsymbol{v}\cdot\boldsymbol{t}_2)+\partial_n(\boldsymbol{v}\cdot\boldsymbol{n})=\operatorname{div}_f\Pi_f\boldsymbol{v}+\partial_n(\boldsymbol{v}\cdot\boldsymbol{n}),$$

where $\Pi_f=I-\boldsymbol{n}\boldsymbol{n}^{\intercal}$ is the projection to the plane containing face f. If $\Pi_f\boldsymbol{v}$ is known, then $\partial_n(\boldsymbol{v}\cdot\boldsymbol{n})=\operatorname{div}\boldsymbol{v}-\operatorname{div}_f\Pi_f\boldsymbol{v}$ can be determined. For normal derivatives, exchange the ordering of derivative, i.e. write $\partial_n^j(\operatorname{div}\boldsymbol{v})=\operatorname{div}\partial_n^j\boldsymbol{v}$ and apply the above argument to conclude DoFs on $\partial_n^j(\boldsymbol{v}\cdot\boldsymbol{n})$ for $j=1,\ldots,r_2^f$ are redundant. Notice that DoF on $\boldsymbol{v}\cdot\boldsymbol{n}$, i.e. for $j=0$, is still needed as $\operatorname{div}\boldsymbol{v}$ only gives constraint on derivatives.

The situation on edges is more complicated. We write the normal derivative as D_n^{α} with $\alpha=(\alpha_1,\alpha_2)\in\mathbb{T}_j^1(e)$ for $j=0,1,\ldots,r_2^e$. As $\operatorname{div}\boldsymbol{v}=\partial_t(\boldsymbol{v}\cdot\boldsymbol{t})+\partial_{n_1}(\boldsymbol{v}\cdot$

$\boldsymbol{n}_1) + \partial_{n_2}(\boldsymbol{v}\cdot\boldsymbol{n}_2)$, for each $\beta\in\mathbb{T}^1_j(e)$, $j=0,\ldots,r_3^f$, we can write

$$D_n^\beta(\operatorname{div}\boldsymbol{v}) = \partial_t D_n^\beta(\boldsymbol{v}\cdot\boldsymbol{t}) + \partial_{n_1} D_n^\beta(\boldsymbol{v}\cdot\boldsymbol{n}_1) + \partial_{n_2} D_n^\beta(\boldsymbol{v}\cdot\boldsymbol{n}_2). \tag{41}$$

On edge e, DoFs of $D_n^\beta(\boldsymbol{v}\cdot\boldsymbol{t})$ and $\nabla^j\boldsymbol{v}(\texttt{v})$ will determine the tangential component $D_n^\beta(\boldsymbol{v}\cdot\boldsymbol{t})|_e\in\mathbb{P}_{k-j}(e)$ and consequently $\partial_t D_n^\beta(\boldsymbol{v}\cdot\boldsymbol{t})|_e$. The normal derivative of $\boldsymbol{v}\cdot\boldsymbol{n}_1$ can be written as

$$\partial_{n_1} D_n^\beta(\boldsymbol{v}\cdot\boldsymbol{n}_1) = D_n^\alpha(\boldsymbol{v}\cdot\boldsymbol{n}_1),\quad \alpha=\beta+\epsilon_1, 1\le|\alpha|\le r_3^e+1\ge r_2^e.$$

Providing DoFs on $D_n^\alpha(\boldsymbol{v}\cdot\boldsymbol{n}_1)$ for all $0\le|\alpha|\le r_2^e$, we can then determine the third component in (41) for certain range of lattice nodes α:

$$\partial_{n_2} D_n^\beta(\boldsymbol{v}\cdot\boldsymbol{n}_2) = D_n^\alpha(\boldsymbol{v}\cdot\boldsymbol{n}_2),\quad \alpha=\beta+\epsilon_2, \alpha=(\alpha_1,\alpha_2), \alpha_2\ge 1, 1\le|\alpha|=j\le r_2^e.$$

But the lattice node $(\alpha_1, 0)$ is missing, i.e., DoFs on $\partial_{n_1}^j(\boldsymbol{v}\cdot\boldsymbol{n}_2)$ for $j=0,1,\ldots,$ r_2^e should be still included.

We are in the position to present finite element description of $\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)$. Take $\mathbb{P}_k^3(T)$ as the space of shape functions. The degrees of freedom are

$$\nabla^i\boldsymbol{v}(\texttt{v}),\quad i=0,\ldots,r_2^{\texttt{v}}, \tag{42a}$$

$$\nabla^j\operatorname{div}\boldsymbol{v}(\texttt{v}),\quad j=\max\{r_2^{\texttt{v}},0\},\ldots,r_3^{\texttt{v}}, \tag{42b}$$

$$\int_e \partial_{n_1}^j(\boldsymbol{v}\cdot\boldsymbol{n}_2)\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-2(r_2^{\texttt{v}}+1)+j}(e), 0\le j\le r_2^e, \tag{42c}$$

$$\int_e \frac{\partial^j(\boldsymbol{v}\cdot\boldsymbol{t})}{\partial n_1^i\partial n_2^{j-i}}\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-2(r_2^{\texttt{v}}+1)+j}(e), 0\le i\le j\le r_2^e, \tag{42d}$$

$$\int_e \frac{\partial^j(\boldsymbol{v}\cdot\boldsymbol{n}_1)}{\partial n_1^i\partial n_2^{j-i}}\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-2(r_2^{\texttt{v}}+1)+j}(e), 0\le i\le j\le r_2^e, \tag{42e}$$

$$\int_e \frac{\partial^j(\operatorname{div}\boldsymbol{v})}{\partial n_1^i\partial n_2^{j-i}}\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-1-2(r_3^{\texttt{v}}+1)+j}(e), 0\le i\le j\le r_3^e, \tag{42f}$$

$$\int_f \boldsymbol{v}\cdot\boldsymbol{n}\,q\,\mathrm{d}S,\quad q\in\mathbb{P}_0(f)\oplus(\mathbb{B}_k(f;\begin{pmatrix} r_2^{\texttt{v}}\\ r_2^e\end{pmatrix})/\mathbb{R}), \tag{42g}$$

$$\int_f \partial_n^j(\boldsymbol{v}\cdot\boldsymbol{t}_\ell)\,q\,\mathrm{d}S,\quad q\in\mathbb{B}_{k-j}(f;\begin{pmatrix} r_2^{\texttt{v}}\\ r_2^e\end{pmatrix}-j), 0\le j\le r_2^f, \ell=1,2, \tag{42h}$$

$$\int_f \partial_n^j(\operatorname{div}\boldsymbol{v})\,q\,\mathrm{d}S,\quad q\in\mathbb{B}_{k-1-j}(f;\begin{pmatrix} r_3^{\texttt{v}}\\ r_3^e\end{pmatrix}-j), 0\le j\le r_3^f, \tag{42i}$$

$$\int_T \operatorname{div}\boldsymbol{v}\,q\,\mathrm{d}x,\quad q\in\mathbb{B}_{k-1}(\boldsymbol{r}_3)/\mathbb{R}, \tag{42j}$$

$$\int_T \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}x,\quad \boldsymbol{q}\in\mathbb{B}_k^{\operatorname{div}}(\boldsymbol{r}_2)\cap\ker(\operatorname{div}) \tag{42k}$$

for each $\texttt{v}\in\Delta_0(T)$, $e\in\Delta_1(T)$ and $f\in\Delta_2(T)$.

Lemma 4.9. *Let $\boldsymbol{r}_3\ge\boldsymbol{r}_2\ominus 1$ be two smoothness vectors and $k\ge\max\{2r_2^{\texttt{v}}+1, r_2^{\texttt{v}}+2, 3(r_2^e+1), 2r_3^{\texttt{v}}+2, 4r_3^f+5, (r_3^e+r_3^f+5)[r_3^{\texttt{v}}=0]\}$. The DoFs* (42) *are unisolvent for $\mathbb{P}_k^3(T)$.*

Proof. The number of DoFs (42b), (42f), (42i), and (42j) to determine $\operatorname{div}\boldsymbol{v}\in\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$ is $\dim\mathbb{P}_{k-1}(T)-1-4\binom{r_2^{\texttt{v}}+2}{3}$, which is a constant independent of $\boldsymbol{r}_3$. Hence the number of DoFs (42a)-(42k) is also constant with respect to $\boldsymbol{r}_3$. To count the dimension, we only need to consider case $\boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$. Now the number of DoFs (42c)-(42f) equals that of

$$\int_e \frac{\partial^j \boldsymbol{v}}{\partial n_1^i \partial n_2^{j-i}} \cdot \boldsymbol{q} \,\mathrm{d}s, \quad e \in \Delta_1(T), \boldsymbol{q} \in \mathbb{P}^3_{k-2(r_2^{\mathrm{v}}+1)+j}(e), 0 \leq i \leq j \leq r_2^e.$$

As a result the number of DoFs (42a)-(42k) equals $\dim \mathbb{P}_k^3(T)$.

Take $\boldsymbol{v} \in \mathbb{P}_k^3(T)$ and assume all the DoFs (42a)-(42k) vanish. The vanishing DoF (42g) implies $\operatorname{div} \boldsymbol{v} \in L_0^2(T)$. By the vanishing DoFs (42a)-(42b), (42e) and (42i)-(42j), we get $\operatorname{div} \boldsymbol{v} = 0$. Since $\operatorname{div} \boldsymbol{v} = \partial_t(\boldsymbol{v}\cdot\boldsymbol{t}) + \partial_{n_1}(\boldsymbol{v}\cdot\boldsymbol{n}_1) + \partial_{n_2}(\boldsymbol{v}\cdot\boldsymbol{n}_2)$ for each edge e and $\operatorname{div} \boldsymbol{v} = \partial_n(\boldsymbol{v}\cdot\boldsymbol{n}) + \operatorname{div}_f(\Pi_f \boldsymbol{v})$ for each face f, it follows from the vanishing DoFs (42a), (42c)-(42e), and (42g)-(42h) that $\boldsymbol{v} \in \mathbb{B}_k^{\operatorname{div}}(\boldsymbol{r}_2) \cap \ker(\operatorname{div})$. Therefore $\boldsymbol{v} = \boldsymbol{0}$ holds from the vanishing DoF (42k). ∎

A basis of $\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3)$ can be generated through computer programming. See Remark 3.7 and Remark 3.15.

Define global $H(\operatorname{div})$-conforming finite element space

$$\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3) : \boldsymbol{v}|_T \in \mathbb{P}_k^3(T)\ \forall\, T \in \mathcal{T}_h, \text{ all the DoFs (42a)-(42i) are single-valued}\}.$$

When $\boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$, we have $\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_2 \ominus 1) = \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2)$. Although the DoFs defining these two finite element spaces are in different forms, from the proof of Lemma 4.9, they can express each other by linear combinations.

When $\boldsymbol{r}_3 \geq \boldsymbol{r}_2 \ominus 1$, we have

$$\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) \subseteq \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_2 \ominus 1) = \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2).$$

Namely additional smoothness on $\operatorname{div} \boldsymbol{v}$ is imposed in space $\mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3)$.

Theorem 4.10. *Let $\boldsymbol{r}_2 \geq -1$ satisfy* (30) *and $\boldsymbol{r}_3 \geq \boldsymbol{r}_2 \ominus 1$. Assume $k \geq \max\{2r_2^{\mathrm{v}}+1, r_2^{\mathrm{v}}+2, 3(r_2^e+1), 2r_3^{\mathrm{v}}+2, 4r_3^f+5, (r_3^e+r_3^f+5)[r_3^{\mathrm{v}}=0]\}$. It holds that*

$$\operatorname{div} \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) = \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3). \tag{43}$$

Proof. The condition $k \geq \max\{2r_2^{\mathrm{v}}+1, 3(r_2^e+1)\}$ ensures that $\dim \mathbb{B}_k(f; \begin{pmatrix} r_2^{\mathrm{v}} \\ r_2^e \end{pmatrix}) \geq 1$ for all cases except $\boldsymbol{r}_2 = (0,-1,-1)$, and condition $k \geq r_2^{\mathrm{v}}+2$ is required for case $\boldsymbol{r}_2 = (0,-1,-1)$. The condition $k \geq \max\{2r_3^{\mathrm{v}}+2, 4r_3^f+5, (r_3^e+r_3^f+5)[r_3^{\mathrm{v}}=0]\}$ guarantees $\dim \mathbb{B}_{k-1}(\boldsymbol{r}_3) \geq 1$. It is apparent that $\operatorname{div} \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) \subseteq \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$. We are going to prove the div operator is surjective.

For $p \in \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) \subset H^{r_3^f+1}(\Omega)$, there exists a $\boldsymbol{u} \in \boldsymbol{H}^{r_3^f+2}(\Omega;\mathbb{R}^3)$ such that $\operatorname{div} \boldsymbol{u} = p$. Take $\boldsymbol{v} = (v_1, v_2, v_3) \in \mathbb{V}_k^{\operatorname{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3)$ such that all DoFs (42a)-(42k) vanish except

$$
\begin{aligned}
\nabla^j(\partial_1 v_1)(\mathrm{v}) &= \nabla^j p(\mathrm{v}), && j=0,\ldots,r_2^{\mathrm{v}}-1,\\
\nabla^j \operatorname{div}\boldsymbol{v}(\mathrm{v}) &= \nabla^j p(\mathrm{v}), && j=\max\{r_2^{\mathrm{v}},0\},\ldots,r_3^{\mathrm{v}},\\
\int_e \frac{\partial^j(\operatorname{div}\boldsymbol{v})}{\partial n_1^i\partial n_2^{j-i}}\, q\,\mathrm{d}s &= \int_e \frac{\partial^j p}{\partial n_1^i\partial n_2^{j-i}}\, q\,\mathrm{d}s, && q\in\mathbb{P}_{k-1-2(r_3^{\mathrm{v}}+1)+j}(e), 0\le i\le j\le r_3^e,\\
\int_f \boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S &= \int_f \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S,\\
\int_f \partial_n^j(\operatorname{div}\boldsymbol{v})\; q\,\mathrm{d}S &= \int_f \partial_n^j p\; q\,\mathrm{d}S, && q\in\mathbb{B}_{k-1-j}(f;\begin{pmatrix} r_3^{\mathrm{v}}\\ r_3^e\end{pmatrix}-j), 0\le j\le r_3^f,\\
\int_T \operatorname{div}\boldsymbol{v}\, q\,\mathrm{d}x &= \int_T p\, q\,\mathrm{d}x, && q\in\mathbb{B}_{k-1}(\boldsymbol{r}_3)/\mathbb{R},
\end{aligned}
$$

for all $\mathrm{v}\in\Delta_0(\mathcal{T}_h)$, $e\in\Delta_1(\mathcal{T}_h)$, $f\in\Delta_2(\mathcal{T}_h)$ and $T\in\mathcal{T}_h$. Then it holds $\operatorname{div}\boldsymbol{v}=p$. ☐

Example 4.11. Taking $k\ge 5$, $\boldsymbol{r}_2=-1$, and $\boldsymbol{r}_3=0$, we get a stable pair for mixed Poisson problem but with continuous displacement. That is we can construct a subspace of $\mathrm{ND}_k^{(2)}/\mathrm{BDM}_k$ space with the range of div is continuous. The degree $k\ge 5$ is to ensure $\dim\mathbb{B}_{k-1}(T,\boldsymbol{r}_3)\ge 1$.

Example 4.12 (3D Falk-Neilan Stokes element)**.** Taking $k\ge 6$, $\boldsymbol{r}_2=(2,1,0)$, and $\boldsymbol{r}_3=(1,0,0)$, we get a stable Stokes-pair with continuous pressure element

$$
(\mathbb{V}_k^{\operatorname{div}}(\begin{pmatrix}2\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}),\mathbb{V}_{k-1}^{L^2}(\begin{pmatrix}1\\0\\0\end{pmatrix})),
$$

which is a generalization of the two-dimensional Falk-Neilan Stokes element constructed in [26] to three dimensions.

5. Finite element de Rham and Stokes complexes

In this section we shall construct several finite element de Rham and Stokes complexes.

