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Convergence and optimality of an adaptive modified weak Galerkin finite element method

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Abstract

An adaptive modified weak Galerkin method (AmWG) for an elliptic problem is studied in this article, in addition to its convergence and optimality. The modified weak Galerkin bilinear form is simplified without the need of the skeletal variable, and the approximation space is chosen as the discontinuous polynomial space as in the discontinuous Galerkin method. Upon a reliable residual-based *a posteriori* error estimator, an adaptive algorithm is proposed together with its convergence and quasi-optimality proved for the lowest order case. The primary tool is to bridge the connection between the modified weak Galerkin method and the Crouzeix–Raviart nonconforming finite element. Unlike the traditional convergence analysis for methods with a discontinuous polynomial approximation space, the convergence of AmWG is penalty parameter free. Numerical results are presented to support the theoretical results.

KEYWORDS

a posteriori error estimation, adaptive methods, convergence, modified weak Galerkin, optimality

1 | INTRODUCTION

Consider the following model second-order elliptic problem

$$\begin{aligned} -\nabla \cdot (A\nabla u) &= f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded polygonal or polyhedral domain in \mathbb{R}^d , $d = 2, 3$. Assume that an initial conforming partition \mathcal{T}_0 of Ω exists and for all $\tau \in \mathcal{T}_0$, the coefficient A is assumed to be a piecewise constant with respect to this partition.

Weak Galerkin (WG) is a novel numerical method for solving partial differential equations in which classical differential operators (such as gradient, divergence, curl) are approximated in a weak sense. WG method was initially introduced in [1-3] for the second-order elliptic problem. Since then, the WG method has successfully found its way to many applications, for example, elliptic interface problems [4], Helmholtz equations [5-7], biharmonic equations [8, 9], Navier-Stokes equations [10, 11] and so forth. In particular, Wang et al. [12] introduced a modified weak Galerkin (mWG) method for the Poisson equation. The mWG method has been successfully applied to, such as parabolic problem [13], Signorini and obstacle problem [14], and Stokes equations [15].

The solution to (1.1) may contain singularities. To approximate problem (1.1) efficiently, the general practice is to adopt adaptivity by designing an adaptive finite element cycle through the help of the *a posteriori* error estimators, a bulk marking strategy, and certain local refinement techniques. As examples non-convergent adaptive algorithms [16] may fail to produce the desired approximation even with additional iterations, the convergence analysis of an adaptive algorithm is of fundamental importance for ensuring that the correct approximation is obtained. It theoretically guarantees that the correct approximation will be obtained, especially if one wants to avoid the situation when more computational resources may go wasted after iterative refinements.

The convergence theory of adaptive finite element methods is relatively mature, see [17] and the references therein. Nevertheless, few research results exist for the *a posteriori* error estimates for WG methods. Chen et al. [18] presented the *a posteriori* error estimates for second-order elliptic problems; Zhang and Chen [19] proposed a residual-type error estimator and proved global upper and lower bounds of the WG method for second-order elliptic problems in a discrete H^1 -norm; Li et al. [20] introduced a simple *a posteriori* error estimator which can be applied to general meshes such as hybrid, polytopal and those with hanging nodes for second-order elliptic problems; Mu [21] presented an *a posteriori* error estimate for the second-order elliptic interface problems; Zheng and Xie [22] discussed a residual-based *a posteriori* error estimator for the Stokes problem. There are only a few research results for *a posteriori* error estimates for mWG methods. Zhang and Lin [23] proposed an *a posteriori* error estimator for the second-order elliptic problems. More recently, Tang et al. proposed an adaptive mWG for $\mathbf{H}(\mathbf{curl})$ -elliptic problems in [24].

This article aims to prove the optimal convergence of an AmWG algorithm for the second-order elliptic problem (1.1). Different from the mWG originally introduced in [12], we simplify the mWG as follows: for a weak function $v = \{v_0, v_b\}$, the edge/face term is not independent anymore as we choose $v_b = Q_b\{v_0\}$, that is, v_b is obtained through averaging the interior discontinuous variable v_0 and then projected through Q_b to a one-degree-lower polynomial space. Compared with interior-penalty discontinuous Galerkin (IPDG) (see e.g., [25]), the mWG is stable without choosing a sufficiently large penalty parameter. This simplification ($Q_b\{v_0\}$ opposed to $\{v_0\}$ in [12]) brings extra difficulty to the analysis of convergence. The reason is that a simple part of the workflow presented in [26], by decomposing the discontinuous approximation space into a continuous subspace and its orthogonal complement, will introduce a penalty parameter that is not originally in the mWG discretization (cf. [27]). To our best knowledge, there is no literature on the convergence of adaptive mWG methods with the skeletal variable being one degree lower than the internal variable.