5.1. Exactness of a complex of finite dimensional spaces.

Lemma 5.1. *Let $\mathcal{P}$ and $\mathcal{V}_i$ be finite-dimensional linear spaces for $i=0,\ldots,3$ and*

$$
\mathcal{P}\xrightarrow{\subset}\mathcal{V}_0\xrightarrow{\mathrm{d}_0}\mathcal{V}_1\xrightarrow{\mathrm{d}_1}\mathcal{V}_2\xrightarrow{\mathrm{d}_2}\mathcal{V}_3\to 0 \tag{44}
$$

be a complex. Assume three out of the four conditions for the exactness of the complex hold

$$
\begin{aligned}
\mathcal{P} &= \mathcal{V}_0\cap\ker(\mathrm{d}_0),\\
\mathrm{d}_0\mathcal{V}_0 &= \mathcal{V}_1\cap\ker(\mathrm{d}_1),\\
\mathrm{d}_1\mathcal{V}_1 &= \mathcal{V}_2\cap\ker(\mathrm{d}_2),\\
\mathrm{d}_2\mathcal{V}_2 &= \mathcal{V}_3,
\end{aligned}
$$

and the dimensions satisfy

$$
\dim\mathcal{P}-\dim\mathcal{V}_0+\dim\mathcal{V}_1-\dim\mathcal{V}_2+\dim\mathcal{V}_3=0, \tag{45}
$$

then complex (44) *is exact.*

Proof. Since $\dim \mathcal{V}_i = \dim \mathcal{V}_i \cap \ker(\mathrm{d}_i) + \dim \mathrm{d}_i\mathcal{V}_i$ for $i = 0, 1, 2$, by (45) we have

$$\begin{aligned}
&- (\dim \mathcal{V}_0 \cap \ker(\mathrm{d}_0) - \dim \mathcal{P}) + (\dim \mathcal{V}_1 \cap \ker(\mathrm{d}_1) - \dim \mathrm{d}_0\mathcal{V}_0)\\
&- (\dim \mathcal{V}_2 \cap \ker(\mathrm{d}_2) - \dim \mathrm{d}_1\mathcal{V}_1) + (\dim \mathcal{V}_3 - \dim \mathrm{d}_2\mathcal{V}_2)\\
=& \dim \mathcal{P} - \dim \mathcal{V}_0 + \dim \mathcal{V}_1 - \dim \mathcal{V}_2 + \dim \mathcal{V}_3 = 0.
\end{aligned}$$

On the other hand, by assumption, three of four numbers $\dim \mathcal{V}_0 \cap \ker(\mathrm{d}_0) - \dim \mathcal{P}$, $\dim \mathcal{V}_1 \cap \ker(\mathrm{d}_1) - \dim \mathrm{d}_0\mathcal{V}_0$, $\dim \mathcal{V}_2 \cap \ker(\mathrm{d}_2) - \dim \mathrm{d}_1\mathcal{V}_1$ and $\dim \mathcal{V}_3 - \dim \mathrm{d}_2\mathcal{V}_2$ are zeros. Thus all the four numbers vanish, that is

$$\begin{aligned}
\dim \mathcal{V}_0 \cap \ker(\mathrm{d}_0) &= \dim \mathcal{P}, & \dim \mathcal{V}_1 \cap \ker(\mathrm{d}_1) &= \dim \mathrm{d}_0\mathcal{V}_0,\\
\dim \mathcal{V}_2 \cap \ker(\mathrm{d}_2) &= \dim \mathrm{d}_1\mathcal{V}_1, & \dim \mathcal{V}_3 &= \dim \mathrm{d}_2\mathcal{V}_2,
\end{aligned}$$

as required. $\square$

A polynomial de Rham complex on tetrahedron T is, for $k \geq 1$,

$$\mathbb{R} \xrightarrow{\subset} \mathbb{P}_{k+2}(T) \xrightarrow{\text{grad}} \mathbb{P}_{k+1}^3(T) \xrightarrow{\text{curl}} \mathbb{P}_k^3(T) \xrightarrow{\text{div}} \mathbb{P}_{k-1}(T) \to 0. \tag{46}$$

By Lemma 5.1, the exactness of polynomial complex (46) can be verified by the identity

$$1 - \binom{k+5}{3} + 3\binom{k+4}{3} - 3\binom{k+3}{3} + \binom{k+2}{3} = 0, \tag{47}$$

and the fact

$$\begin{aligned}
\mathbb{R} &= \ker(\text{grad}),\\
\mathbb{P}_{k+1}^3(T) \cap \ker(\text{curl}) &= \text{grad}\, \mathbb{P}_{k+2}(T),\\
\text{div}\, \mathbb{P}_k^3(T) &= \mathbb{P}_{k-1}(T).
\end{aligned}$$

The first two are trivial and the last one can be proved by $\text{div}(\boldsymbol{x}\mathbb{P}_{k-1}(T)) = \mathbb{P}_{k-1}(T)$.

5.2. **Graft de Rham complexes.** We first present an abstract definition of space

$$\mathbb{V}_k^{\text{curl}}(\boldsymbol{r}_1) = \mathbb{V}_k^3(\boldsymbol{r}_1) \cap \boldsymbol{H}(\text{curl}, \Omega).$$

We call $(\boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$ a valid de Rham smoothness sequence if the following sequence, with a sufficiently large degree k,

$$\mathbb{R} \xrightarrow{\subset} \mathbb{V}_{k+2}^{\text{grad}}(\boldsymbol{r}_0) \xrightarrow{\text{grad}} \mathbb{V}_{k+1}^{\text{curl}}(\boldsymbol{r}_1) \xrightarrow{\text{curl}} \mathbb{V}_k^{\text{div}}(\boldsymbol{r}_2) \xrightarrow{\text{div}} \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) \to 0 \tag{48}$$

is an exact Hilbert complex. For a smoothness vector $\hat{\boldsymbol{r}}_2 \geq \boldsymbol{r}_2$, we can define the subspace

$$\mathbb{V}_{k+1}^{\text{curl}}(\boldsymbol{r}_1, \hat{\boldsymbol{r}}_2) = \{\boldsymbol{v} \in \mathbb{V}_{k+1}^{\text{curl}}(\boldsymbol{r}_1) : \text{curl}\, \boldsymbol{v} \in \mathbb{V}_k^{\text{div}}(\hat{\boldsymbol{r}}_2) \cap \ker(\text{div})\}.$$

Such space is well-defined as $\mathbb{V}_k^{\text{div}}(\hat{\boldsymbol{r}}_2) \cap \ker(\text{div}) \subseteq \mathbb{V}_k^{\text{div}}(\boldsymbol{r}_2) \cap \ker(\text{div})$ and $(\boldsymbol{r}_0, \boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$ is a valid de Rham smoothness sequence implies $\text{curl}\, \mathbb{V}_{k+1}^{\text{curl}}(\boldsymbol{r}_1) = \mathbb{V}_k^{\text{div}}(\boldsymbol{r}_2) \cap \ker(\text{div})$.

Recall that $(\boldsymbol{r}_2, \boldsymbol{r}_3, k)$ is called div stable pair if $\text{div}\, \mathbb{V}_k^{\text{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) = \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$ for a sufficiently large degree k.

Theorem 5.2. *Assume* $(\boldsymbol{r}_0,\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3)$ *is a valid de Rham smoothness sequence and* $(\hat{\boldsymbol{r}}_2,\hat{\boldsymbol{r}}_3,k)$ *is div stable with* $\hat{\boldsymbol{r}}_2\geq \boldsymbol{r}_2$, $\hat{\boldsymbol{r}}_3\geq\boldsymbol{r}_3$. *Then* $(\boldsymbol{r}_0,\boldsymbol{r}_1,\hat{\boldsymbol{r}}_2,\hat{\boldsymbol{r}}_3)$ *is also a valid de Rham smoothness sequence in the sense that the following complex*

$$\mathbb{R}\xrightarrow{\subset}\mathbb{V}^{\rm grad}_{k+2}(\boldsymbol{r}_0)\xrightarrow{\rm grad}\mathbb{V}^{\rm curl}_{k+1}(\boldsymbol{r}_1,\hat{\boldsymbol{r}}_2)\xrightarrow{\rm curl}\mathbb{V}^{\rm div}_{k}(\hat{\boldsymbol{r}}_2,\hat{\boldsymbol{r}}_3)\xrightarrow{\rm div}\mathbb{V}^{L^2}_{k-1}(\hat{\boldsymbol{r}}_3)\to 0$$

is exact.

Proof. Exactness of (48) implies $\ker(\operatorname{div})\cap\mathbb{V}^{\rm div}_k(\boldsymbol{r}_2)=\operatorname{curl}\mathbb{V}^{\rm curl}_{k+1}(\boldsymbol{r}_1)$. Then $\ker(\operatorname{div})\cap\mathbb{V}^{\rm div}_k(\hat{\boldsymbol{r}}_2)\subseteq\ker(\operatorname{div})\cap\mathbb{V}^{\rm div}_k(\boldsymbol{r}_2)=\operatorname{curl}\mathbb{V}^{\rm curl}_{k+1}(\boldsymbol{r}_1)$, by which we get

$$\ker(\operatorname{div})\cap\mathbb{V}^{\rm div}_k(\hat{\boldsymbol{r}}_2)=\operatorname{curl}\mathbb{V}^{\rm curl}_{k+1}(\boldsymbol{r}_1,\hat{\boldsymbol{r}}_2).$$

As in $\mathbb{V}^{\rm curl}_{k+1}(\boldsymbol{r}_1,\hat{\boldsymbol{r}}_2)$, only the range of curl operator is changed, the relation $\mathbb{V}^{\rm curl}_{k+1}(\boldsymbol{r}_1,\hat{\boldsymbol{r}}_2)\cap\ker(\operatorname{curl})=\operatorname{grad}\mathbb{V}^{\rm grad}_{k+2}(\boldsymbol{r}_0)$ still holds. The relation $\operatorname{div}\mathbb{V}^{\rm div}_k(\hat{\boldsymbol{r}}_2,\hat{\boldsymbol{r}}_3)=\mathbb{V}^{L^2}_{k-1}(\hat{\boldsymbol{r}}_3)$ is from the assumption $(\hat{\boldsymbol{r}}_2,\hat{\boldsymbol{r}}_3,k)$ is div stable. $\square$

In Example 5.3, we shall further simplify the notation by presenting the smoothness vectors only and skip the space notation which should be clear from the context.

Example 5.3. The standard de Rham complex is $(\mathbf{0},\mathbf{-1},\mathbf{-1},\mathbf{-1})$. Take the stable div pair $\hat{\boldsymbol{r}}_2=(2,1,0),\hat{\boldsymbol{r}}_3=(1,0,-1)$, we obtain the finite element Stokes complex

$$\mathbb{R}\xrightarrow{\subset}\begin{pmatrix}0\\0\\0\end{pmatrix}\xrightarrow{\rm grad}\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\xrightarrow{\rm curl}\begin{pmatrix}2\\1\\0\end{pmatrix}\xrightarrow{\rm div}\begin{pmatrix}1\\0\\-1\end{pmatrix}\to 0.$$

Use $\hat{\boldsymbol{r}}_2=(2,1,0),\hat{\boldsymbol{r}}_3=(1,0,0)$, we get another finite element Stokes complex ending with a continuous element

$$\mathbb{R}\xrightarrow{\subset}\begin{pmatrix}0\\0\\0\end{pmatrix}\xrightarrow{\rm grad}\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\xrightarrow{\rm curl}\begin{pmatrix}2\\1\\0\end{pmatrix}\xrightarrow{\rm div}\begin{pmatrix}1\\0\\0\end{pmatrix}\to 0.$$

As $\mathbb{V}^{\rm div}_k(\hat{\boldsymbol{r}}_2)\subset\boldsymbol{H}^1(\Omega;\mathbb{R}^3)$, the space $\mathbb{V}^{\rm curl}_{k+1}(\boldsymbol{r}_1,\hat{\boldsymbol{r}}_2)\subset\boldsymbol{H}(\operatorname{grad}\operatorname{curl},\Omega):=\{\boldsymbol{v}\in\boldsymbol{H}(\operatorname{curl},\Omega),\operatorname{curl}\boldsymbol{v}\in\boldsymbol{H}^1(\Omega;\mathbb{R}^3)\}$ which can be used to discretize the quad-curl problem [52, 56]. Some non-conforming finite element discretizations of Stokes complex can be found in [35, 36].

5.3. $H(\operatorname{curl})$**-conforming finite elements.** Next we give a finite element description for $\mathbb{V}^{\rm curl}_{k+1}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ with $\boldsymbol{r}_2\geq\boldsymbol{r}_1\ominus 1$. We should keep DoFs for $\operatorname{curl}\boldsymbol{v}\in\mathbb{V}^{\rm div}_k(\boldsymbol{r}_2)$, combine DoFs for $\mathbb{V}^3_{k+1}(\boldsymbol{r}_1)$, and eliminate linearly dependent ones.

On edge e, we choose frame $\{\boldsymbol{t},\boldsymbol{n}_1,\boldsymbol{n}_2\}$ and to facilitate the calculation simplify as (x_0,x_1,x_2). Then $\boldsymbol{u}=(u_0,u_1,u_2)^{\intercal}$ with $u_0=\boldsymbol{u}\cdot\boldsymbol{t},u_1=\boldsymbol{u}\cdot\boldsymbol{n}_1,u_2=\boldsymbol{u}\cdot\boldsymbol{n}_2$, and $\operatorname{curl}\boldsymbol{u}=(\partial_1u_2-\partial_2u_1,\partial_2u_0-\partial_0u_2,\partial_0u_1-\partial_1u_0)^{\intercal}$. Apply the normal derivative D^α_n to $\operatorname{curl}\boldsymbol{u}$, we obtain, for $\alpha\in\mathbb{T}^1_j(e)$, $j=0,\ldots,r^e_2$,

$$D_n^\alpha\operatorname{curl}\boldsymbol{u}=\big(D_n^{(\alpha_1+1,\alpha_2)}u_2-D_n^{(\alpha_1,\alpha_2+1)}u_1, \tag{49}$$

$$D_n^{(\alpha_1,\alpha_2+1)}u_0-\partial_0D_n^{(\alpha_1,\alpha_2)}u_2, \tag{50}$$

$$\partial_0D_n^{(\alpha_1,\alpha_2)}u_1-D_n^{(\alpha_1+1,\alpha_2)}u_0\big)^{\intercal}. \tag{51}$$

DoFs on $D^\alpha_n\operatorname{curl}\boldsymbol{u}$ are given and thus $D^\alpha_n\operatorname{curl}\boldsymbol{u}$ is considered as known on edge e. Include DoFs on $D^\alpha_nu_1, 0\leq|\alpha|\leq r^e_1$, then $\partial_0D^\alpha_nu_1$ is known on edge e. Linear combination with (49), we can determine $D^\alpha_nu_2$, $1\leq|\alpha|\leq r^e_1,\alpha_1\geq 1$ but $\partial^j_{n_2}u_2$, $j=0,\ldots,r^e_1$ are left and thus should be included in the DoF. Linear combination with (51), we can determine $D_n^\alpha u_0$ for $1 \leq |\alpha| \leq r_1^e$ with $\alpha_1 \geq 1$. Linear combination with (50), we also know $D_n^\alpha u_0$ for $1 \leq |\alpha| \leq r_1^e$ with $\alpha_2 \geq 1$. So only $\alpha = (0,0)$ is left. Namely DoF on $u_0 = \boldsymbol{u} \cdot \boldsymbol{t}$ should be included explicitly.

We then move to faces and present formulae on the normal and tangential component of $\operatorname{curl} \boldsymbol{u}$. For smooth scalar function v and face f with unit normal vector $\boldsymbol{n}$, define surface gradient

$$\nabla_f v := \Pi_f(\nabla v) = \nabla v - (\partial_n v)\boldsymbol{n}.$$

For smooth vector function $\boldsymbol{u}$, define surface rotation

$$\operatorname{rot}_f \boldsymbol{u} := (\boldsymbol{n} \times \nabla) \cdot \boldsymbol{u} = (\operatorname{curl} \boldsymbol{u}) \cdot \boldsymbol{n}.$$

Clearly it holds $\operatorname{rot}_f \boldsymbol{u} = \operatorname{rot}_f(\Pi_f \boldsymbol{u})$.

Lemma 5.4. *On face f, for smooth enough function $\boldsymbol{u}$, it holds that*

$$\boldsymbol{n} \cdot (\nabla \times \boldsymbol{u}) = \operatorname{rot}_f(\Pi_f \boldsymbol{u}), \tag{52}$$

$$\boldsymbol{n} \times (\nabla \times \boldsymbol{u}) = \nabla_f(\boldsymbol{u} \cdot \boldsymbol{n}) - \partial_n(\Pi_f \boldsymbol{u}), \tag{53}$$

$$\partial_n^j(\boldsymbol{n} \times (\nabla \times \boldsymbol{u})) = \nabla_f(\partial_n^j \boldsymbol{u} \cdot \boldsymbol{n}) - \Pi_f \partial_n^{j+1} \boldsymbol{u}, \quad j \geq 0. \tag{54}$$

Proof. Identity (52) is indeed the definition of rot_f. By a direct computation, (53) follows from

$$\begin{aligned} \boldsymbol{n} \times (\nabla \times \boldsymbol{u}) &= \nabla(\boldsymbol{u} \cdot \boldsymbol{n}) - \partial_n \boldsymbol{u} \\ &= \nabla_f(\boldsymbol{u} \cdot \boldsymbol{n}) + \partial_n(\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} - \big(\partial_n(\Pi_f \boldsymbol{u}) + \partial_n(\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n}\big) \\ &= \nabla_f(\boldsymbol{u} \cdot \boldsymbol{n}) - \partial_n(\Pi_f \boldsymbol{u}). \end{aligned}$$

Exchange partial derivatives ∂_n^j with $\boldsymbol{n} \times (\nabla \times \cdot)$ to get (54). $\square$

We always include DoFs for $\operatorname{curl} \boldsymbol{u}$. So by (52) $\operatorname{rot}_f \Pi_f \boldsymbol{u}$ can be determined. By the Helmholtz decomposition of a vector function on the face, the tangential component $\Pi_f \boldsymbol{u}$ can be determined by $\operatorname{rot}_f \Pi_f \boldsymbol{u}$ and the moment with $\operatorname{grad}_f \mathbb{B}_{k+2}(f; \begin{pmatrix} r_1^{\mathrm{v}} \\ r_1^e \end{pmatrix} + 1)$. The normal derivative of the normal component $\partial_n^j(\boldsymbol{u} \cdot \boldsymbol{n})$, for $j = 0, 1, \ldots, r_1^f$, will be included as DoFs. Then $\nabla_f(\partial_n^j \boldsymbol{u} \cdot \boldsymbol{n})$ can be computed on face. Thanks to (54), the normal derivative of the tangential component $\partial_n^j \Pi_f \boldsymbol{u}$ can be determined.