To conquer this difficulty, by introducing an interpolation operator $I_{\mathcal{T}}$ onto the Crouzeix–Raviart type nonconforming finite element space $V^{\text{nc}}(\mathcal{T})$, we bound the stabilization term and prove an *a posteriori* error estimate in the discrete H^1 -norm. One main ingredient in the convergence analysis of a standard adaptive procedure is the orthogonality of the error to the finite element space. However,

such an orthogonality does not hold for mWG approximations. Instead, a quasi-orthogonality result is established. Hu and Xu [28] defined a canonical interpolation operator for the lowest Crouzeix–Raviart type nonconforming finite element space and established the quasi-orthogonality property for both the velocity and the pressure in the Stokes problem. The main observation is that the modified weak gradient for a function $v_{\mathcal{T}} = \{v_0, Q_b\{v_0\}\}$ is equal to the elementwise gradient of the interpolant $I_{\mathcal{T}}^{\text{CR}} v_0$, namely $\nabla_w v_{\mathcal{T}} = \nabla_h I_{\mathcal{T}}^{\text{CR}} v_0$ and we can derive the desired quasi-orthogonality property for the lowest order (P_1-P_0) mWG.

Another key component to establish the optimality of the adaptive algorithm is the localized discrete upper bound for the *a posteriori* error estimator. By using a prolongation operator introduced in [28], we are able to derive the discrete reliability and use it to prove the optimality of the convergence.

For the *a posteriori* error analysis of mWG approximations, we mainly follow Bonito and Nochetto [26] and Chen et al. [18]. For the analysis of the convergence and the optimality of adaptive procedure, we mainly use the Hu and Xu [28] and Huang and Xu [29]. We do not claim any originality on the proof of convergence and optimality. Instead, the main contribution of this article is to bound the stabilization term by the element-wise residual and flux jump, as well as to establish a quasi-orthogonality and a discrete upper bound which are important ingredients on the convergence theory of adaptive finite element methods.

The rest of this article is organized as follows. In Section 2, the definitions of weak gradient and discrete weak gradient are introduced, as well as the modified weak Galerkin finite element spaces and the corresponding bilinear form $a_{\mathcal{T}}(\cdot, \cdot)$. In Section 3, a residual-type error estimator is constructed, and its reliability and efficiency are shown. In Section 4, we introduce an adaptive modified weak Galerkin method (AmWG) and prove its convergence and optimality. Some numerical examples are presented in Section 5 to verify the theoretical results.

2 | NOTATION AND PRELIMINARY

The goal of this section is to present the modified weak Galerkin (mWG) formulation for (1.1). First, the standard weak Galerkin method is reviewed, then an mWG finite element space and the discretization thereof are introduced.

2.1 | Weak Galerkin methods

Given a polygonal/polyhedral element K with boundary ∂K , the notation $v = \{v_0, v_b\}$ defines a weak function on K such that $v_0 \in L^2(K)$ and $v_b \in L^2(\partial K)$. Subsequently, the weak function space on K is defined as

$$W(K) = \{v = \{v_0, v_b\} : v_0 \in L^2(K), v_b \in L^2(\partial K)\}.$$

Let $P_{\ell}(K)$ be the set of polynomials on K with degree no more than ℓ ($\ell \geq 1$). The discrete weak gradient operator $\nabla_{w,K,\ell}(\cdot)$ is defined for polynomial functions. With slight abuse of notation, when no ambiguity arises, we shall denote $\nabla_{w,K}(\cdot)$ as $\nabla_w(\cdot)$, which should be clear from the context.

Definition 2.1 ([30, Definition 2.1]). The discrete weak gradient operator, denoted by $\nabla_{w,K,\ell}$, is defined as the unique polynomial $\nabla_{w,K,\ell} v \in (P_{\ell-1}(K))^d$ satisfying the following equation for $v \in W(K)$

$$(\nabla_{w,K,\ell} v, \mathbf{q})_K = -(v_0, \nabla \cdot \mathbf{q})_K + \langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial K}, \quad \forall \mathbf{q} \in (P_{\ell-1}(K))^d. \quad (2.1)$$

In the definition above, \mathbf{n} is the outward normal direction to ∂K , $(v_0, \nabla \cdot \mathbf{q})_K = \int_K v_0 (\nabla \cdot \mathbf{q}) \, dK$ is the $L^2(K)$ -inner product of v_0 and $\nabla \cdot \mathbf{q}$, and $\langle v_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial K} = \int_{\partial K} v_b (\mathbf{q} \cdot \mathbf{n}) \, ds$ is the $L^2(\partial K)$ -inner product of v_b and $\mathbf{q} \cdot \mathbf{n}$. The differential operators involved are well-defined when restricted to one element. In the context of the gradient operator defined across multiple elements in \mathcal{T} , the elementwise gradient ∇_h is introduced, that is, $(\nabla_h v)|_K := \nabla(v|_K)$ in element $K \in \mathcal{T}$.

In the rest of the article, we restrict ourselves to a shape-regular triangulation \mathcal{T} of Ω . \mathcal{E} denotes the set of all the edges or faces in \mathcal{T} , and \mathcal{E}^{int} is the set of all the interior edges or faces. Denote $|\cdot|$ the d -dimensional Lebesgue measure. For $\tau \in \mathcal{T}$, its associated patch as $\omega(\tau) = \bigcup_{\tau' \cap \tau \neq \emptyset} \tau'$. For a set $\mathcal{R} \subseteq \mathcal{T}$, its associated patch element patch is $\omega(\mathcal{R}) = \bigcup_{\tau \in \mathcal{R}} \omega(\tau)$.

Given a positive integer $\ell \geq 1$, the ℓ th order weak Galerkin finite element space on \mathcal{T} is defined as follows:

$$V^{\text{wg}}(\mathcal{T}) := \{v = \{v_0, v_b\} : v_0|_{\tau} \in P_{\ell}(\tau), v_b|_e \in P_{\ell-1}(e), e \in \mathcal{E}, \tau \in \mathcal{T}\}, \quad (2.2)$$

and the one with zero boundary condition:

$$V_0^{\text{wg}}(\mathcal{T}) = \{v : v \in V^{\text{wg}}(\mathcal{T}), v_b = 0 \text{ on } \partial\Omega\}. \quad (2.3)$$

Note that v_b is single-valued on each e .

For $v = \{v_0, v_b\} \in V^{\text{wg}}(\mathcal{T})$, $w = \{w_0, w_b\} \in V^{\text{wg}}(\mathcal{T})$, the discrete bilinear form for the variational form of problem (1.1) is defined as

$$a^{\text{wg}}(v, w) := \sum_{\tau \in \mathcal{T}} (A \nabla_w v, \nabla_w w)_{\tau} + \sum_{\tau \in \mathcal{T}} h_{\tau}^{-1} \langle Q_b v_0 - v_b, Q_b w_0 - w_b \rangle_{\partial\tau}, \quad (2.4)$$

where the weak gradient $\nabla_w = \nabla_{w, \tau, \ell-1}$, and Q_b is the L^2 projection from $L^2(e)$ to $P_{\ell-1}(e)$ on an $e \in \mathcal{E}$. Again when it is clear from the context, we shall omit the degree of the polynomial involved in the projection operator.

The weak Galerkin discretization is: to find a $u_h = \{u_0, u_b\} \in V_0^{\text{wg}}(\mathcal{T})$ satisfying

$$a^{\text{wg}}(u_h, v) = (f, v_0), \quad \forall v = \{v_0, v_b\} \in V_0^{\text{wg}}(\mathcal{T}). \quad (2.5)$$

The WG discretization (2.5) is well-posed as $a^{\text{wg}}(\cdot, \cdot)$ defines an inner product on the space $V_0^{\text{wg}}(\mathcal{T})$.

For completion and the convenience of our readers, we include a short argument here to show that

$$\|v\|_1 = (a^{\text{wg}}(v, v))^{1/2}, \quad (2.6)$$

defines a norm in $V_0^{\text{wg}}(\mathcal{T})$. Assume that $\|v\|_1 = 0$, then $\nabla_w v|_{\tau} = 0$ on every element $\tau \in \mathcal{T}$ and $Q_b v_0 = v_b$ on every $e \subset \partial\tau$, then by definition (2.1)

$$\begin{aligned} 0 &= (\nabla_w v, \nabla v_0)_{\tau} = -(v_0, \Delta v_0)_{\tau} + \langle v_b, \nabla v_0 \cdot \mathbf{n} \rangle_{\partial\tau} \\ &= -(v_0, \Delta v_0)_{\tau} + \langle Q_b v_0, \nabla v_0 \cdot \mathbf{n} \rangle_{\partial\tau} \\ &= -(v_0, \Delta v_0)_{\tau} + \langle v_0, \nabla v_0 \cdot \mathbf{n} \rangle_{\partial\tau} \\ &= (\nabla v_0, \nabla v_0)_{\tau}, \end{aligned}$$