We are in the position to present a finite element description for the space $\mathbb{V}_{k+1}^{\operatorname{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2)$. Take $\mathbb{P}_{k+1}^3(T)$ as the space of shape functions. The degrees of freedom are

$$\nabla^i \boldsymbol{v}(\mathtt{v}), \quad i=0,\ldots,r_1^{\mathtt{v}}, \tag{55a}$$

$$\nabla^j (\operatorname{curl}\boldsymbol{v})(\mathtt{v}), \quad j=\max\{r_1^{\mathtt{v}},0\},\ldots,r_2^{\mathtt{v}}, \tag{55b}$$

$$\int_e \boldsymbol{v}\cdot\boldsymbol{t}\, q\,\mathrm{d}s, \quad q\in\mathbb{P}_{k-1-2r_1^{\mathtt{v}}}(e), \tag{55c}$$

$$\int_e \frac{\partial^j(\boldsymbol{v}\cdot\boldsymbol{n}_1)}{\partial n_1^i\partial n_2^{j-i}}\, q\,\mathrm{d}s, \quad q\in\mathbb{P}_{k-1-2r_1^{\mathtt{v}}+j}(e), 0\leq i\leq j\leq r_1^e, \tag{55d}$$

$$\int_e \partial_{n_2}^j(\boldsymbol{v}\cdot\boldsymbol{n}_2)\, q\,\mathrm{d}s, \quad q\in\mathbb{P}_{k-1-2r_1^{\mathtt{v}}+j}(e), 0\leq j\leq r_1^e, \tag{55e}$$

$$\int_e \partial_{n_1}^j((\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n}_2)\, q\,\mathrm{d}s, \quad q\in\mathbb{P}_{k-2(r_2^{\mathtt{v}}+1)+j}(e), 0\leq j\leq r_2^e, \tag{55f}$$

$$\int_e \frac{\partial^j((\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{t})}{\partial n_1^i\partial n_2^{j-i}}\, q\,\mathrm{d}s, \quad q\in\mathbb{P}_{k-2(r_2^{\mathtt{v}}+1)+j}(e), 0\leq i\leq j\leq r_2^e, \tag{55g}$$

$$\int_e \frac{\partial^j((\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n}_1)}{\partial n_1^i\partial n_2^{j-i}}\, q\,\mathrm{d}s, \quad q\in\mathbb{P}_{k-2(r_2^{\mathtt{v}}+1)+j}(e), 0\leq i\leq j\leq r_2^e, \tag{55h}$$

$$\int_f (\Pi_f\boldsymbol{v})\cdot\boldsymbol{q}\,\mathrm{d}S, \quad \boldsymbol{q}\in\operatorname{grad}_f\mathbb{B}_{k+2}(f;\begin{pmatrix} r_1^{\mathtt{v}}\\ r_1^e\end{pmatrix}+1), \tag{55i}$$

$$\int_f \partial_n^j(\boldsymbol{v}\cdot\boldsymbol{n})\, q\,\mathrm{d}S, \quad q\in\mathbb{B}_{k+1-j}(f;\begin{pmatrix} r_1^{\mathtt{v}}\\ r_1^e\end{pmatrix}-j), 0\leq j\leq r_1^f, \tag{55j}$$

$$\int_f (\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n}\, q\,\mathrm{d}S, \quad q\in\mathbb{B}_{k}(f;\begin{pmatrix} r_2^{\mathtt{v}}\\ r_2^e\end{pmatrix})/\mathbb{R}, \tag{55k}$$

$$\int_f \partial_n^j((\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{t}_\ell)\; q\,\mathrm{d}S, \quad q\in\mathbb{B}_{k-j}(f;\begin{pmatrix} r_2^{\mathtt{v}}\\ r_2^e\end{pmatrix}-j), 0\leq j\leq r_2^f, \ell=1,2, \tag{55l}$$

$$\int_T (\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{q}\,\mathrm{d}x, \quad \boldsymbol{q}\in\mathbb{B}_k^{\operatorname{div}}(\boldsymbol{r}_2)\cap\ker(\operatorname{div}), \tag{55m}$$

$$\int_T \boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}x, \quad \boldsymbol{q}\in\operatorname{grad}\mathbb{B}_{k+2}(\boldsymbol{r}_1+1) \tag{55n}$$

for each $\mathtt{v}\in\Delta_0(T)$, $e\in\Delta_1(T)$ and $f\in\Delta_2(T)$.

Lemma 5.5. *Assume k is a large enough integer satisfying $k\geq\max\{2r_1^{\mathtt{v}}+1, 2r_2^{\mathtt{v}}+1, r_2^{\mathtt{v}}+2, 3(r_2^e+1)\}$. The DoFs* (55a)-(55n) *are unisolvent for $\mathbb{P}_{k+1}^3(T)$.*

Proof. Since $\nabla(\operatorname{curl}\boldsymbol{v})$ is trace-free, the number of DoF (55b) at one vertex is

$$3\binom{r_2^{\mathtt{v}}+3}{3}-3\binom{r_1^{\mathtt{v}}+2}{3}-\left(\binom{r_2^{\mathtt{v}}+2}{3}-\binom{r_1^{\mathtt{v}}+1}{3}\right).$$

Then thanks to the proof of Lemma 4.9, the sum of the number of DoFs (55b), (55f)-(55h) and (55k)-(55m) is

$$\begin{aligned}
&3\dim\mathbb{P}_k(T)-\left(\dim\mathbb{P}_{k-1}(T)-1-4\binom{r_2^{\mathrm{v}}+2}{3}\right)-4-12\binom{r_2^{\mathrm{v}}+3}{3}\\
&+12\binom{r_2^{\mathrm{v}}+3}{3}-12\binom{r_1^{\mathrm{v}}+2}{3}-4\binom{r_2^{\mathrm{v}}+2}{3}+4\binom{r_1^{\mathrm{v}}+1}{3}\\
&=3\dim\mathbb{P}_k(T)-\dim\mathbb{P}_{k-1}(T)-3-12\binom{r_1^{\mathrm{v}}+2}{3}+4\binom{r_1^{\mathrm{v}}+1}{3},
\end{aligned}\tag{56}$$

which is constant with respect to $\boldsymbol{r}_2$. Hence the sum of the number of DoFs (55a)-(55n) is also constant with respect to $\boldsymbol{r}_2$. It suffices to consider case $\boldsymbol{r}_2=\boldsymbol{r}_1\ominus 1$ to count the dimension. Now the number of DoFs (55e)-(55h) equals that of

$$\int_e \frac{\partial^j(\boldsymbol{v}\cdot\boldsymbol{t})}{\partial n_1^i\partial n_2^{j-i}}\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-1-2r_1^{\mathrm{v}}+j}(e), 0\le i\le j, 1\le j\le r_1^e,\tag{57}$$

$$\int_e \frac{\partial^j(\boldsymbol{v}\cdot\boldsymbol{n}_2)}{\partial n_1^i\partial n_2^{j-i}}\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-1-2r_1^{\mathrm{v}}+j}(e), 0\le i\le j\le r_1^e.\tag{58}$$

As a result the number of DoFs (55a)-(55n) equals $\dim\mathbb{P}_{k+1}^3(T)$.

Take $\boldsymbol{v}\in\mathbb{P}_{k+1}^3(T)$ and assume all the DoFs (55a)-(55n) vanish. The vanishing DoF (55c) implies $(\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n}|_f\in L_0^2(f)$ for $f\in\Delta_2(T)$. By the vanishing DoFs (55a)-(55b), (55f)-(55h) and (55k)-(55m), we get $\operatorname{curl}\boldsymbol{v}=\boldsymbol{0}$.

For edge $e\in\Delta_1(T)$ with frame $\{\boldsymbol{t},\boldsymbol{n}_1,\boldsymbol{n}_2\}$, we have

$$\begin{aligned}
\operatorname{curl}\boldsymbol{v}&=\operatorname{curl}((\boldsymbol{v}\cdot\boldsymbol{t})\boldsymbol{t}+(\boldsymbol{v}\cdot\boldsymbol{n}_1)\boldsymbol{n}_1+(\boldsymbol{v}\cdot\boldsymbol{n}_2)\boldsymbol{n}_2)\\
&=-(\boldsymbol{t}\times\nabla)(\boldsymbol{v}\cdot\boldsymbol{t})-(\boldsymbol{n}_1\times\nabla)(\boldsymbol{v}\cdot\boldsymbol{n}_1)-(\boldsymbol{n}_2\times\nabla)(\boldsymbol{v}\cdot\boldsymbol{n}_2),
\end{aligned}$$

which combined with $\operatorname{curl}\boldsymbol{v}=\boldsymbol{0}$ implies

$$(\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{t}=\partial_{n_1}(\boldsymbol{v}\cdot\boldsymbol{n}_2)-\partial_{n_2}(\boldsymbol{v}\cdot\boldsymbol{n}_1)=0,\tag{59}$$

$$(\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n}_1=\partial_{n_2}(\boldsymbol{v}\cdot\boldsymbol{t})-\partial_{t}(\boldsymbol{v}\cdot\boldsymbol{n}_2)=0,\tag{60}$$

$$(\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n}_2=\partial_{t}(\boldsymbol{v}\cdot\boldsymbol{n}_1)-\partial_{n_1}(\boldsymbol{v}\cdot\boldsymbol{t})=0.\tag{61}$$

Then it follows from the vanishing DoFs (55d)-(55e) that (57)-(58) vanish.

Similarly for face $f\in\Delta_2(T)$ with frame $\{\boldsymbol{n},\boldsymbol{t}_1,\boldsymbol{t}_2\}$, we have

$$\begin{aligned}
\operatorname{curl}\boldsymbol{v}&=\operatorname{curl}((\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}+(\boldsymbol{v}\cdot\boldsymbol{t}_1)\boldsymbol{t}_1+(\boldsymbol{v}\cdot\boldsymbol{t}_2)\boldsymbol{t}_2)\\
&=-(\boldsymbol{n}\times\nabla)(\boldsymbol{v}\cdot\boldsymbol{n})-(\boldsymbol{t}_1\times\nabla)(\boldsymbol{v}\cdot\boldsymbol{t}_1)-(\boldsymbol{t}_2\times\nabla)(\boldsymbol{v}\cdot\boldsymbol{t}_2),
\end{aligned}$$

which combined with $\operatorname{curl}\boldsymbol{v}=\boldsymbol{0}$ implies

$$\begin{aligned}
(\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n}&=\partial_{t_1}(\boldsymbol{v}\cdot\boldsymbol{t}_2)-\partial_{t_2}(\boldsymbol{v}\cdot\boldsymbol{t}_1)=0,\\
(\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{t}_1&=\partial_{t_2}(\boldsymbol{v}\cdot\boldsymbol{n})-\partial_{n}(\boldsymbol{v}\cdot\boldsymbol{t}_2)=0,\\
(\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{t}_2&=\partial_{n}(\boldsymbol{v}\cdot\boldsymbol{t}_1)-\partial_{t_1}(\boldsymbol{v}\cdot\boldsymbol{n})=0.
\end{aligned}$$

Then it follows from the vanishing DoFs (55i)-(55j) that

$$(\partial_n^j\boldsymbol{v})|_f=\boldsymbol{0}\quad\text{for }\ 0\le j\le r_1^f.$$

Finally $\boldsymbol{v}=\boldsymbol{0}$ holds from the vanishing DoF (55n). □

Define the global $H(\mathrm{curl})$-conforming finite element space

$$\mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3) : \boldsymbol{v}|_T \in \mathbb{P}_{k+1}^3(T) \text{ for each } T \in \mathcal{T}_h, \text{ and all the DoFs (55a)-(55l) are single-valued}\}.$$

By the proof of Lemma 5.5, we have

$$\mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) \subseteq \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1), \quad \mathrm{curl}\, \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) \subseteq \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3).$$

We illustrate $\mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) \subset \boldsymbol{H}(\mathrm{curl}, \Omega)$. First consider $r_2^e \geq 0$. For $\boldsymbol{v} \in \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2)$, DoFs (55a)-(55b) and (55f)-(55h) determine $(\nabla^j(\mathrm{curl}\, \boldsymbol{v}))|_e$ for edge e and $j = 0, \ldots, r_2^e$. Due to identities (59)-(61), DoFs (55a) and (55c)-(55e) then determine $(\boldsymbol{v} \cdot \boldsymbol{t})|_e$ and $(\nabla^j \boldsymbol{v})|_e$ for $j = 0, \ldots, r_1^e$. Finally, $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega)$ follows from DoFs (55i) and (55k).

When $r_2^e = -1$, we have $r_1^e \in \{-1, 0\}$. For $\boldsymbol{v} \in \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2)$, DoFs (55a) and (55c)-(55e) determine $(\boldsymbol{v} \cdot \boldsymbol{t})|_e$ for $r_1^e = -1$ and $\boldsymbol{v}|_e$ for $r_1^e = 0$. Then $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, \Omega)$ follows from DoFs (55i) and (55k).

5.4. **Finite element de Rham and Stokes complexes with decay smoothness.** We consider the decay smoothness sequence with lower bound -1

$$\boldsymbol{r}_0, \quad \boldsymbol{r}_1 = \boldsymbol{r}_0 \ominus 1, \quad \boldsymbol{r}_2 = \boldsymbol{r}_1 \ominus 1, \quad \boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1.$$

We first consider the case $r_0^f = m \geq 2$ so that $r_2^f \geq 0$. Then $\boldsymbol{r}_i = \boldsymbol{r}_{i-1} - 1 \geq 0$ for $i = 1, 2$. The polynomial degree starts from $k \geq 8m + 1 \geq 17$. In the result below, we explicitly present constraints on $\boldsymbol{r}_2$ to emphasize the space $\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2)$ is continuous and in $H^1(\Omega)$.

Lemma 5.6. *Let* $r_2^f \geq 0, r_2^e \geq 2\, r_2^f + 2, r_2^{\mathrm{v}} \geq 2\, r_2^e + 2, k \geq 2r_2^{\mathrm{v}} + 3$, *and let* $\boldsymbol{r}_0 = \boldsymbol{r}_1 + 1, \boldsymbol{r}_1 = \boldsymbol{r}_2 + 1, \boldsymbol{r}_3 = \boldsymbol{r}_2 - 1$. *Write*

$$\dim \mathbb{V}_{k+2}^{\mathrm{grad}}(\mathcal{T}_h; \boldsymbol{r}_0) = C_{00}|\Delta_0(\mathcal{T}_h)| + C_{01}|\Delta_1(\mathcal{T}_h)| + C_{02}|\Delta_2(\mathcal{T}_h)| + C_{03}|\Delta_3(\mathcal{T}_h)|,$$
$$\dim \mathbb{V}_{k+1}^{\mathrm{curl}}(\mathcal{T}_h; \boldsymbol{r}_1) = C_{10}|\Delta_0(\mathcal{T}_h)| + C_{11}|\Delta_1(\mathcal{T}_h)| + C_{12}|\Delta_2(\mathcal{T}_h)| + C_{13}|\Delta_3(\mathcal{T}_h)|,$$
$$\dim \mathbb{V}_{k}^{\mathrm{div}}(\mathcal{T}_h; \boldsymbol{r}_2) = C_{20}|\Delta_0(\mathcal{T}_h)| + C_{21}|\Delta_1(\mathcal{T}_h)| + C_{22}|\Delta_2(\mathcal{T}_h)| + C_{23}|\Delta_3(\mathcal{T}_h)|,$$
$$\dim \mathbb{V}_{k-1}^{L^2}(\mathcal{T}_h; \boldsymbol{r}_3) = C_{30}|\Delta_0(\mathcal{T}_h)| + C_{31}|\Delta_1(\mathcal{T}_h)| + C_{32}|\Delta_2(\mathcal{T}_h)| + C_{33}|\Delta_3(\mathcal{T}_h)|.$$

Then

$$C_{ij} = \binom{3}{i} C_j(k + 2 - i, \boldsymbol{r}_i), \quad i, j = 0, 1, 2, 3,$$

where $C_i(k, \boldsymbol{r})$ *is defined in Lemma* 3.11, *satisfy the alternating sum identity*

$$C_{0i} - C_{1i} + C_{2i} - C_{3i} = (-1)^i, \quad i = 0, 1, 2, 3.$$

Proof. For the column of $|\Delta_0(\mathcal{T}_h)|$, by Lemma 3.11 and (47) with $k = r_2^{\mathrm{v}}$,

$$C_{00} - C_{10} + C_{20} - C_{30} = \binom{r_2^{\mathrm{v}} + 5}{3} - 3\binom{r_2^{\mathrm{v}} + 4}{3} + 3\binom{r_2^{\mathrm{v}} + 3}{3} - \binom{r_2^{\mathrm{v}} + 2}{3} = 1.$$

For the column of $|\Delta_1(\mathcal{T}_h)|$, by Lemma 3.11 and (47) with $k = r_2^e - 1$,

$$\begin{aligned} &C_{01} - C_{11} + C_{21} - C_{31} \\ &= (k + r_2^e - 2r_2^{\mathrm{v}} - 1)\left[\binom{r_2^e + 4}{2} - 3\binom{r_2^e + 3}{2} + 3\binom{r_2^e + 2}{2} - \binom{r_2^e + 1}{2}\right] \\ &\quad - \binom{r_2^e + 4}{3} + 3\binom{r_2^e + 3}{3} - 3\binom{r_2^e + 2}{3} + \binom{r_2^e + 1}{3} = -1. \end{aligned}$$

For the column of $|\Delta_2(\mathcal{T}_h)|$, by Lemma 3.11 and (47),

$$\begin{aligned}
& C_{02} - C_{12} + C_{22} - C_{32} \\
&= \binom{k+5}{3} - 3\binom{k+4}{3} + 3\binom{k+3}{3} - \binom{k+2}{3} \\
&\quad - 3\left[\binom{r^{\mathrm{v}}+5}{3} - 3\binom{r^{\mathrm{v}}+4}{3} + 3\binom{r^{\mathrm{v}}+3}{3} - \binom{r^{\mathrm{v}}+2}{3}\right] \\
&\quad - 3\left[\binom{k-2r^{\mathrm{v}}-3}{3} - 3\binom{k-2r^{\mathrm{v}}-2}{3} + 3\binom{k-2r^{\mathrm{v}}-1}{3} - \binom{k-2r^{\mathrm{v}}}{3}\right] \\
&= 1 - 3 + 3 = 1.
\end{aligned}$$

For the column of $|\Delta_3(\mathcal{T}_h)|$, applying (47) again,

$$\begin{aligned}
& C_{03} - C_{13} + C_{23} - C_{33} \\
&= \binom{k+5}{3} - 3\binom{k+4}{3} + 3\binom{k+3}{3} - \binom{k+2}{3} - 4(C_{00} - C_{10} + C_{20} - C_{30}) \\
&\quad - 6(C_{01} - C_{11} + C_{21} - C_{31}) - 4(C_{02} - C_{12} + C_{22} - C_{32}) \\
&= 1 - 4 + 6 - 4 = -1.
\end{aligned}$$

This ends the proof. ∎

We summarize the coefficients C_{ij} in Table 2.