which implies $\nabla v_0 = 0$ on τ . As a result, v is constant on τ . Moreover, $Q_b v_0 = v_b$ on each $\partial\tau$. Knowing $v_b = 0$ on $\partial\Omega$. By an argument of continuation, $v_0 = v_b = 0$. Therefore, $\|\cdot\|_1$ defines a norm and $a^{\text{wg}}(\cdot, \cdot)$ defines an inner product on the space $V_0^{\text{wg}}(\mathcal{T})$.

The proof above reveals that the boundary part v_b can be set as a polynomial with degree one less than that of the interior one v_0 since its presence is only in $\langle v_b, \nabla v_0 \cdot \mathbf{n} \rangle_{\partial\tau}$.

2.2 | Modified weak Galerkin finite element

Let $e \in \mathcal{E}^{\text{int}}$ be the common edge/face shared by two elements $\tau_1, \tau_2 \in \mathcal{T}$, and denote by $\omega(e) = \tau_1 \cup \tau_2$. We assume that globally each e is associated with a fixed unit normal vector \mathbf{n}_e . When $d = 2$, we can get e 's tangential vector by $\mathbf{t}_e = \langle n_{e,2}, -n_{e,1} \rangle$, which is obtained by rotating \mathbf{n}_e clockwise by $\pi/2$. Without loss of generality, the \mathbf{n}_e is assumed pointing from τ_1 to τ_2 . Denote by \mathbf{n}_1 and \mathbf{n}_2 the outer unit normal vectors with respect to τ_1 and τ_2 , respectively. For a smooth enough scalar function w , we define its average and jump on e by

$$\begin{aligned} \{w\}_e &= (w|_{\tau_1} + w|_{\tau_2})/2, \quad \text{for } e \in \mathcal{E}^{\text{int}}; \\ \llbracket w \rrbracket_e &= w|_{\tau_1} - w|_{\tau_2}, \quad \text{for } e \in \mathcal{E}^{\text{int}}; \\ \llbracket w \rrbracket_e &= \{w\}_e = w, \quad \text{for } e \subset \partial\Omega. \end{aligned}$$

Similarly, for an admissible vector function \mathbf{w} , we have

$$\begin{aligned} \{\mathbf{w}\}_e &= (\mathbf{w}|_{\tau_1} + \mathbf{w}|_{\tau_2})/2, \quad \text{for } e \in \mathcal{E}^{\text{int}}; \\ \llbracket \mathbf{w} \cdot \mathbf{n}_e \rrbracket_e &= \mathbf{w}|_{\tau_1} \cdot \mathbf{n}_1 + \mathbf{w}|_{\tau_2} \cdot \mathbf{n}_2, \quad \text{for } e \in \mathcal{E}^{\text{int}}; \\ \{\mathbf{w}\}_e &= \mathbf{w}, \quad \llbracket \mathbf{w} \cdot \mathbf{n}_e \rrbracket_e = \mathbf{w} \cdot \mathbf{n}_e, \quad \text{for } e \subset \partial\Omega. \end{aligned}$$

In the definitions above, $w|_{\tau}$ and $\mathbf{w}|_{\tau}$'s values on e are defined for those spaces which yield a well-defined trace, respectively.

Specifically, in discretization (2.5), v_0 is chosen to be in the discontinuous polynomial space

$$V^{\text{DG}}(\mathcal{T}) := \{v : v|_{\tau} \in P_{\ell}(\tau), \tau \in \mathcal{T}\},$$

and the edge/face term $v_b = Q_b\{v_0\}$. Now a weak function is $v = \{v_0, Q_b\{v_0\}\}$, which offers a continuous embedding of $V^{\text{DG}}(\mathcal{T}) \hookrightarrow V^{\text{WG}}(\mathcal{T})$ with respect to norm (2.6), which will be shown equivalent to the modified IP norm associated with (2.10). The modified weak Galerkin finite element space for \mathcal{T} is then defined as

$$V(\mathcal{T}) := \{v_{\mathcal{T}} = \{v_0, v_b\} : v_0 \in V^{\text{DG}}(\mathcal{T}); v_b = Q_b\{v_0\}, e \in \mathcal{E}\}, \tag{2.7}$$

and

$$V_0(\mathcal{T}) = \{v_{\mathcal{T}} : v_{\mathcal{T}} \in V(\mathcal{T}), Q_b\{v_0\} = 0 \text{ on } \partial\Omega\}. \tag{2.8}$$

Remark 2.2. The original modified weak Galerkin function space was introduced in [12]. The edge/face term v_b is chosen as $\{v_0\}|_e \in P_{\ell}(e)$ in [12], while $v_b \equiv Q_b\{v_0\} \in P_{\ell-1}(e)$ here to match the reduced-order weak Galerkin scheme in [30].

The definition of the weak gradient is modified accordingly as follows.

Definition 2.3. The modified weak gradient operator acting on any $v_{\mathcal{T}} \in V(\mathcal{T})$, denoted by $\nabla_{w,\tau}$ on $\tau \in \mathcal{T}$, is defined as the unique polynomial in $(P_{\ell-1}(\tau))^d$, $d = 2, 3$ such that its inner product with any $\mathbf{q} \in (P_{\ell-1}(\tau))^d$ satisfies the following equation:

$$(\nabla_{w,\tau} v_{\mathcal{T}}, \mathbf{q})_{\tau} = -(v_0, \nabla \cdot \mathbf{q})_{\tau} + \langle Q_b\{v_0\}, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial\tau}. \tag{2.9}$$

When the triangulation is clear from the context, we shall abbreviate the notation $\nabla_{w,\tau}$ as ∇_w .

Let $e \in \mathcal{E}^{\text{int}}$ be the common edge/face shared by $\tau_1, \tau_2 \in \mathcal{T}$. By using the mWG space (2.7) in the WG bilinear form (2.4), one can simplify the stabilization term on e for $v_{\mathcal{T}} = \{v_0, Q_b\{v_0\}\} \in V(\mathcal{T})$ and $w_{\mathcal{T}} = \{w_0, Q_b\{w_0\}\} \in V(\mathcal{T})$, henceforth the same argument applies to other edges/faces.

$$\begin{aligned}
& \langle Q_b(v_0 - \{v_0\}), Q_b(w_0 - \{w_0\}) \rangle_e \\
&= \left\langle Q_b \left(v_0|_{\tau_1} - \frac{v_0|_{\tau_1} + v_0|_{\tau_2}}{2} \right), Q_b \left(w_0|_{\tau_1} - \frac{w_0|_{\tau_1} + w_0|_{\tau_2}}{2} \right) \right\rangle_e \\
&= \left\langle Q_b \left(\frac{v_0|_{\tau_1} - v_0|_{\tau_2}}{2} \right), Q_b \left(\frac{w_0|_{\tau_1} - w_0|_{\tau_2}}{2} \right) \right\rangle_e \\
&= \frac{1}{4} \langle Q_b \llbracket v_0 \rrbracket, Q_b \llbracket w_0 \rrbracket \rangle_e.
\end{aligned}$$

Consequently, for $v_{\mathcal{T}}, w_{\mathcal{T}} \in V(\mathcal{T})$, the associated bilinear form is defined as

$$\begin{aligned}
a_{\mathcal{T}}(v_{\mathcal{T}}, w_{\mathcal{T}}) &:= \sum_{\tau \in \mathcal{T}} (A \nabla_w v_{\mathcal{T}}, \nabla_w w_{\mathcal{T}})_{\tau} + 4 \sum_{\tau \in \mathcal{T}} h_{\tau}^{-1} \langle Q_b(v_0 - \{v_0\}), Q_b(w_0 - \{w_0\}) \rangle_{\partial\tau} \\
&= \sum_{\tau \in \mathcal{T}} (A \nabla_w v_{\mathcal{T}}, \nabla_w w_{\mathcal{T}})_{\tau} + \sum_{\tau \in \mathcal{T}} h_{\tau}^{-1} \langle Q_b \llbracket v_0 \rrbracket, Q_b \llbracket w_0 \rrbracket \rangle_{\partial\tau}. \tag{2.10}
\end{aligned}$$

A modified weak Galerkin (mWG) approximation is then to seek $u_{\mathcal{T}} \in V_0(\mathcal{T})$ satisfying

$$a_{\mathcal{T}}(u_{\mathcal{T}}, v_{\mathcal{T}}) = (f, v_0), \quad \forall v_{\mathcal{T}} \in V_0(\mathcal{T}). \tag{2.11}$$

It is clear that the modified weak Galerkin finite element scheme (2.11) is also well-posed since the problem is solved in a subspace (the embedding of the DG space) of the original WG space. If nonhomogeneous Dirichlet boundary condition $u = g$ on $\partial\Omega$ is presented, $u_{\mathcal{T}}$ can be decomposed to the sum of two parts, the first part satisfies the approximation problem above. For the second part, the boundary contribution is $\{0, Q_b g\}$ which can be moved to the right-hand side in addition to the source term. For other treatments of the boundary data such as interpolations, see, for example, [31].