TABLE 2. Dimensions of finite element spaces

	$\mathbb{V}^{\mathrm{grad}}_{k+2}(\boldsymbol{r}_0)$	$\mathbb{V}^{\mathrm{curl}}_{k+1}(\boldsymbol{r}_1)$	$\mathbb{V}^{\mathrm{div}}_{k}(\boldsymbol{r}_2)$	$\mathbb{V}^{L^2}_{k-1}(\boldsymbol{r}_3)$	$\sum_{i=0}^{3}(-1)^i C_{ij}$
$\lvert\Delta_0(\mathcal{T}_h)\rvert$	$C_0(k+2,\boldsymbol{r}_0)$	$3C_0(k+1,\boldsymbol{r}_1)$	$3C_0(k,\boldsymbol{r}_2)$	$C_0(k-1,\boldsymbol{r}_3)$	1
$\lvert\Delta_1(\mathcal{T}_h)\rvert$	$C_1(k+2,\boldsymbol{r}_0)$	$3C_1(k+1,\boldsymbol{r}_1)$	$3C_1(k,\boldsymbol{r}_2)$	$C_1(k-1,\boldsymbol{r}_3)$	-1
$\lvert\Delta_2(\mathcal{T}_h)\rvert$	$C_2(k+2,\boldsymbol{r}_0)$	$3C_2(k+1,\boldsymbol{r}_1)$	$3C_2(k,\boldsymbol{r}_2)$	$C_2(k-1,\boldsymbol{r}_3)$	1
$\lvert\Delta_3(\mathcal{T}_h)\rvert$	$C_3(k+2,\boldsymbol{r}_0)$	$3C_3(k+1,\boldsymbol{r}_1)$	$3C_3(k,\boldsymbol{r}_2)$	$C_3(k-1,\boldsymbol{r}_3)$	-1

We first consider the case $r_2^f \geq 0$ so that $\mathbb{V}^{\mathrm{div}}_k(\boldsymbol{r}_2) \subset \boldsymbol{H}^1(\Omega;\mathbb{R}^3)$ and present the following finite element Stokes complex.

Theorem 5.7. *Let $r_2^f \geq 0, r_2^e \geq 2\,r_2^f + 2, r_2^{\mathrm{v}} \geq 2\,r_2^e + 2, k \geq 2r_2^{\mathrm{v}} + 3$, and let $\boldsymbol{r}_0 = \boldsymbol{r}_1 + 1, \boldsymbol{r}_1 = \boldsymbol{r}_2 + 1, \boldsymbol{r}_3 = \boldsymbol{r}_2 - 1$. The finite element Stokes complex*

$$\mathbb{R} \xrightarrow{\subset} \mathbb{V}^{\mathrm{grad}}_{k+2}(\boldsymbol{r}_0) \xrightarrow{\mathrm{grad}} \mathbb{V}^{\mathrm{curl}}_{k+1}(\boldsymbol{r}_1) \xrightarrow{\mathrm{curl}} \mathbb{V}^{\mathrm{div}}_{k}(\boldsymbol{r}_2) \xrightarrow{\mathrm{div}} \mathbb{V}^{L^2}_{k-1}(\boldsymbol{r}_3) \to 0 \tag{62}$$

is exact.

Proof. By construction (62) is a complex, and

$$\mathrm{grad}\,\mathbb{V}^{\mathrm{grad}}_{k+2}(\boldsymbol{r}_0) = \mathbb{V}^{\mathrm{curl}}_{k+1}(\boldsymbol{r}_1) \cap \ker(\mathrm{curl}).$$

Thanks to (31), $\operatorname{div} \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) = \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$. By Lemma 5.6 and the Euler's formula,

$$\begin{aligned}&1 - \dim \mathbb{V}_{k+2}^{\mathrm{grad}}(\boldsymbol{r}_0) + \dim \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1) - \dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) + \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)\\ &= 1 - |\Delta_0(\mathcal{T}_h)| + |\Delta_1(\mathcal{T}_h)| - |\Delta_2(\mathcal{T}_h)| + |\Delta_3(\mathcal{T}_h)| = 0.\end{aligned}$$

Therefore the exactness of complex (62) follows from Lemma 5.1. ☐

We then consider the case $r_2^f = -1$ and thus $\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) \subset \boldsymbol{H}(\mathrm{div}, \Omega)$ only.

Theorem 5.8. *Let $\boldsymbol{r}_2$ satisfy* (30) *with $r_2^f = -1$, and let $\boldsymbol{r}_0 = \boldsymbol{r}_1 + 1 \geq 0, \boldsymbol{r}_2 = \boldsymbol{r}_1 \ominus 1, \boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$ be smoothness vectors. Assume $k \geq \max\{2r_2^{\mathrm{v}}+1, r_2^{\mathrm{v}}+2, 3(r_2^e+1)\}$. The finite element de Rham complex*

$$\mathbb{R} \xrightarrow{\subset} \mathbb{V}_{k+2}^{\mathrm{grad}}(\boldsymbol{r}_0) \xrightarrow{\mathrm{grad}} \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1) \xrightarrow{\mathrm{curl}} \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) \xrightarrow{\mathrm{div}} \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) \to 0 \tag{63}$$

is exact.

Proof. By construction (63) is a complex, and

$$\operatorname{grad} \mathbb{V}_{k+2}^{\mathrm{grad}}(\boldsymbol{r}_0) = \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1) \cap \ker(\mathrm{curl}).$$

Thanks to (31), $\operatorname{div} \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) = \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$. Then we count the dimensions. By comparing DoFs of $\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2)$ and $\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$,

$$\begin{aligned}&\dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) - \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)\\ &= |\Delta_0(\mathcal{T}_h)| \left(3\binom{r_2^{\mathrm{v}}+3}{3} - \binom{r_2^{\mathrm{v}}+2}{3}\right) + |\Delta_1(\mathcal{T}_h)| \sum_{j=0}^{r_2^e} (2j+3)(k - 2r_2^{\mathrm{v}} - 1 + j)\\ &\quad + |\Delta_2(\mathcal{T}_h)| \dim \mathbb{B}_k(f; \begin{pmatrix} r_2^{\mathrm{v}} \\ r_2^e \end{pmatrix}) + |\Delta_3(\mathcal{T}_h)| (\dim \mathbb{B}_k^{\mathrm{div}}(\boldsymbol{r}_2) \cap \ker(\mathrm{div}) - 1).\end{aligned}$$

Then it follows from DoFs (55a)-(55n) of space $\mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1)$ that

$$\begin{aligned}&\dim \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1) - \dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) + \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)\\ &= |\Delta_0(\mathcal{T}_h)| \left(3\binom{r_1^{\mathrm{v}}+3}{3} - 3\binom{r_1^{\mathrm{v}}+2}{3} + \binom{r_1^{\mathrm{v}}+1}{3}\right)\\ &\quad + |\Delta_1(\mathcal{T}_h)| (k - 2r_1^{\mathrm{v}}) + |\Delta_1(\mathcal{T}_h)| \sum_{j=0}^{r_1^e} (j+2)(k - 2r_1^{\mathrm{v}} + j)\\ &\quad + |\Delta_2(\mathcal{T}_h)| \dim \mathbb{B}_{k+2}(f; \begin{pmatrix} r_0^{\mathrm{v}} \\ r_0^e \end{pmatrix}) + \chi(r_1^f = 0) |\Delta_2(\mathcal{T}_h)| \dim \mathbb{B}_{k+1}(f; \begin{pmatrix} r_1^{\mathrm{v}} \\ r_1^e \end{pmatrix})\\ &\quad - |\Delta_2(\mathcal{T}_h)| + |\Delta_3(\mathcal{T}_h)| (\dim \mathbb{B}_{k+2}(\boldsymbol{r}_0) + 1).\end{aligned}$$

As a result, by the Euler's formula,

$$\begin{aligned}&1 - \dim \mathbb{V}_{k+2}^{\mathrm{grad}}(\boldsymbol{r}_0) + \dim \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1) - \dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) + \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)\\ &= 1 - |\Delta_0(\mathcal{T}_h)| + |\Delta_1(\mathcal{T}_h)| - |\Delta_2(\mathcal{T}_h)| + |\Delta_3(\mathcal{T}_h)| = 0.\end{aligned}$$

Therefore the exactness of complex (63) follows from Lemma 5.1. ☐

5.5. **Finite element de Rham and Stokes complex with inequality constraint.** We consider the most general case with inequality constraint on smoothness vectors.

Theorem 5.9. *Let* $\boldsymbol{r}_0 \geq 0, \boldsymbol{r}_1 = \boldsymbol{r}_0 - 1, \boldsymbol{r}_2 \geq \boldsymbol{r}_1 \ominus 1, \boldsymbol{r}_3 \geq \boldsymbol{r}_2 \ominus 1$ *be smoothness vectors. Assume* $(\boldsymbol{r}_2, \boldsymbol{r}_3, k)$ *is div stable. Assume* $k \geq \max\{2r_1^{\mathrm{v}} + 1, 2r_2^{\mathrm{v}} + 1, r_2^{\mathrm{v}} + 2, 3(r_2^e + 1), 2r_3^{\mathrm{v}} + 2, 4r_3^f + 5, (r_3^e + r_3^f + 5)[r_3^{\mathrm{v}} = 0]\}$. *Then the finite element complex*

$$\mathbb{R} \xrightarrow{\subset} \mathbb{V}_{k+2}^{\mathrm{grad}}(\boldsymbol{r}_0) \xrightarrow{\mathrm{grad}} \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) \xrightarrow{\mathrm{curl}} \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) \xrightarrow{\mathrm{div}} \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) \to 0 \tag{64}$$

is exact.

Proof. By construction (64) is a complex, and

$$\mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) \cap \ker(\mathrm{curl}) = \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1) \cap \ker(\mathrm{curl}) = \mathrm{grad}\, \mathbb{V}_{k+2}^{\mathrm{grad}}(\boldsymbol{r}_0).$$

Thanks to (43), $\mathrm{div}\, \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) = \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$. By Lemma 5.1, it suffices to prove

$$\dim \mathbb{V}_{k+2}^{\mathrm{grad}}(\boldsymbol{r}_0) - \dim \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) + \dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) - \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) = 1. \tag{65}$$

Through comparing DoFs, we find that $\dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) - \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$ is constant with respect to $\boldsymbol{r}_3$, which means

$$\dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) - \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) = \dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) - \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_2 \ominus 1).$$

Similarly, since $\dim \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) - \dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) + \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_2 \ominus 1)$ is constant with respect to $\boldsymbol{r}_2$, we have

$$\begin{aligned} &\dim \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) - \dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2) + \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_2 \ominus 1) \\ &= \dim \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1) - \dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_1 \ominus 1) + \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_1 \ominus 2). \end{aligned}$$

Combining the last two identities yields

$$\begin{aligned} &- \dim \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1, \boldsymbol{r}_2) + \dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2, \boldsymbol{r}_3) - \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) \\ &= - \dim \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1) + \dim \mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_1 \ominus 1) - \dim \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_1 \ominus 2). \end{aligned}$$

Therefore (65) follows from Theorem 5.7 and Theorem 5.8. ∎

Example 5.10. Taking $\boldsymbol{r}_0 = (1, 0, 0)$, $\boldsymbol{r}_1 = \boldsymbol{r}_0 - 1$, $\boldsymbol{r}_2 = \boldsymbol{r}_1 \ominus 1$, $\boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$ and $k \geq 1$, we obtain the Hermite family finite element de Rham complex in [21]

$$\mathbb{R} \xrightarrow{\subset} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\mathrm{grad}} \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \xrightarrow{\mathrm{curl}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \xrightarrow{\mathrm{div}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \to 0.$$

Taking $\boldsymbol{r}_0 = (2, 1, 0)$, $\boldsymbol{r}_1 = \boldsymbol{r}_0 - 1$, $\boldsymbol{r}_2 = \boldsymbol{r}_1 \ominus 1$, $\boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$ and $k \geq 3$, we obtain the Argyris family finite element de Rham complex in [21]

$$\mathbb{R} \xrightarrow{\subset} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\mathrm{grad}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \xrightarrow{\mathrm{curl}} \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \xrightarrow{\mathrm{div}} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \to 0.$$

Example 5.11. Taking $\boldsymbol{r}_0 = (4, 2, 1)$, $\boldsymbol{r}_1 = \boldsymbol{r}_0 - 1$, $\boldsymbol{r}_2 = \boldsymbol{r}_1 - 1$, $\boldsymbol{r}_3 = \boldsymbol{r}_2 \ominus 1$ and $k \geq 7$, the finite element de Rham complex

$$\mathbb{R} \xrightarrow{\subset} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \xrightarrow{\mathrm{grad}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\mathrm{curl}} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \xrightarrow{\mathrm{div}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \to 0$$

can be used to discretize the decoupled formulation of the biharmonic equation in three dimensions in [14, Section 3.2].

Example 5.12. Taking $\boldsymbol{r}_0 = (4, 2, 1)$, $\boldsymbol{r}_2 = (2, 1, 0)$, $\boldsymbol{r}_1 = \boldsymbol{r}_0 - 1$, $\boldsymbol{r}_3 = \boldsymbol{r}_2 - 1$ and $k \geq 7$, we obtain the Stokes complex in [41]

$$\mathbb{R} \xrightarrow{\subset} \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \xrightarrow{\text{grad}} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{curl}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{div}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \to 0.$$

Remark 5.13. Based on smooth scalar finite elements in arbitrary dimension in Appendix A, it is doable to construct finite element de Rham complexes with various smoothness in arbitrary dimension. However, it is difficult to prove the exactness of the resulting finite element complexes. Recent work [27] on split meshes might be helpful.

5.6. **Commutative diagram.** To construct a commutative diagram for finite element complex (64), we adjust DoFs (17a)-(17d) of $\mathbb{V}^{\text{grad}}_{k+2}(\boldsymbol{r}_0)$ in consideration of DoFs (55a)-(55n) of $\mathbb{V}^{\text{curl}}_{k+1}(\boldsymbol{r}_1, \boldsymbol{r}_2)$. We present new DoFs for $\mathbb{V}^{\text{grad}}_{k+2}(\boldsymbol{r}_0)$ as follows:

$$\nabla^j u(\texttt{v}), \quad j = 0, \ldots, r_0^{\texttt{v}}, \tag{66a}$$

$$\int_e \partial_t u\, q \,\mathrm{d}s, \quad q \in \mathbb{P}_{k-2r_0^{\texttt{v}}+1}(e)/\mathbb{R}, \tag{66b}$$

$$\int_e \frac{\partial^j u}{\partial n_1^i \partial n_2^{j-i}}\, q \,\mathrm{d}s, \quad q \in \mathbb{P}_{k-2r_0^{\texttt{v}}+j}(e), 0 \leq i \leq j, 1 \leq j \leq r_0^e, \tag{66c}$$

$$\int_f (\operatorname{grad}_f u) \cdot \boldsymbol{q} \,\mathrm{d}S, \quad \boldsymbol{q} \in \operatorname{grad}_f \mathbb{B}_{k+2}(f; \begin{pmatrix} r_0^{\texttt{v}} \\ r_0^e \end{pmatrix}), \tag{66d}$$

$$\int_f \partial_n^j u\, q \,\mathrm{d}S, \quad q \in \mathbb{B}_{k+2-j}(f; \begin{pmatrix} r_0^{\texttt{v}} \\ r_0^e \end{pmatrix} - j), 1 \leq j \leq r_0^f, \tag{66e}$$

$$\int_T (\operatorname{grad} u) \cdot \boldsymbol{q} \,\mathrm{d}x, \quad \boldsymbol{q} \in \operatorname{grad} \mathbb{B}_{k+2}(\boldsymbol{r}_0) \tag{66f}$$

for each $\texttt{v} \in \Delta_0(T)$, $e \in \Delta_1(T)$ and $f \in \Delta_2(T)$.

Lemma 5.14. *The DoFs* (66a)-(66f) *are unisolvent for* $\mathbb{P}_{k+2}(T)$.

Proof. By comparing DoFs (17a)-(17d) and DoFs (66a)-(66f), the number of DoFs (66a)-(66f) equals $\dim \mathbb{P}_{k+2}(T)$.

Assume $u \in \mathbb{P}_{k+2}(T)$ and all the DoFs (66a)-(66f) vanish. The vanishing DoFs (66a)-(66b) imply $\int_e u\, q \,\mathrm{d}s = 0$ for $q \in \mathbb{P}_{k-2r_0^{\texttt{v}}}(e)$. Thanks to Theorem 3.6, we get from the vanishing DoFs (66a) and (66c) that $\nabla^j u|_e = 0$ for $0 \leq j \leq r_0^e$ and $e \in \Delta_1(T)$. Then $u|_f \in \mathbb{B}_{k+2}(f; \begin{pmatrix} r_0^{\texttt{v}} \\ r_0^e \end{pmatrix})$ for $f \in \Delta_2(T)$, which combined with (66d) yields $u|_f = 0$. Applying Theorem 3.6 again, it follows from the vanishing DoF (66e) that $\partial_n^j u|_f = 0$ for $0 \leq j \leq r_0^f$, i.e. $u \in \mathbb{B}_{k+2}(\boldsymbol{r}_0)$. Thus $u = 0$ holds from the vanishing DoF (66f). ☐

Define $I_h^{\text{grad}}, I_h^{\text{curl}}, I_h^{\text{div}}$, and $I_h^{L^2}$ as the canonical interpolation operators using the DoFs.