Remark 2.4. In the classic IPDG formulation (see e.g., [26]), for $v_0, w_0 \in V^{\text{DG}}(\mathcal{T})$, the bilinear form is

$$\begin{aligned}
a_{\mathcal{T}}^{\text{IPDG}}(v_0, w_0) &= \sum_{\tau \in \mathcal{T}} (A \nabla_h v_0, \nabla_h w_0)_{\tau} - \sum_{e \in \mathcal{E}} \langle \llbracket v_0 \rrbracket, \{A \nabla_h w_0 \cdot \mathbf{n}\} \rangle_e \\
&\quad - \sum_{e \in \mathcal{E}} \langle \llbracket w_0 \rrbracket, \{A \nabla_h v_0 \cdot \mathbf{n}\} \rangle_e + \sum_{\tau \in \mathcal{T}} \mu h_{\tau}^{-1} \langle \llbracket v_0 \rrbracket, \llbracket w_0 \rrbracket \rangle_{\partial\tau}. \tag{2.12}
\end{aligned}$$

In comparison, in (2.10), using the definition of the weak gradient, several times of the integration by parts, and the assumption that A is a piecewise constant, for $v_{\mathcal{T}} = \{v_0, Q_b \{v_0\}\}$, $w_{\mathcal{T}} = \{w_0, Q_b \{w_0\}\} \in V(\mathcal{T})$, an equivalent bilinear form of (2.10) reads

$$\begin{aligned}
a_{\mathcal{T}}(v_{\mathcal{T}}, w_{\mathcal{T}}) &= \sum_{\tau \in \mathcal{T}} (A \nabla_h v_0, \nabla_h w_0)_{\tau} - \sum_{e \in \mathcal{E}} \langle \llbracket v_0 \rrbracket, \{A \nabla_h w_0 \cdot \mathbf{n}\} \rangle_e \\
&\quad - \sum_{e \in \mathcal{E}} \langle \llbracket w_0 \rrbracket, \{A \nabla_w v_{\mathcal{T}} \cdot \mathbf{n}\} \rangle_e + \sum_{\tau \in \mathcal{T}} h_{\tau}^{-1} \langle Q_b \llbracket v_0 \rrbracket, Q_b \llbracket w_0 \rrbracket \rangle_{\partial\tau}. \tag{2.13}
\end{aligned}$$

The only difference is that one $\nabla_h v_0$ in (2.12) is replaced by $\nabla_w v_{\mathcal{T}}$ in (2.13). This minor change leads to a major improvement that the mWG (2.11) is automatically coercive and continuous as the bilinear form $a_{\mathcal{T}}(\cdot, \cdot)$ induces a norm. While in IPDG (2.12), the penalty parameter μ being sufficiently large is a necessary condition to achieve the coercivity both theoretically and numerically [32].

The following quantity is well-defined for $H_0^1(\Omega) + V_0(\mathcal{T})$ and defines a mesh dependent norm on $V_0(\mathcal{T})$ (for similar results see, e.g., [30])

$$\| \! \| \! \| v_{\mathcal{T}} \| \! \| \! \|_{\mathcal{T}}^2 := a_{\mathcal{T}}(v_{\mathcal{T}}, v_{\mathcal{T}}) = \left\| A^{1/2} \nabla_w v_{\mathcal{T}} \right\|_{\mathcal{T}}^2 + \sum_{\tau \in \mathcal{T}} h_{\tau}^{-1} \left\| Q_b \llbracket v_0 \rrbracket \right\|_{\partial\tau}^2. \tag{2.14}$$

We note that (2.14) defines a norm on $V_0(\mathcal{T})$ naturally, since $V_0(\mathcal{T})$ is a subspace of the bigger space $V_0^{\text{WG}}(\mathcal{T})$ and $\|\cdot\|_{\mathcal{T}}$ is equivalent to (2.6) restricted to this subspace; see (2.6) and the proof afterwards.