Corollary 5.15. *With the same setting as in Theorem* 5.9, *the following diagram is commutative.*

$$\begin{array}{ccccccccccc}
\mathbb{R} & \xrightarrow{\subset} & \mathcal{C}^\infty(\Omega) & \xrightarrow{\mathrm{grad}} & \mathcal{C}^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{\mathrm{curl}} & \mathcal{C}^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{\mathrm{div}} & \mathcal{C}^\infty(\Omega) & \longrightarrow & 0 \\
 & & \big\downarrow I_h^{\mathrm{grad}} & & \big\downarrow I_h^{\mathrm{curl}} & & \big\downarrow I_h^{\mathrm{div}} & & \big\downarrow I_h^{L^2} & & \\
\mathbb{R} & \xrightarrow{\subset} & \mathbb{V}_{k+2}^{\mathrm{grad}}(\boldsymbol{r}_0) & \xrightarrow{\mathrm{grad}} & \mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1,\boldsymbol{r}_2) & \xrightarrow{\mathrm{curl}} & \mathbb{V}_{k}^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3) & \xrightarrow{\mathrm{div}} & \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3) & \longrightarrow & 0.
\end{array}$$

Proof.

Step 1. We first prove

$$\operatorname{div}(I_h^{\mathrm{div}}\boldsymbol{v}) = I_h^{L^2}(\operatorname{div}\boldsymbol{v}) \quad \forall\ \boldsymbol{v}\in\mathcal{C}^\infty(\Omega;\mathbb{R}^3). \tag{67}$$

By comparing DoFs (17a)-(17d) for $\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$, and DoFs (42a)-(42b), (42f) and (42i)-(42j) for $\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)$, it suffices to prove

$$\int_T \operatorname{div}(I_h^{\mathrm{div}}\boldsymbol{v})\,\mathrm{d}x = \int_T \operatorname{div}\boldsymbol{v}\,\mathrm{d}x \quad \forall\ T\in\mathcal{T}_h,$$

which is an immediate result of DoF (42g).

Step 2. Next we prove

$$\operatorname{curl}(I_h^{\mathrm{curl}}\boldsymbol{v}) = I_h^{\mathrm{div}}(\operatorname{curl}\boldsymbol{v}) \quad \forall\ \boldsymbol{v}\in\mathcal{C}^\infty(\Omega;\mathbb{R}^3). \tag{68}$$

By (67), we have $\operatorname{div}(\operatorname{curl}(I_h^{\mathrm{curl}}\boldsymbol{v})) = \operatorname{div}(I_h^{\mathrm{div}}(\operatorname{curl}\boldsymbol{v})) = 0$. By comparing DoFs (42a), (42c)-(42e), (42g)-(42h) and (42k) for $\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)$, and DoFs (55a)-(55b), (55f)-(55h) and (55k)-(55m) for $\mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1,\boldsymbol{r}_2)$, it suffices to prove

$$\int_f \operatorname{curl}(I_h^{\mathrm{curl}}\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}S = \int_f (\operatorname{curl}\boldsymbol{v})\cdot\boldsymbol{n}\,\mathrm{d}S \quad \forall\ f\in\Delta_2(\mathcal{T}_h),$$

which is an immediate result of DoF (55c).

Step 3. Finally we prove

$$\operatorname{grad}(I_h^{\mathrm{grad}}u) = I_h^{\mathrm{curl}}(\operatorname{grad}u) \quad \forall\ u\in\mathcal{C}^\infty(\Omega). \tag{69}$$

By (68), we have $\operatorname{curl}(\operatorname{grad}(I_h^{\mathrm{grad}}u)) = \operatorname{curl}(I_h^{\mathrm{curl}}(\operatorname{grad}u)) = \mathbf{0}$. By comparing DoFs (55a), (55c)-(55e), (55i)-(55j) and (55n) for $\mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1,\boldsymbol{r}_2)$, and DoFs (66a)-(66f) for $\mathbb{V}_{k+2}^{\mathrm{grad}}(\boldsymbol{r}_0)$, it suffices to prove

$$\int_e \partial_t(I_h^{\mathrm{grad}}u)\,\mathrm{d}s = \int_e \partial_t u\,\mathrm{d}s \quad \forall\ e\in\Delta_1(\mathcal{T}_h),$$

which is an immediate result of DoF (66a).

Combining (67)-(69) will end the proof. □

5.7. **The first kind finite elements.** Firstly, we construct the first type $H(\mathrm{div})$-conforming finite elements for $\boldsymbol{r}_2$ satisfying (30) and $\boldsymbol{r}_3 \geq \boldsymbol{r}_2 \ominus 1$. Take $\mathbb{P}_{k,-}^{\mathrm{div}}(T;\mathbb{R}^3) := \mathbb{P}_{k-1}^3(T) + \boldsymbol{x}\mathbb{H}_{k-1}(T)$ as the space of shape functions, where $\mathbb{H}_{k-1}(T) := \mathbb{P}_{k-1}(T)\backslash\mathbb{P}_{k-2}(T)$ is the homogenous polynomial space of degree $k-1$. Using the fact $\operatorname{div}(\boldsymbol{x}q) = (k+3)q$ for all $q\in\mathbb{H}_{k-1}(T)$, we know $\operatorname{div}\mathbb{P}_{k,-}^{\mathrm{div}}(T;\mathbb{R}^3) = \mathbb{P}_{k-1}(T)$. So, among DoFs (42) for $\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)$, we can keep DoFs for $\operatorname{div}\boldsymbol{v}$. As $\boldsymbol{x}\cdot\boldsymbol{n}|_f$ is constant, the normal trace on faces remains in $\mathbb{P}_{k-1}(f)$. Also as the added subspace only contributes to the range of div operator,

$$\mathbb{P}^{\rm div}_{k,-}(T;\mathbb{R}^3)\cap\ker(\operatorname{div})=\mathbb{P}^3_{k-1}(T)\cap\ker(\operatorname{div}).$$

We use DoFs (42) for $\mathbb{V}^{\rm div}_{k-1}(T;\boldsymbol{r}_2,\boldsymbol{r}_3)$, denoted by $\mathrm{DoF}^{\rm div}_{k-1}(T;\boldsymbol{r}_2,\boldsymbol{r}_3)$, but increase the DoFs for $\operatorname{div}\boldsymbol{v}$ from $k-2$ to $k-1$. Based on these observations, we propose the following degrees of freedom:

$$\nabla^i\boldsymbol{v}(\mathrm{v}),\quad i=0,\ldots,r_2^{\mathrm{v}}, \tag{70a}$$

$$\nabla^j\operatorname{div}\boldsymbol{v}(\mathrm{v}),\quad j=\max\{r_2^{\mathrm{v}},0\},\ldots,r_3^{\mathrm{v}}, \tag{70b}$$

$$\int_e\partial^j_{n_1}(\boldsymbol{v}\cdot\boldsymbol{n}_2)\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-1-2(r_2^{\mathrm{v}}+1)+j}(e),0\leq j\leq r_2^e, \tag{70c}$$

$$\int_e\frac{\partial^j(\boldsymbol{v}\cdot\boldsymbol{t})}{\partial n_1^i\partial n_2^{j-i}}\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-1-2(r_2^{\mathrm{v}}+1)+j}(e),0\leq i\leq j\leq r_2^e, \tag{70d}$$

$$\int_e\frac{\partial^j(\boldsymbol{v}\cdot\boldsymbol{n}_1)}{\partial n_1^i\partial n_2^{j-i}}\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-1-2(r_2^{\mathrm{v}}+1)+j}(e),0\leq i\leq j\leq r_2^e, \tag{70e}$$

$$\int_e\frac{\partial^j(\operatorname{div}\boldsymbol{v})}{\partial n_1^i\partial n_2^{j-i}}\,q\,\mathrm{d}s,\quad q\in\mathbb{P}_{k-1-2(r_3^{\mathrm{v}}+1)+j}(e),0\leq i\leq j\leq r_3^e, \tag{70f}$$

$$\int_f\boldsymbol{v}\cdot\boldsymbol{n}\,q\,\mathrm{d}S,\quad q\in\mathbb{P}_0(f)\oplus(\mathbb{B}_{k-1}(f;\begin{pmatrix}r_2^{\mathrm{v}}\\ r_2^e\end{pmatrix})/\mathbb{R}), \tag{70g}$$

$$\int_f\partial^j_n(\boldsymbol{v}\cdot\boldsymbol{t}_\ell)\;q\,\mathrm{d}S,\quad q\in\mathbb{B}_{k-1-j}(f;\begin{pmatrix}r_2^{\mathrm{v}}\\ r_2^e\end{pmatrix}-j),0\leq j\leq r_2^f,\ell=1,2, \tag{70h}$$

$$\int_f\partial^j_n(\operatorname{div}\boldsymbol{v})\;q\,\mathrm{d}S,\quad q\in\mathbb{B}_{k-1-j}(f;\begin{pmatrix}r_3^{\mathrm{v}}\\ r_3^e\end{pmatrix}-j),0\leq j\leq r_3^f, \tag{70i}$$

$$\int_T\operatorname{div}\boldsymbol{v}\,q\,\mathrm{d}x,\quad q\in\mathbb{B}_{k-1}(\boldsymbol{r}_3)/\mathbb{R}, \tag{70j}$$

$$\int_T\boldsymbol{v}\cdot\boldsymbol{q}\,\mathrm{d}x,\quad \boldsymbol{q}\in\mathbb{B}^{\rm div}_{k-1}(\boldsymbol{r}_2)\cap\ker(\operatorname{div}) \tag{70k}$$

for each $\mathrm{v}\in\Delta_0(T)$, $e\in\Delta_1(T)$ and $f\in\Delta_2(T)$.

Lemma 5.16. *Let $\boldsymbol{r}_3\geq\boldsymbol{r}_2\ominus 1$ be two smoothness vectors. Assume $k\geq\max\{2r_2^{\mathrm{v}}+2,3r_2^e+4,2r_3^{\mathrm{v}}+2,4r_3^f+5,3[r_2^{\mathrm{v}}=0],(r_3^e+r_3^f+5)[r_3^{\mathrm{v}}=0]\}$. The DoFs (70) are unisolvent for $\mathbb{P}^{\rm div}_{k,-}(T;\mathbb{R}^3)=\mathbb{P}^3_{k-1}(T)+\boldsymbol{x}\mathbb{H}_{k-1}(T)$.*

Proof. When $k=1$, we have $\boldsymbol{r}_2=\boldsymbol{r}_3=-1$, and DoFs (70) are reduced to (70g), which is exactly DoFs of the lowest order Raviart-Thomas element in [39,46]. Then we consider $k\geq 2$.

The number of DoFs (70b), (70f), (70i), and (70j) to determine $\operatorname{div}\boldsymbol{v}\in\mathbb{V}^{L^2}_{k-1}(\boldsymbol{r}_3)$ is $\dim\mathbb{P}_{k-1}(T)-1-4\binom{r_2^{\mathrm{v}}+2}{3}$. While in $\mathrm{DoF}^{\rm div}_{k-1}(T;\boldsymbol{r}_2,\boldsymbol{r}_3)$, the part to determine $\operatorname{div}\boldsymbol{v}\in\mathbb{V}^{L^2}_{k-2}(\boldsymbol{r}_3)$ is $\dim\mathbb{P}_{k-2}(T)-1-4\binom{r_2^{\mathrm{v}}+2}{3}$. So the number of DoFs added is $\dim\mathbb{P}_{k-1}(T)-\dim\mathbb{P}_{k-2}(T)=\dim\mathbb{H}_{k-1}(T)$ which matches the increase of the dimension of spaces $\dim\mathbb{P}^{\rm div}_{k,-}(T;\mathbb{R}^3)-\dim\mathbb{P}^3_{k-1}(T)$.

Take $\boldsymbol{v}\in\mathbb{P}^{\rm div}_{k,-}(T;\mathbb{R}^3)$ and assume all the DoFs (70) vanish. By the proof of Lemma 4.9, the vanishing DoFs (70a)-(70j) imply $\operatorname{div}\boldsymbol{v}=0$. Then $\boldsymbol{v}\in\mathbb{P}^3_{k-1}(T)$. Finally apply Lemma 4.9 to conclude $\boldsymbol{v}=\boldsymbol{0}$. □

Define the global $H(\mathrm{div})$-conforming finite element space

$$\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3) = \{\boldsymbol{v}\in \boldsymbol{L}^2(\Omega;\mathbb{R}^3) : \boldsymbol{v}|_T \in \mathbb{P}_{k-1}^3(T) + \boldsymbol{x}\mathbb{H}_{k-1}(T) \\ \text{for each } T\in\mathcal{T}_h, \text{ and all the DoFs (70) are single-valued}\}.$$

As $\mathrm{tr}^{\mathrm{div}}\,\mathbb{P}_{k,-}^{\mathrm{div}}(T;\mathbb{R}^3) = \mathrm{tr}^{\mathrm{div}}\,\mathbb{P}_{k-1}^3(T)$, the normal component $\boldsymbol{v}\cdot\boldsymbol{n}|_f$ on each face is still continuous. So $\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3)\subset \boldsymbol{H}(\mathrm{div},\Omega)$. The tangential component is one degree higher and the current DoFs cannot ensure the required continuity. Namely even $r_2^f\geq 0$ $\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3)\not\subseteq \boldsymbol{H}^1(\Omega;\mathbb{R}^3)$. We use superscript $^-$ in $\boldsymbol{r}_2^-$ to denote this deficiency when $\boldsymbol{r}_2\geq 0$. Nevertheless for $k\geq 2$ we have

$$\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3)\cap\ker(\mathrm{div}) = \mathbb{V}_{k-1}^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)\cap\ker(\mathrm{div})\subset \boldsymbol{C}^{r_2^f}(\Omega;\mathbb{R}^3)\cap\boldsymbol{H}^{r_2^f+1}(\Omega;\mathbb{R}^3),$$

and thus

$$\mathrm{curl}\,\mathbb{V}_k^{\mathrm{curl}}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3)\cap\ker(\mathrm{div}). \tag{71}$$

By construction $\mathrm{div}\,\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3)\subseteq \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$. Since

$$\begin{aligned}\dim\mathrm{div}\,\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3) &= \dim\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3) - \dim\left(\mathbb{V}_{k-1}^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)\cap\ker(\mathrm{div})\right)\\ &= \dim\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3) - \dim\mathbb{V}_{k-1}^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3) + \dim\mathbb{V}_{k-2}^{L^2}(\boldsymbol{r}_3)\\ &= \dim\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3),\end{aligned}$$

it holds

$$\mathrm{div}\,\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3) = \mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3). \tag{72}$$

Next we construct the first type $H(\mathrm{curl})$-conforming finite elements for $\boldsymbol{r}_2\geq \boldsymbol{r}_1\ominus 1\geq -1$. Take $\mathbb{P}_{k+1,-}^{\mathrm{curl}}(T;\mathbb{R}^3) := \mathbb{P}_k^3(T)\oplus \boldsymbol{x}\times\mathbb{H}_k^3(T) = \mathbb{P}_{k+1}^3(T)\backslash\,\mathrm{grad}\,\mathbb{H}_{k+2}(T)$ as the space of shape functions. Locally $\mathrm{curl}\,\mathbb{P}_{k+1,-}^{\mathrm{curl}}(T;\mathbb{R}^3) = \ker(\mathrm{div})\cap\mathbb{P}_k^3(T)$ by the polynomial de Rham complex. So we take (55) for $\mathrm{DoF}_{k+1}^{\mathrm{curl}}(T;\boldsymbol{r}_2,\boldsymbol{r}_3)$ but decrease the degree of polynomial for DoFs of $\boldsymbol{v}$. The degrees of freedom are

$$\nabla^i \boldsymbol{v}(\mathrm{v}), \quad i = 0, \ldots, r_1^{\mathrm{v}}, \tag{73a}$$

$$\nabla^j (\operatorname{curl} \boldsymbol{v})(\mathrm{v}), \quad j = \max\{r_1^{\mathrm{v}}, 0\}, \ldots, r_2^{\mathrm{v}}, \tag{73b}$$

$$\int_e \boldsymbol{v} \cdot \boldsymbol{t}\, q \,\mathrm{d}s, \quad q \in \mathbb{P}_{k-2-2r_1^{\mathrm{v}}}(e), \tag{73c}$$

$$\int_e \frac{\partial^j (\boldsymbol{v} \cdot \boldsymbol{n}_1)}{\partial n_1^i \partial n_2^{j-i}}\, q \,\mathrm{d}s, \quad q \in \mathbb{P}_{k-2-2r_1^{\mathrm{v}}+j}(e), 0 \leq i \leq j \leq r_1^e, \tag{73d}$$

$$\int_e \partial_{n_2}^j (\boldsymbol{v} \cdot \boldsymbol{n}_2)\, q \,\mathrm{d}s, \quad q \in \mathbb{P}_{k-2-2r_1^{\mathrm{v}}+j}(e), 0 \leq j \leq r_1^e, \tag{73e}$$

$$\int_e \partial_{n_1}^j ((\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n}_2)\, q \,\mathrm{d}s, \quad q \in \mathbb{P}_{k-2(r_2^{\mathrm{v}}+1)+j}(e), 0 \leq j \leq r_2^e, \tag{73f}$$

$$\int_e \frac{\partial^j ((\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{t})}{\partial n_1^i \partial n_2^{j-i}}\, q \,\mathrm{d}s, \quad q \in \mathbb{P}_{k-2(r_2^{\mathrm{v}}+1)+j}(e), 0 \leq i \leq j \leq r_2^e, \tag{73g}$$

$$\int_e \frac{\partial^j ((\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n}_1)}{\partial n_1^i \partial n_2^{j-i}}\, q \,\mathrm{d}s, \quad q \in \mathbb{P}_{k-2(r_2^{\mathrm{v}}+1)+j}(e), 0 \leq i \leq j \leq r_2^e, \tag{73h}$$

$$\int_f (\Pi_f \boldsymbol{v}) \cdot \boldsymbol{q} \,\mathrm{d}S, \quad \boldsymbol{q} \in \operatorname{grad}_f \mathbb{B}_{k+1}(f; \begin{pmatrix} r_1^{\mathrm{v}} \\ r_1^e \end{pmatrix} + 1), \tag{73i}$$

$$\int_f \partial_n^j (\boldsymbol{v} \cdot \boldsymbol{n})\, q \,\mathrm{d}S, \quad q \in \mathbb{B}_{k-j}(f; \begin{pmatrix} r_1^{\mathrm{v}} \\ r_1^e \end{pmatrix} - j), 0 \leq j \leq r_1^f, \tag{73j}$$

$$\int_f (\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{n}\, q \,\mathrm{d}S, \quad q \in \mathbb{B}_k(f; \begin{pmatrix} r_2^{\mathrm{v}} \\ r_2^e \end{pmatrix})/\mathbb{R}, \tag{73k}$$

$$\int_f \partial_n^j ((\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{t}_\ell)\; q \,\mathrm{d}S, \quad q \in \mathbb{B}_{k-j}(f; \begin{pmatrix} r_2^{\mathrm{v}} \\ r_2^e \end{pmatrix} - j), 0 \leq j \leq r_2^f, \ell = 1, 2, \tag{73l}$$

$$\int_T (\operatorname{curl} \boldsymbol{v}) \cdot \boldsymbol{q} \,\mathrm{d}x, \quad \boldsymbol{q} \in \mathbb{B}_k^{\operatorname{div}}(\boldsymbol{r}_2) \cap \ker(\operatorname{div}), \tag{73m}$$

$$\int_T \boldsymbol{v} \cdot \boldsymbol{q} \,\mathrm{d}x, \quad \boldsymbol{q} \in \operatorname{grad} \mathbb{B}_{k+1}(\boldsymbol{r}_1 + 1) \tag{73n}$$

for each $\mathrm{v} \in \Delta_0(T)$, $e \in \Delta_1(T)$ and $f \in \Delta_2(T)$.

Lemma 5.17. *Assume k is a large enough integer satisfying $k \geq 0$ for $\boldsymbol{r}_1 = \boldsymbol{r}_2 = -1$, and $k \geq \max\{2r_1^{\mathrm{v}} + 2, 2r_2^{\mathrm{v}} + 1, r_2^{\mathrm{v}} + 2, 3(r_2^e + 1)\}$ for other cases. The DoFs* (73) *are unisolvent for $\mathbb{P}_{k+1,-}^{\operatorname{curl}}(T; \mathbb{R}^3)$.*

Proof. When $k = 0$, we have $\boldsymbol{r}_1 = \boldsymbol{r}_2 = -1$, and DoFs (73) are reduced to (73c), which are exactly DoFs of the lowest order Nédélec element in [39]. Then we consider $k \geq 1$.

Thanks to (56) and the polynomial de Rham complex (46), the number of DoFs (73b), (73f)-(73h) and (73k)-(73m) is

$$3 \dim \mathbb{P}_{k+1}(T) - \dim \mathbb{P}_{k+2}(T) - 2 - 12\binom{r_1^{\mathrm{v}} + 2}{3} + 4\binom{r_1^{\mathrm{v}} + 1}{3}.$$

Hence the number of DoFs (73) is

$$\begin{aligned}&3\dim\mathbb{P}_k(T)+(3\dim\mathbb{P}_{k+1}(T)-\dim\mathbb{P}_{k+2}(T))-(3\dim\mathbb{P}_k(T)-\dim\mathbb{P}_{k+1}(T))\\&=\dim\mathbb{P}^3_{k+1}(T)-(\dim\mathbb{P}_{k+2}(T)-\dim\mathbb{P}_{k+1}(T)),\end{aligned}$$

which matches the dimension of $\mathbb{P}^3_{k+1}(T)\backslash\operatorname{grad}\mathbb{H}_{k+2}(T)$.

Take $\boldsymbol{v}\in\mathbb{P}^{\rm curl}_{k+1,-}(T;\mathbb{R}^3)$ and assume all DoFs (73) vanish. By the proof of Lemma 5.5, the vanishing DoFs (73a)-(73c), (73f)-(73h) and (73k)-(73m) imply $\operatorname{curl}\boldsymbol{v}=\boldsymbol{0}$. Then $\boldsymbol{v}\in\mathbb{P}^3_k(T)$. Finally apply Lemma 5.5 to conclude $\boldsymbol{v}=\boldsymbol{0}$. ☐

Define the global $H(\mathrm{curl})$-conforming finite element space

$$\begin{aligned}\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)=\{&\boldsymbol{v}\in\boldsymbol{L}^2(\Omega;\mathbb{R}^3):\boldsymbol{v}|_T\in\mathbb{P}^3_k(T)\oplus\boldsymbol{x}\times\mathbb{H}^3_k(T)\\&\text{for each } T\in\mathcal{T}_h,\ \text{and all the DoFs (73) are single-valued}\}.\end{aligned}$$

Apparently, $\mathbb{P}^{\rm curl}_{k+1,-}(T;\mathbb{R}^3)\cdot\boldsymbol{t}=\mathbb{P}^3_k(T)\cdot\boldsymbol{t}$ on each edge. Introduce the trace operator of curl as $\operatorname{tr}^{\rm curl}_f\boldsymbol{v}=\boldsymbol{n}\times\boldsymbol{v}|_f$ on face f. Since

$$\operatorname{tr}^{\rm curl}_f(\boldsymbol{x}\times\boldsymbol{q})=(\boldsymbol{q}\cdot\boldsymbol{n})\boldsymbol{x}-(\boldsymbol{x}\cdot\boldsymbol{n})\boldsymbol{q}=(\boldsymbol{q}\cdot\boldsymbol{n})\Pi_f\boldsymbol{x}-(\boldsymbol{x}\cdot\boldsymbol{n})\Pi_f\boldsymbol{q}\in\mathbb{P}_k(f;\mathbb{R}^2)+\mathbb{P}_k(f)\Pi_f\boldsymbol{x}$$

for $\boldsymbol{q}\in\mathbb{P}^3_k(T)$, we have $\operatorname{tr}^{\rm curl}_f\mathbb{P}^{\rm curl}_{k+1,-}(T;\mathbb{R}^3)=\mathbb{P}_k(f;\mathbb{R}^2)+\mathbb{P}_k(f)\Pi_f\boldsymbol{x}$. Then the single-valued DoFs (73a), (73c), (73i) and (73k) ensure that the tangential component $\Pi_f\boldsymbol{v}$ on each face is still continuous, which means $\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)\subset\boldsymbol{H}(\mathrm{curl},\Omega)$. The normal component is one degree higher and the current DoFs cannot ensure the required continuity. Namely $\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)\not\subseteq\boldsymbol{H}^1(\Omega;\mathbb{R}^3)$ even when $r_1^f\geq 0$. By $\mathbb{P}^{\rm curl}_{k+1,-}(T;\mathbb{R}^3)\cap\ker(\mathrm{curl})=\operatorname{grad}\mathbb{P}_{k+1}(T)$, for $\boldsymbol{v}\in\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)\cap\ker(\mathrm{curl})$, it follows $\boldsymbol{v}|_T\in\operatorname{grad}\mathbb{P}_{k+1}(T)=\mathbb{P}_k(T)\cap\ker(\mathrm{curl})$ for each $T\in\mathcal{T}_h$. Then for $k\geq 1$ we have

$$\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)\cap\ker(\mathrm{curl})=\mathbb{V}^{\mathrm{curl}}_{k}(\boldsymbol{r}_1,\boldsymbol{r}_2)\cap\ker(\mathrm{curl})\subset\boldsymbol{C}^{r_1^f}(\Omega;\mathbb{R}^3),$$

and thus

$$\operatorname{grad}\mathbb{V}^{\mathrm{grad}}_{k+1}(\boldsymbol{r}_1+1)=\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)\cap\ker(\mathrm{curl}).\tag{74}$$

Clearly it holds

$$\operatorname{curl}\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)\subseteq\mathbb{V}^{\mathrm{div}}_{k}(\boldsymbol{r}_2,\boldsymbol{r}_2\ominus 1)\cap\ker(\mathrm{div})=\operatorname{curl}\mathbb{V}^{\mathrm{curl}}_{k+1}(\boldsymbol{r}_1,\boldsymbol{r}_2).$$

Since

$$\begin{aligned}\dim\operatorname{curl}\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)&=\dim\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)-\dim\left(\mathbb{V}^{\mathrm{curl}}_{k}(\boldsymbol{r}_1,\boldsymbol{r}_2)\cap\ker(\mathrm{curl})\right)\\&=\dim\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)-\dim\mathbb{V}^{\mathrm{curl}}_{k}(\boldsymbol{r}_1,\boldsymbol{r}_2)\\&\quad+\dim\operatorname{curl}\mathbb{V}^{\mathrm{curl}}_{k}(\boldsymbol{r}_1,\boldsymbol{r}_2)\\&=\dim\operatorname{curl}\mathbb{V}^{\mathrm{curl}}_{k+1}(\boldsymbol{r}_1,\boldsymbol{r}_2),\end{aligned}$$

it holds

$$\operatorname{curl}\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)=\mathbb{V}^{\mathrm{div}}_{k}(\boldsymbol{r}_2,\boldsymbol{r}_3)\cap\ker(\mathrm{div})\tag{75}$$

with $\boldsymbol{r}_3\geq\boldsymbol{r}_2\ominus 1$.

Example 5.18. Space $\mathbb{V}^{\mathrm{curl},-}_{k+1}(\boldsymbol{-1},\begin{pmatrix}2\\1\\0\end{pmatrix})$ for $k\geq 6$ is the one constructed in [52].

With spaces $\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3)$ and $\mathbb{V}_{k+1}^{\mathrm{curl},-}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)$ for $\boldsymbol{r}_2\geq\boldsymbol{r}_1\ominus 1$ and $\boldsymbol{r}_3\geq\boldsymbol{r}_2\ominus 1$, employing (71)-(72) and (74)-(75), we can construct more finite element de Rham complexes involving these spaces:

$$\mathbb{R}\xrightarrow{\subset}\mathbb{V}_{k+1}^{\mathrm{grad}}(\boldsymbol{r}_1+1)\xrightarrow{\mathrm{grad}}\mathbb{V}_{k+1}^{\mathrm{curl},-}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)\xrightarrow{\mathrm{curl}}\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)\xrightarrow{\mathrm{div}}\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)\to 0,$$

$$\mathbb{R}\xrightarrow{\subset}\mathbb{V}_{k+1}^{\mathrm{grad}}(\boldsymbol{r}_1+1)\xrightarrow{\mathrm{grad}}\mathbb{V}_k^{\mathrm{curl}}(\boldsymbol{r}_1,\boldsymbol{r}_2)\xrightarrow{\mathrm{curl}}\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3)\xrightarrow{\mathrm{div}}\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)\to 0,$$

$$\mathbb{R}\xrightarrow{\subset}\mathbb{V}_k^{\mathrm{grad}}(\boldsymbol{r}_1+1)\xrightarrow{\mathrm{grad}}\mathbb{V}_k^{\mathrm{curl},-}(\boldsymbol{r}_1^-,\boldsymbol{r}_2)\xrightarrow{\mathrm{curl}}\mathbb{V}_k^{\mathrm{div},-}(\boldsymbol{r}_2^-,\boldsymbol{r}_3)\xrightarrow{\mathrm{div}}\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)\to 0.$$

6. Conclusion and future work

We have constructed the finite element de Rham complex

$$\mathbb{R}\xrightarrow{\subset}\mathbb{V}_{k+2}^{\mathrm{grad}}(\boldsymbol{r}_0)\xrightarrow{\mathrm{grad}}\mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1,\boldsymbol{r}_2)\xrightarrow{\mathrm{curl}}\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)\xrightarrow{\mathrm{div}}\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)\to 0, \tag{76}$$

with various smoothness at vertices, edges, and faces. Comparing with 2D results in [17,32], the non-trivial parts are the div stability $\operatorname{div}\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)=\mathbb{V}_{k-1}^{L^2}(\boldsymbol{r}_3)$ and DoFs for face elements $\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)$ and edge elements $\mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1,\boldsymbol{r}_2)$. In 2D [17,32], the div stability can be proved using the dimension count and the edge element is simply a rotation of a face element.

The developed tools (geometric decomposition of simplicial lattice, barycentric calculus, and the $t-n$ decomposition) and the approach to construct finite element spaces $\mathbb{V}_{k+1}^{\mathrm{curl}}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ and $\mathbb{V}_k^{\mathrm{div}}(\boldsymbol{r}_2,\boldsymbol{r}_3)$ will shed light on the unified construction of finite element Hessian, elasticity, and divdiv complexes via the Bernstein-Gelfand-Gelfand (BGG) framework developed by Arnold and Hu [6]. Finite element Hessian complexes, elasticity complexes, and divdiv complexes have been constructed recently case by case in [15,18–20,29–31,33]. Our goal is to extend the BGG construction to finite element complexes and thus unify these scattered results and produce more in a systematical way. In our recent work [17], we have achieved this goal in two dimensions but extension to three dimensions is non-trivial. We will report our finding in a forthcoming paper [13].

Appendix A. Smooth finite elements in arbitrary dimension

In a recent work [32], Hu, Lin and Wu have constructed a C^m-conforming finite element on simplexes in arbitrary dimension. It unifies the scattered results [1,9,50] in two dimensions, [38,51,53] in three dimensions, and [55] in four dimensions. In this appendix, we use the simplicial lattice to give a geometric decomposition of the finite element spaces constructed in [32] and consequently give a different construction of HLW element. The smoothness at subsimplexes is exponentially increasing as the dimension decreases

$$r_n=0,\;\; r_{n-1}=m,\;\; r_\ell\geq 2r_{\ell+1}\;\text{ for }\ell=n-2,\dots,0.$$

And the degree of polynomial $k\geq 2r_0+1\geq 2^n m+1$. The key in [32] is a non-overlapping decomposition of the simplicial lattice in which each component will be used to determine the normal derivatives on lower sub-simplexes.

Our approach is closely related to the multivariate splines on triangulations [23, 37]. For example, construction of C^m element in $n=2,3$, but not arbitrary $n\geq 2$, can be also found in the book [37, Section 8.1 for 2D and Section 18.11 for 3D]. The major difference between HLW element and the multivariate splines, which is also the art of designing finite elements, is the choice of DoFs. In the multivariate splines, DoFs are chosen as function values or derivatives at some points, as the major question studied there is the interpolation of data, while the integral form on subsimplexes proposed in [32] enables us to prove the unisolvence easily and has the advantage for constructing finite element de Rham complexes as we have done in three dimensions.

A.1. **Important relation.** The first important relation is: for $\alpha \in \mathbb{T}^n_k, \beta \in \mathbb{N}^{1:n}$, we have

$$D^\beta \lambda^\alpha|_f = 0, \quad \text{ if } \operatorname{dist}(\alpha, f) > |\beta|.$$

Namely the polynomial λ^α vanishes on f to order $\operatorname{dist}(\alpha, f)$. See Lemma 2.4.

The second one is the one-to-one mapping of the space $\operatorname{span}\{\lambda^\alpha = \lambda_f^{\alpha_f}\lambda_{f^*}^{\alpha_{f^*}}, \alpha \in L(f,s), i.e., \alpha \in \mathbb{T}^n_k, |\alpha_{f^*}| = s\}$ to the following DoFs, by changing α_{f^*} to β,

$$\int_f \frac{\partial^\beta u}{\partial n_f^\beta}\, \lambda_f^{\alpha_f}\, \mathrm{d}s \quad \forall\ \alpha_f \in \mathbb{T}^\ell_{k-s}(f), \beta \in \mathbb{N}^{1:n-\ell}, |\beta| = s.$$

See Lemma A.4 for a proof of this statement.

A.2. **Decomposition of the simplicial lattice.** We explain the requirement $r_{\ell-1} \geq 2r_\ell$.

Lemma A.1. *Let T be an n-dimensional simplex. For $\ell = 1, \ldots, n-1$, if $r_{\ell-1} \geq 2r_\ell$, the subsets $\{D(f, r_\ell)\backslash \left[\cup_{e\in\Delta_{\ell-1}(f)} D(e, r_{\ell-1})\right], f \in \Delta_\ell(T)\}$ are disjoint.*

Proof. Consider two different subsimplices $f, \tilde{f} \in \Delta_\ell(T)$. The dimension of their intersection is at most $\ell-1$. Therefore $f \cap \tilde{f} \subseteq e$ for some $e \in \Delta_{\ell-1}(f)$. Then $e^* \subseteq (f\cap\tilde{f})^* = f^* \cup \tilde{f}^*$. For $\alpha \in D(f, r_\ell)\cap D(\tilde{f}, r_\ell)$, we have $|\alpha_{e^*}| \leq |\alpha_{f^*}| + |\alpha_{\tilde{f}^*}| \leq 2r_\ell \leq r_{\ell-1}$. Therefore we have shown the intersection region $D(f, r_\ell) \cap D(\tilde{f}, r_\ell) \subseteq \cup_{e\in\Delta_{\ell-1}(f)} D(e, r_{\ell-1})$ and the result follows. ☐

Next we remove $D(e, r_i)$ from $D(f, r_\ell)$ for all $e \in \Delta_i(T)$ and $i = 0, 1, \ldots, \ell-1$.