With slight abuse of notation, we are interested in estimating the following error using computable quantities and designing a convergent adaptive algorithm to reduce its magnitude successively:

$$\|u - u_{\mathcal{T}}\|_{\mathcal{T}}^2 := \left\| A^{1/2}(\nabla u - \nabla_w u_{\mathcal{T}}) \right\|^2 + \sum_{\tau \in \mathcal{T}} h_{\tau}^{-1} \left\| Q_b \llbracket u - u_0 \rrbracket \right\|_{\partial\tau}^2. \tag{2.15}$$

3 | A POSTERIORI ERROR ANALYSIS

In this section, we shall prove the reliability and efficiency of a residual-type error estimator. For $\tau \in \mathcal{T}$, we define the element-wise error estimator as

$$\begin{aligned} \eta^2(\nabla_w v_{\mathcal{T}}, \tau) &:= h_{\tau}^2 A_{\tau}^{-1} \|R(\nabla_w v_{\mathcal{T}})\|_{0,\tau}^2 \\ &+ \sum_{e \subset \partial\tau} h_{\tau} \int_e \left((A_e^{\max})^{-1} J_{n,e}^2(A \nabla_w v_{\mathcal{T}}) + A_e^{\min} J_{t,e}^2(\nabla_w v_{\mathcal{T}}) \right) ds, \end{aligned} \tag{3.1}$$

where $A_e^{\max} := \max\{A_{\tau_1}, A_{\tau_2}\}$ and $A_e^{\min} := \min\{A_{\tau_1}, A_{\tau_2}\}$ for $\tau_1, \tau_2 \in \omega(e)$. The element residual is defined as

$$R(\nabla_w v_{\mathcal{T}}) = f + \nabla \cdot (A \nabla_w v_{\mathcal{T}}),$$

and the normal jump of the weak flux is defined as

$$J_{n,e}(A \nabla_w v_{\mathcal{T}}) := \begin{cases} \llbracket A \nabla_w v_{\mathcal{T}} \cdot \mathbf{n}_e \rrbracket_e, & \text{if } e \in \mathcal{E}^{\text{int}} \\ 0, & \text{otherwise.} \end{cases}$$

For the tangential jumps, when $d = 2$:

$$J_{t,e}(\nabla_w v_{\mathcal{T}}) := \begin{cases} \llbracket \nabla_w v_{\mathcal{T}} \cdot \mathbf{t}_e \rrbracket_e, & \text{if } e \in \mathcal{E}^{\text{int}} \\ 0, & \text{otherwise,} \end{cases}$$

and when $d = 3$

$$J_{t,e}(\nabla_w v_{\mathcal{T}}) := \begin{cases} \llbracket \nabla_w v_{\mathcal{T}} \times \mathbf{n}_e \rrbracket_e, & \text{if } e \in \mathcal{E}^{\text{int}} \\ 0, & \text{otherwise.} \end{cases}$$

Then the error estimator for the set $\mathcal{M} \subseteq \mathcal{T}$ is defined as

$$\eta^2(\nabla_w v_{\mathcal{T}}, \mathcal{M}) = \sum_{\tau \in \mathcal{M}} \eta^2(\nabla_w v_{\mathcal{T}}, \tau). \tag{3.2}$$

In computing the error estimator, only the information of $\nabla_w v_{\mathcal{T}}$ is used. Thus, we opt for a notation of $\eta(\nabla_w v_{\mathcal{T}}, \mathcal{M})$ instead of $\eta(v_{\mathcal{T}}, \mathcal{M})$. In Remark 4.1, some further explanation is given with regard to this choice of the notation for the error estimator in the context of the convergence analysis.

3.1 | A space decomposition

In this section, a space decomposition is first introduced to bridge the past result of the full degree jump $\llbracket v_0 \rrbracket$ to $Q_b \llbracket v_0 \rrbracket$ with one less degree. Then, similar to [26], a partial orthogonality for the mWG approximation $u_{\mathcal{T}}$ is introduced to enable the insertion of continuous interpolants to prove the reliability.