Lemma A.2. *Given integer $m \geq 0$, let non-negative integer array $\boldsymbol{r} = (r_0, r_1, \ldots, r_n)$ satisfy*

$$r_n = 0, \;\; r_{n-1} = m, \;\; r_\ell \geq 2r_{\ell+1} \;\; \text{for } \ell = n-2, \ldots, 0.$$

Let $k \geq 2r_0 + 1 \geq 2^n m + 1$. For $\ell = 1, \ldots, n-1$,

$$D(f, r_\ell)\backslash \left[\bigcup_{i=0}^{\ell-1} \bigcup_{e\in\Delta_i(f)} D(e, r_i)\right] = D(f, r_\ell)\backslash \left[\bigcup_{i=0}^{\ell-1} \bigcup_{e\in\Delta_i(T)} D(e, r_i)\right]. \tag{77}$$

Proof. In (12), the relation $\supseteq$ is obvious as $\Delta_i(f) \subseteq \Delta_i(T)$.

To prove $\subseteq$, it suffices to show for $\alpha \in D(f, r_\ell)\backslash \left[\bigcup_{i=0}^{\ell-1} \bigcup_{e\in\Delta_i(f)} D(e, r_i)\right]$, it is not in $D(e, r_i)$ for $e \in \Delta_i(T)$ and $e \notin \Delta_i(f)$.

By definition,

$$|\alpha_{f^*}| \leq r_\ell, \; |\alpha_e| \leq k - r_i - 1 \;\text{ for all } e \in \Delta_i(f), i = 0, \ldots, \ell-1.$$

For each $e \in \Delta_i(T)$ but $e \notin \Delta_i(f)$, the dimension of the intersection $e \cap f$ is at most $i-1$. It follows from $r_j \geq 2r_{j+1}$ and $k \geq 2r_0 + 1$ that: when $i > 0$,

$$|\alpha_e| = |\alpha_{e\cap f}| + |\alpha_{e\cap f^*}| \leq k - r_{i-1} - 1 + r_\ell \leq k - r_i - 1,$$

and when $i = 0$,

$$|\alpha_e| = |\alpha_{e\cap f^*}| \leq r_\ell \leq k - r_i - 1.$$

So $|\alpha_{e^*}| > r_i$. We conclude that $\alpha \notin D(e, r_i)$ for all $e \in \Delta_i(T)$ and (12) follows. $\square$

We are in the position to present a geometric decomposition of the simplicial lattice and polynomial spaces. Again it is a reinterpretation of that in [32] using the distance function introduced in Section 3.

Theorem A.3. *Given integer $m \geq 0$, let non-negative integer array $\boldsymbol{r} = (r_0, r_1, \ldots, r_n)$ satisfy*

$$r_n = 0,\ \ r_{n-1} = m,\ \ r_\ell \geq 2r_{\ell+1}\ \ \text{for } \ell = n-2, \ldots, 0.$$

Let $k \geq 2r_0 + 1 \geq 2^n m + 1$. Then we have the following direct decomposition of the simplicial lattice on an n-dimensional simplex T:

$$\mathbb{T}_k^n(T) = \oplus_{\ell=0}^n \oplus_{f \in \Delta_\ell(T)} S_\ell(f),$$

where

$$\begin{aligned} S_0(\mathtt{v}) &= D(\mathtt{v}, r_0), \\ S_\ell(f) &= D(f, r_\ell) \backslash \left[\bigcup_{i=0}^{\ell-1} \bigcup_{e \in \Delta_i(f)} D(e, r_i) \right],\ \ell = 1, \ldots, n-1, \\ S_n(T) &= \mathbb{T}_k^n(T) \backslash \left[\bigcup_{i=0}^{n-1} \bigcup_{f \in \Delta_i(T)} D(f, r_i) \right]. \end{aligned}$$

Consequently we have the following geometric decomposition of $\mathbb{P}_k(T)$

$$\mathbb{P}_k(T) = \oplus_{\ell=0}^n \oplus_{f \in \Delta_\ell(T)} \mathbb{P}_k(S_\ell(f)). \tag{78}$$

Proof. First we show that the sets $\{S_\ell(f), f \in \Delta_\ell(T), \ell = 0, \ldots, n\}$ are disjoint. Take two vertices $\mathtt{v}_1, \mathtt{v}_2 \in \Delta_0(T)$. For $\alpha \in D(\mathtt{v}_1, r_0)$, we have $\alpha_{\mathtt{v}_1} \geq k - r_0$. As $\mathtt{v}_1 \subseteq \mathtt{v}_2^*$ and $k \geq 2r_0 + 1$, $|\alpha_{\mathtt{v}_2^*}| \geq \alpha_{\mathtt{v}_1} \geq k - r_0 \geq r_0 + 1$, i.e., $\alpha \notin D(\mathtt{v}_2, r_0)$. Hence $\{S_0(\mathtt{v}), \mathtt{v} \in \Delta_0(T)\}$ are disjoint and $\bigoplus_{\mathtt{v} \in \Delta_0(T)} S_0(\mathtt{v})$ is a disjoint union. By Lemma A.1 and (77), we know $\{S_\ell(f), f \in \Delta_\ell(T), \ell = 0, \ldots, n\}$ are disjoint.

Next we inductively prove

$$\oplus_{i=0}^\ell \oplus_{f \in \Delta_i(T)} S_i(f) = \bigcup_{i=0}^{\ell} \bigcup_{f \in \Delta_i(T)} D(f, r_i) \quad \text{ for } \ \ell = 0, \ldots, n-1.$$

Obviously (15) holds for $\ell = 0$. Assume (15) holds for $\ell < j$. Then

$$\begin{aligned} \oplus_{i=0}^j \oplus_{f \in \Delta_i(T)} S_i(f) &= \oplus_{f \in \Delta_j(T)} S_j(f) \ \oplus \bigcup_{i=0}^{j-1} \bigcup_{e \in \Delta_i(T)} D(e, r_i) \\ &= \oplus_{f \in \Delta_j(T)} \left(D(f, r_j) \backslash \left[\bigcup_{i=0}^{j-1} \bigcup_{e \in \Delta_i(T)} D(e, r_i) \right] \right) \oplus \bigcup_{i=0}^{j-1} \bigcup_{e \in \Delta_i(T)} D(e, r_i) \\ &= \bigcup_{i=0}^{j} \bigcup_{f \in \Delta_i(T)} D(f, r_i). \end{aligned}$$

By induction, (15) holds for $\ell = 0, \ldots, n-1$. Then (13) is true from the definition of $S_n(T)$ and (15). $\square$

We can write out the inequality constraints in $S_\ell(f)$. For $\ell = 1, \ldots, n$,

$$S_\ell(f) = \{\alpha \in \mathbb{T}_k^n : |\alpha_{f^*}| \leq r_\ell, |\alpha_e| \leq k - r_i - 1, \forall e \in \Delta_i(f), i = 0, \ldots, \ell - 1\}. \tag{79}$$

For $\alpha \in S_\ell(f)$, by Lemma 3.1 we also have $\alpha \notin D(\tilde{f}, r_\ell)$ for $\tilde{f} \in \Delta_\ell(T)\backslash\{f\}$, i.e.

$$|\alpha_{\tilde{f}}| \leq k - r_\ell - 1 \quad \forall\, \tilde{f} \in \Delta_\ell(T)\backslash\{f\}. \tag{80}$$

From the implementation point of view, the index set $S_\ell(f)$ can be found by a logic array and set the entry as true when the distance constraint holds.

A.3. **Decomposition of degree of freedoms.** Recall that $L(f, s) = \{\alpha \in \mathbb{T}_k^n, \operatorname{dist}(\alpha, f) = s\}$ consists of lattice nodes s away from f.

Lemma A.4. *Let $\ell = 0, \ldots, n-1$ and $s \leq r_\ell$ be a non-negative integer. Given $f \in \Delta_\ell(T)$, let $\{\boldsymbol{n}_f^1, \boldsymbol{n}_f^2, \ldots, \boldsymbol{n}_f^{n-\ell}\}$ be $n - \ell$ vectors spanning the normal plane of f. The polynomial space $\mathbb{P}_k(S_\ell(f) \cap L(f, s))$ is uniquely determined by DoFs*

$$\int_f \frac{\partial^\beta u}{\partial n_f^\beta}\, \lambda_f^{\alpha_f}\, \mathrm{d}s \quad \forall\, \alpha \in S_\ell(f), |\alpha_f| = k - s, \beta \in \mathbb{N}^{1:n-\ell}, |\beta| = s. \tag{81}$$

Proof. A basis of $\mathbb{P}_k(S_\ell(f) \cap L(f, s))$ is $\{\lambda^\alpha = \lambda_f^{\alpha_f} \lambda_{f^*}^{\alpha_{f^*}}, \alpha \in S_\ell(f), |\alpha_{f^*}| = s\}$ and thus the dimensions match (by mapping α_{f^*} to β).

We choose a basis of the normal plane $\{\boldsymbol{n}_f^1, \boldsymbol{n}_f^2, \ldots, \boldsymbol{n}_f^{n-\ell}\}$ s.t. it is dual to the vectors $\{\nabla\lambda_{f^*(1)}, \nabla\lambda_{f^*(2)}, \ldots,\}$, i.e., $\nabla\lambda_{f^*(i)} \cdot \boldsymbol{n}_f^j = \delta_{i,j}$ for $i, j = 1, \ldots, n - \ell$. Then we have the duality

$$\frac{\partial^\beta}{\partial n_f^\beta}(\lambda_{f^*}^{\alpha_{f^*}}) = \beta!\delta(\alpha_{f^*}, \beta), \quad \alpha_{f^*}, \beta \in \mathbb{N}^{1:n-\ell}, |\alpha_{f^*}| = |\beta| = s, \tag{82}$$

which can be proved easily by induction on s. When T is the reference simplex $\hat{T}$, $\lambda_i = x_i$ and $\nabla\lambda_i = -\boldsymbol{e}_i$, (82) is the calculus result $D_{n_f}^\beta \boldsymbol{x}_{f^*}^{\alpha_{f^*}} = \beta!\delta(\alpha_{f^*}, \beta)$.

Assume $u = \sum c_{\alpha_f, \alpha_{f^*}} \lambda_f^{\alpha_f} \lambda_{f^*}^{\alpha_{f^*}} \in \mathbb{P}_k(S_\ell(f) \cap L(f, s))$. If the derivative is not fully applied to the component $\lambda_{f^*}^{\alpha_{f^*}}$, then there is a term $\lambda_{f^*}^\gamma$ with $|\gamma| > 0$ left and $\lambda_i^\gamma|_f = 0$ for $i \in f^*$. So for any $\beta \in \mathbb{N}^{1:n-\ell}$ and $|\beta| = s$,

$$\frac{\partial^\beta u}{\partial n_f^\beta}|_f = \beta! \sum_{\alpha \in S_\ell(f), |\alpha_f| = k - s} c_{\alpha_f, \beta} \lambda_f^{\alpha_f}.$$

The vanishing DoF (81) implies $\sum_{\alpha \in S_\ell(f), |\alpha_f| = k-s} c_{\alpha_f, \beta} \lambda_f^{\alpha_f}|_f = 0$. Hence $c_{\alpha_f, \beta} = 0$ for all $|\alpha_f| = k - s, \alpha \in S_\ell(f)$. As β is arbitrary, we conclude all coefficients $c_{\alpha_f, \alpha_{f^*}} = 0$ and thus $u = 0$. ☐

For $u \in \mathbb{P}_k(S_\ell(f) \cap L(f, s))$ and $\beta \in \mathbb{N}^{1:n-\ell}$ with $|\beta| < s$, by Lemma 2.4, $\frac{\partial^\beta u}{\partial n_f^\beta}|_f = 0$. Applying the operator $\frac{\partial^\beta (\cdot)}{\partial n_f^\beta}|_f$ to the direct decomposition $\mathbb{P}_k(S_\ell(f)) = \bigoplus_{s=0}^{r_\ell} \mathbb{P}_k(S_\ell(f) \cap L(f, s))$ will possess a block lower triangular structure and leads to the following unisolvence result. We refer to [17] for a clear illustration in 2D.

Lemma A.5. *Let $\ell = 0, \ldots, n-1$. The polynomial space $\mathbb{P}_k(S_\ell(f))$ is uniquely determined by DoFs*

$$\int_f \frac{\partial^\beta u}{\partial n_f^\beta}\, \lambda_f^{\alpha_f}\, \mathrm{d}s \quad \forall\, \alpha \in S_\ell(f), |\alpha_f| = k - s, \beta \in \mathbb{N}^{1:n-\ell}, |\beta| = s, s = 0, \ldots, r_\ell.$$

Together with decomposition (78) of the polynomial space, we obtain the following result.

Theorem A.6 (Theorem 1.1 in [32]). *Given integer $m \geq 0$, let non-negative integer array $\boldsymbol{r} = (r_0, r_1, \ldots, r_n)$ satisfy*

$$r_n = 0, \;\; r_{n-1} = m, \;\; r_\ell \geq 2r_{\ell+1} \;\; \textit{for } \ell = n-2, \ldots, 0.$$

Let $k \geq 2r_0 + 1 \geq 2^n m + 1$. Then the shape function $\mathbb{P}_k(T)$ is uniquely determined by the following DoFs

$$D^\alpha u(\mathtt{v}) \quad \alpha \in \mathbb{N}^{1:n}, |\alpha| \leq r_0, \mathtt{v} \in \Delta_0(T), \tag{83}$$

$$\int_f \frac{\partial^\beta u}{\partial n_f^\beta} \lambda_f^{\alpha_f} \,\mathrm{d}s \quad \alpha \in S_\ell(f), |\alpha_f| = k - s, \beta \in \mathbb{N}^{1:n-\ell}, |\beta| = s, \tag{84}$$
$$f \in \Delta_\ell(T), \ell = 1, \ldots, n-1, s = 0, \ldots, r_\ell,$$

$$\int_T u\lambda^\alpha \,\mathrm{d}x \quad \alpha \in S_n(T). \tag{85}$$

Proof. Thanks to the decomposition (78), the dimensions match. Take $u \in \mathbb{P}_k(T)$ satisfy all the DoFs (83)-(85) vanish. We are going to show $u = 0$.

For $\alpha \in S_\ell(f)$ and $e \in \Delta_i(T)$ with $i \leq \ell$ and $e \neq f$, by (79) and (80) we have $|\alpha_{e^*}| \geq r_i + 1$, hence $\frac{\partial^\beta \lambda^\alpha}{\partial n_e^\beta}|_e = 0$ for $\beta \in \mathbb{N}^{1:n-i}$ with $|\beta| \leq r_i$. Again this tells us that applying the operator $\frac{\partial^\beta (\cdot)}{\partial n_f^\beta}|_f$ to the direct decomposition $\mathbb{P}_k(T) = \bigoplus_{\ell=0}^n \bigoplus_{f \in \Delta_\ell(T)} \mathbb{P}_k(S_\ell(f))$ will produce a block lower triangular structure. Then apply Lemma A.5, we conclude $u \in \mathbb{P}_k(S_n(T))$, which together with the vanishing DoF (85) gives $u = 0$. $\square$

Remark A.7. For $\alpha \in S_\ell(f)$, by (79) we have $|\alpha_e| \leq k - r_{\ell-1} - 1$ for all $e \in \Delta_{\ell-1}(f)$, then $\alpha_f \geq r_{\ell-1} + 1 - |\alpha_{f^*}|$, and

$$\lambda^\alpha = \lambda_{f^*}^{\alpha_{f^*}} \lambda_f^{\alpha_f} = \lambda_{f^*}^{\alpha_{f^*}} \lambda_f^{r_{\ell-1}+1-|\alpha_{f^*}|} \lambda_f^{\alpha_f - (r_{\ell-1}+1) + |\alpha_{f^*}|}.$$

Using $\alpha_f - (r_{\ell-1}+1) + |\alpha_{f^*}|$ as the new index, DoFs (84)-(85) can be replaced by

$$\int_f \frac{\partial^\beta u}{\partial n_f^\beta} \lambda_f^\alpha \,\mathrm{d}s \quad \beta \in \mathbb{N}^{1:n-\ell}, |\beta| = s, s = 0, \ldots, r_\ell, \;\; \alpha \in \mathbb{T}^\ell_{k-(\ell+1)(r_{\ell-1}+1)+\ell s},$$
$$|\alpha_e| \leq k - r_i - 1 - (i+1)(r_{\ell-1}+1-s), \forall e \in \Delta_i(f), i = 0, \ldots, \ell-2,$$
$$f \in \Delta_\ell(T), \ell = 1, \ldots, n-1,$$
$$\int_T u\lambda^\alpha \,\mathrm{d}x \quad \alpha \in \mathbb{T}^n_{k-(n+1)(m+1)},$$
$$|\alpha_e| \leq k - r_i - 1 - (i+1)(m+1), \forall e \in \Delta_i(T), i = 0, \ldots, n-2.$$

Namely we can remove bubble functions in the test function space.

A.4. **Smooth scalar finite elements in arbitrary dimension.** Given a triangulation $\mathcal{T}_h$, the finite element space is obtained by asking the DoFs depending on the subsimplex only.

Theorem A.8 (Theorem 3.3 in [32]). *Given integer $m \geq 0$, let non-negative integer array $\boldsymbol{r} = (r_0, r_1, \ldots, r_n)$ satisfy*

$$r_n = 0, \;\; r_{n-1} = m, \;\; r_\ell \geq 2r_{\ell+1} \;\; \textit{for } \ell = n-2, \ldots, 0.$$

Let $k \geq 2r_0 + 1 \geq 2^n m + 1$. *The following DoFs*

$$D^\alpha u(\mathtt{v}) \quad \alpha \in \mathbb{N}^{1:n}, |\alpha| \leq r_0, \mathtt{v} \in \Delta_0(\mathcal{T}_h), \tag{86}$$

$$\int_f \frac{\partial^\beta u}{\partial n_f^\beta} \lambda_f^{\alpha_f} \,\mathrm{d}s \quad \alpha \in S_\ell(f), |\alpha_f| = k - s, \beta \in \mathbb{N}^{1:n-\ell}, |\beta| = s, s = 0, \ldots, r_\ell, \tag{87}$$
$$f \in \Delta_\ell(\mathcal{T}_h), \ell = 1, \ldots, n-1,$$

$$\int_T u\lambda^\alpha \,\mathrm{d}x \quad \alpha \in S_n(T), T \in \mathcal{T}_h, \tag{88}$$

will define a finite element space

$$V_h = \{u \in C^m(\Omega) : \text{ DoFs } (86) - (87) \text{ are single valued}, u|_T \in \mathbb{P}_k(T), \forall T \in \mathcal{T}_h\}.$$

Proof. Restricted to one simplex T, by Theorem A.6, DoFs (86)-(88) will define a function u s.t. $u|_T \in \mathbb{P}_k(T)$. We only need to verify $u \in C^m(\Omega)$. It suffices to prove $\frac{\partial^i u}{\partial n_F^i}|_F \in \mathbb{P}_{k-i}(F)$, for all $i = 0, \ldots, m$ and all $F \in \Delta_{n-1}(T)$, are uniquely determined by (86)-(87) on F.

Let $w = \frac{\partial^i u}{\partial n_F^i}|_F \in \mathbb{P}_{k-i}(F)$. Consider the modified index sequence $\boldsymbol{r}_F^i = (r_0 - i, r_1 - i, \ldots, r_{n-2} - i, 0)$ and degree $k^i = k - i$. Then $k^i, \boldsymbol{r}_F^i$ satisfies the condition in Theorem 3.3 and we obtain a direct decomposition of $\mathbb{T}_{k-i}^{n-1}(F) = \bigoplus_{\ell=0}^{n-1} \bigoplus_{f\in\Delta_\ell(F)} S_\ell^F(f)$, where

$$S_0^F(\mathtt{v}) = D(\mathtt{v}, r_0 - i) \cap \mathbb{T}_{k-i}^{n-1}(F),$$
$$S_\ell^F(f) = (D(f, r_\ell - i) \cap \mathbb{T}_{k-i}^{n-1}(F)) \backslash \left[\bigoplus_{i=0}^{\ell-1} \bigoplus_{e\in\Delta_i(F)} S_i^F(e)\right], \ \ell = 1, \ldots, n-2,$$
$$S_{n-1}^F(F) = \mathbb{T}_{k-i}^{n-1}(F) \backslash \left[\bigoplus_{\ell=0}^{n-2} \bigoplus_{f\in\Delta_\ell(F)} S_\ell^F(f)\right].$$

The DoFs (86)-(87) related to w are

$$D_F^\alpha w(\mathtt{v}) \quad \alpha \in \mathbb{N}^{1:n-1}, |\alpha| \leq r_0 - i, \mathtt{v} \in \Delta_0(F),$$
$$\int_f \frac{\partial^\beta w}{\partial n_{F,f}^\beta} \lambda_f^{\alpha_f} \,\mathrm{d}s \quad \alpha \in S_\ell^F(f), |\alpha_f| = k - i - s, \beta \in \mathbb{N}^{1:n-1-\ell}, |\beta| = s,$$
$$f \in \Delta_\ell(F), \ell = 1, \ldots, n-2, s = 0, \ldots, r_\ell - i,$$
$$\int_F w\lambda^\alpha \,\mathrm{d}x \quad \alpha \in S_{n-1}^F(F),$$

where $D_F w$ is the tangential derivative of w, $n_{F,f}$ is the normal vector of f but tangential to F. Clearly the modified sequence $\boldsymbol{r}_i^F$ still satisfies constraints required in Theorem A.6. We can apply Theorem A.6 with the shape function space $\mathbb{P}_{k-i}(F)$ to conclude w is uniquely determined on F. Thus the result $u \in C^m(\Omega)$ follows. ☐

Counting the dimension of V_h is hard and not necessary. The cardinality of $S_\ell(f)$ is difficult to describe due to the inequality constraints. In the implementation, compute the distance of lattice nodes to subsimplexes and use a logic array to find out $S_\ell(f)$.

References

[1] J. Argyris, I. Fried, and D. Scharpf, *The TUBA family of plate elements for the matrix displacement method*, Aero. J. Roy. Aero. Soc. **72** (1968), 701–709. 102

[2] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus: from Hodge theory to numerical stability*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), no. 2, 281–354, DOI 10.1090/S0273-0979-10-01278-4. MR2594630 ↑55

[3] D. N. Arnold, *Finite Element Exterior Calculus*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 93, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2018, DOI 10.1137/1.9781611975543.ch1. MR3908678 ↑55

[4] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numer. **15** (2006), 1–155, DOI 10.1017/S0962492906210018. MR2269741 ↑55

[5] D. N. Arnold, R. S. Falk, and R. Winther, *Geometric decompositions and local bases for spaces of finite element differential forms*, Comput. Methods Appl. Mech. Engrg. **198** (2009), no. 21-26, 1660–1672, DOI 10.1016/j.cma.2008.12.017. MR2517938 ↑58, 59, 62

[6] D. N. Arnold and K. Hu, *Complexes from complexes*, Found. Comput. Math. **21** (2021), no. 6, 1739–1774, DOI 10.1007/s10208-021-09498-9. MR4343022 ↑102

[7] F. Bonizzoni and G. Kanschat, *H^1-conforming finite element cochain complexes and commuting quasi-interpolation operators on Cartesian meshes*, Calcolo **58** (2021), no. 2, Paper No. 18, 29, DOI 10.1007/s10092-021-00409-6. MR4244253 ↑57

[8] F. Bonizzoni and G. Kanschat, *A tensor-product finite element cochain complex with arbitrary continuity*, Preprint, `arXiv:2207.00309`, 2022. 57

[9] J. H. Bramble and M. Zlámal, *Triangular elements in the finite element method*, Math. Comp. **24** (1970), 809–820, DOI 10.2307/2004615. MR282540 ↑102

[10] F. Brezzi, J. Douglas Jr., R. Durán, and M. Fortin, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math. **51** (1987), no. 2, 237–250, DOI 10.1007/BF01396752. MR890035 ↑72

[11] F. Brezzi, J. Douglas Jr., and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math. **47** (1985), no. 2, 217–235, DOI 10.1007/BF01389710. MR799685 ↑72

[12] A. Buffa, C. de Falco, and G. Sangalli, *IsoGeometric Analysis: stable elements for the 2D Stokes equation*, Internat. J. Numer. Methods Fluids **65** (2011), no. 11-12, 1407–1422, DOI 10.1002/fld.2337. MR2808112 ↑57

[13] L. Chen and X. Huang, Complexes from complexes: finite element complexes in three dimensions, arXiv preprint `arXiv:2211.08656`, 2022. 102

[14] L. Chen and X. Huang, *Decoupling of mixed methods based on generalized Helmholtz decompositions*, SIAM J. Numer. Anal. **56** (2018), no. 5, 2796–2825, DOI 10.1137/17M1145872. MR3852718 ↑55, 96

[15] L. Chen and X. Huang, Finite elements for divdiv-conforming symmetric tensors, arXiv preprint `arXiv:2005.01271`, 2020. 102

[16] L. Chen and X. Huang, Geometric decomposition of div-conforming finite element tensors, part I: Vector and matrix functions, arXiv preprint `arXiv:2112.14351`, 2021. 56, 64, 71, 72

[17] L. Chen and X. Huang, Finite element complexes in two dimensions, arXiv preprint `arXiv:2206.00851`, 2022. 55, 56, 102, 105

[18] L. Chen and X. Huang, *A finite element elasticity complex in three dimensions*, Math. Comp. **91** (2022), no. 337, 2095–2127, DOI 10.1090/mcom/3739. MR4451457 ↑102

[19] L. Chen and X. Huang, *Finite elements for* $\operatorname{div}\operatorname{div}$ *conforming symmetric tensors in three dimensions*, Math. Comp. **91** (2022), no. 335, 1107–1142, DOI 10.1090/mcom/3700. MR4405490 ↑102

[20] S. H. Christiansen, J. Gopalakrishnan, J. Guzmán, and K. Hu, A discrete elasticity complex on three-dimensional Alfeld splits, arXiv preprint `arXiv:2009.07744`, 2020. 102

[21] S. H. Christiansen, J. Hu, and K. Hu, *Nodal finite element de Rham complexes*, Numer. Math. **139** (2018), no. 2, 411–446, DOI 10.1007/s00211-017-0939-x. MR3802677 ↑57, 95

[22] S. H. Christiansen and K. Hu, *Generalized finite element systems for smooth differential forms and Stokes' problem*, Numer. Math. **140** (2018), no. 2, 327–371, DOI 10.1007/s00211-018-0970-6. MR3851060 ↑57

[23] C. K. Chui and M. J. Lai, *Multivariate vertex splines and finite elements*, J. Approx. Theory **60** (1990), no. 3, 245–343, DOI 10.1016/0021-9045(90)90063-V. MR1042653 ↑102

[24] M. Costabel and A. McIntosh, *On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains*, Math. Z. **265** (2010), no. 2, 297–320, DOI 10.1007/s00209-009-0517-8. MR2609313 ↑81

[25] J. A. Evans and T. J. R. Hughes, *Isogeometric divergence-conforming B-splines for the steady Navier-Stokes equations*, Math. Models Methods Appl. Sci. **23** (2013), no. 8, 1421–1478, DOI 10.1142/S0218202513500139. MR3048532 ↑57

[26] R. S. Falk and M. Neilan, *Stokes complexes and the construction of stable finite elements with pointwise mass conservation*, SIAM J. Numer. Anal. **51** (2013), no. 2, 1308–1326, DOI 10.1137/120888132. MR3045658 ↑86

[27] G. Fu, J. Guzmán, and M. Neilan, *Exact smooth piecewise polynomial sequences on Alfeld splits*, Math. Comp. **89** (2020), no. 323, 1059–1091, DOI 10.1090/mcom/3520. MR4063312 ↑57, 96

[28] J. Guzmán and M. Neilan, *A family of nonconforming elements for the Brinkman problem*, IMA J. Numer. Anal. **32** (2012), no. 4, 1484–1508, DOI 10.1093/imanum/drr040. MR2991835 ↑57

[29] J. Hu and Y. Liang, *Conforming discrete Gradgrad-complexes in three dimensions*, Math. Comp. **90** (2021), no. 330, 1637–1662, DOI 10.1090/mcom/3628. MR4273111 ↑102

[30] J. Hu, Y. Liang, and R. Ma, *Conforming finite element divdiv complexes and the application for the linearized Einstein-Bianchi system*, SIAM J. Numer. Anal. **60** (2022), no. 3, 1307–1330, DOI 10.1137/21M1404235. MR4431967 ↑102

[31] J. Hu, Y. Liang, R. Ma, and M. Zhang, New conforming finite element divdiv complexes in three dimensions, arXiv preprint `arXiv:2204.07895`, 2022. 102

[32] J. Hu, T. Lin, and Q. Wu, *A construction of C^r conforming finite element spaces in any dimension*, `arXiv:2103.14924`, 2021. 56, 64, 102, 103, 104, 106

[33] J. Hu, R. Ma, and M. Zhang, *A family of mixed finite elements for the biharmonic equations on triangular and tetrahedral grids*, Sci. China Math. **64** (2021), no. 12, 2793–2816, DOI 10.1007/s11425-020-1883-9. MR4343068 ↑102

[34] K. Hu, Q. Zhang, and Z. Zhang, *A family of finite element Stokes complexes in three dimensions*, SIAM J. Numer. Anal. **60** (2022), no. 1, 222–243, DOI 10.1137/20M1358700. MR4369574 ↑57

[35] X. Huang, *Nonconforming finite element Stokes complexes in three dimensions*, Sci. China Math. (2023), `https://doi.org/10.1007/s11425-021-2026-7`. 88

[36] X. Huang and C. Zhang, Robust mixed finite element methods for a quad-curl singular perturbation problem, arXiv preprint `arXiv:2206.10864`, 2022. 57, 88

[37] M.-J. Lai and L. L. Schumaker, *Spline Functions on Triangulations*, Encyclopedia of Mathematics and its Applications, vol. 110, Cambridge University Press, Cambridge, 2007, DOI 10.1017/CBO9780511721588. MR2355272 ↑58, 102

[38] M.-J. Lai and L. L. Schumaker, *Trivariate C^r polynomial macroelements*, Constr. Approx. **26** (2007), no. 1, 11–28, DOI 10.1007/s00365-006-0631-x. MR2310685 ↑102

[39] P. Le Tallec, *A mixed finite element approximation of the Navier-Stokes equations*, Numer. Math. **35** (1980), no. 4, 381–404, DOI 10.1007/BF01399007. MR593835 ↑98, 100

[40] J.-C. Nédélec, *A new family of mixed finite elements in $\mathbf{R}^3$*, Numer. Math. **50** (1986), no. 1, 57–81, DOI 10.1007/BF01389668. MR864305 ↑72

[41] M. Neilan, *Discrete and conforming smooth de Rham complexes in three dimensions*, Math. Comp. **84** (2015), no. 295, 2059–2081, DOI 10.1090/S0025-5718-2015-02958-5. MR3356019 ↑57, 82, 96

[42] M. Neilan, *The Stokes complex: a review of exactly divergence-free finite element pairs for incompressible flows*, 75 Years of Mathematics of Computation, Contemp. Math., vol. 754, Amer. Math. Soc., [Providence], RI, [2020] ©2020, pp. 141–158, DOI 10.1090/conm/754/15142. MR4132120 ↑57

[43] M. Neilan and D. Sap, *Stokes elements on cubic meshes yielding divergence-free approximations*, Calcolo **53** (2016), no. 3, 263–283, DOI 10.1007/s10092-015-0148-x. MR3541155 ↑57

[44] M. Neilan and D. Sap, *Macro Stokes elements on quadrilaterals*, Int. J. Numer. Anal. Model. **15** (2018), no. 4-5, 729–745. MR3789588 ↑57

[45] R. A. Nicolaides, *On a class of finite elements generated by Lagrange interpolation. II*, SIAM J. Numer. Anal. **10** (1973), 182–189, DOI 10.1137/0710019. MR317512 ↑58

[46] P.-A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, Mathematical Aspects of Finite Element Methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Lecture Notes in Math., Vol. 606, Springer, Berlin, 1977, pp. 292–315. MR0483555 ↑98

[47] L. R. Scott and M. Vogelius, *Conforming finite element methods for incompressible and nearly incompressible continua*, Large-Scale Computations in Fluid Mechanics, Part 2 (La Jolla, Calif., 1983), Lectures in Appl. Math., vol. 22, Amer. Math. Soc., Providence, RI, 1985, pp. 221–244, DOI 10.1051/m2an/1985190101111. MR818790 ↑73

[48] R. Stenberg, *A nonstandard mixed finite element family*, Numer. Math. **115** (2010), no. 1, 131–139, DOI 10.1007/s00211-009-0272-0. MR2594344 ↑72, 82

[49] X.-C. Tai and R. Winther, *A discrete de Rham complex with enhanced smoothness*, Calcolo **43** (2006), no. 4, 287–306, DOI 10.1007/s10092-006-0124-6. MR2283095 ↑57

[50] A. Ženíšek, *Interpolation polynomials on the triangle*, Numer. Math. **15** (1970), 283–296, DOI 10.1007/BF02165119. MR275014 ↑102

[51] A. Ženíšek, *Tetrahedral finite $C^{(m)}$-elements*, Acta Univ. Carolin. Math. Phys. **15** (1974), no. 1-2, 189–193, DOI 10.1016/0016-0032(64)90077-8. MR395160 ↑102

[52] Q. Zhang and Z. Zhang, *A family of curl-curl conforming finite elements on tetrahedral meshes*, CSIAM Trans. Appl. Math. **1** (2020), no. 4, 639–663. 88, 101

[53] S. Zhang, *A family of 3D continuously differentiable finite elements on tetrahedral grids*, Appl. Numer. Math. **59** (2009), no. 1, 219–233, DOI 10.1016/j.apnum.2008.02.002. MR2474112 ↑102

[54] S. Zhang, *Divergence-free finite elements on tetrahedral grids for $k \geq 6$*, Math. Comp. **80** (2011), no. 274, 669–695, DOI 10.1090/S0025-5718-2010-02412-3. MR2772092 ↑73

[55] S. Y. Zhang, *A family of differentiable finite elements on simplicial grids in four space dimensions* (Chinese, with English and Chinese summaries), Math. Numer. Sin. **38** (2016), no. 3, 309–324. MR3617203 ↑102

[56] B. Zheng, Q. Hu, and J. Xu, *A nonconforming finite element method for fourth order curl equations in $\mathbb{R}^3$*, Math. Comp. **80** (2011), no. 276, 1871–1886, DOI 10.1090/S0025-5718-2011-02480-4. MR2813342 ↑88

Department of Mathematics, University of California at Irvine, Irvine, California 92697

Email address: chenlong@math.uci.edu

School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, People's Republic of China

Email address: huang.xuehai@sufe.edu.cn