For $V^{\text{DG}}(\mathcal{T})$ consisting of discontinuous polynomial of degree ℓ :

$$V^c(\mathcal{T}) \subset V^{\text{nc}}(\mathcal{T}) \subset V^{\text{DG}}(\mathcal{T}), \quad (3.3)$$

where $V^c(\mathcal{T}) := V^{\text{DG}}(\mathcal{T}) \cap H_0^1(\Omega)$ is the continuous Lagrange finite element space. $V^{\text{nc}}(\mathcal{T})$ is a subspace of $V^{\text{DG}}(\mathcal{T})$ so that their quotient space is endowed with the induced topology of the seminorm $\sum_{e \in \mathcal{E}} h_\tau^{-1} \left\| Q_b \llbracket \cdot \rrbracket \right\|_e^2$. When $\ell = 1$, $V^{\text{nc}}(\mathcal{T})$ is simply the well-known Crouzeix–Raviart finite element space [33]. When $\ell \geq 1$, $V^{\text{nc}}(\mathcal{T})$ is a generalization of the Crouzeix–Raviart type nonconforming finite element space [34], which can be viewed a special case of nonconforming virtual element [35] restricted on triangulations:

$$V^{\text{nc}}(\mathcal{T}) := V^{\text{DG}}(\mathcal{T}) \cap H^{1,\text{nc}}(\mathcal{T}), \quad (3.4)$$

where

$$H^{1,\text{nc}}(\mathcal{T}) = \left\{ v \in \prod_{\tau \in \mathcal{T}} H^1(\tau) : \int_e \llbracket v \rrbracket q \, ds = 0 \quad \forall q \in P_{\ell-1}(e), \forall e \in \mathcal{E} \right\}. \quad (3.5)$$

Unlike the virtual element space, where the shape function may not be polynomials, it is shown in [34, 36] that the space $V^{\text{nc}}(\mathcal{T})$ can be obtained from $V^c(\mathcal{T})$ by adding locally supported polynomial bases. More specifically, in [34], the authors constructed a set of nodal basis for a strict subspace of $V^{\text{nc}}(\mathcal{T})$ while satisfying the continuity constraint (3.5). However, to serve the purpose of this article, $V^{\text{nc}}(\mathcal{T})$ is to bridge the proofs, it only suffices to know the existence of a set of unisolvent degrees of freedom, while not explicitly construct the dual basis of it. We refer the reader to [35, Section 3.2] for the degrees of freedom for general polytopes which includes the case of triangulations ($d = 2, 3$) in this article.

When restricted a smaller subspace $V^c(\mathcal{T}) \subset V^{\text{DG}}(\mathcal{T})$, the weak gradient coincides with the piecewise gradient, and the stabilization is vanished which leads to the following partial orthogonality.

Lemma 3.1. *Let u and $u_{\mathcal{T}} \in V_0(\mathcal{T})$ be the solutions of (1.1) and (2.11), respectively, then*

$$(A \nabla u - A \nabla_w u_{\mathcal{T}}, \nabla v^c)_{\mathcal{T}} = 0, \quad \forall v^c \in V^c(\mathcal{T}). \quad (3.6)$$

Proof. It follows from $v^c \in H_0^1(\Omega)$ that $(A \nabla u, \nabla v^c) = (f, v^c)$. In addition, $\llbracket v^c \rrbracket = 0$ for functions when $v^c \in V^c(\mathcal{T}) \subset H_0^1(\Omega)$, thus $Q_b \llbracket v^c \rrbracket = 0$ in $a_{\mathcal{T}}(\cdot, \cdot)$, which further implies $\nabla_w v^c = \nabla v^c$. Lastly since $V^c(\mathcal{T}) \cap H_0^1(\Omega) \subset V_0(\mathcal{T})$ implies that $a_{\mathcal{T}}(u_{\mathcal{T}}, v^c) = (f, v^c)$ for any $v^c \in V^c(\mathcal{T})$, as a result, $(A \nabla_w u, \nabla v^c)_{\mathcal{T}} = (f, v^c)$ and the lemma follows. ■

The following interpolation operator to the nonconforming space will play an important role in the analysis.

Lemma 3.2. *There exist an interpolation operator $I_{\mathcal{T}} : V^{\text{DG}}(\mathcal{T}) \rightarrow V^{\text{nc}}(\mathcal{T})$, which is locally defined and a projection, as well as a constant depending only on the shape regularity of τ such that for all $\tau \in \mathcal{T}$ the following inequality hold: for $|a| = 0, 1$,*

$$\left\| D^a (v_{\mathcal{T}} - I_{\mathcal{T}} v_{\mathcal{T}}) \right\|_{\tau}^2 \lesssim \sum_{e \in \mathcal{E}^{\text{int}}(\omega(\tau))} h_{\tau}^{1-2|a|} \left\| Q_b \llbracket v_{\mathcal{T}} \rrbracket \right\|_e^2, \quad \forall v_{\mathcal{T}} \in V^{\text{DG}}(\mathcal{T}). \quad (3.7)$$

Proof. The proof follows a similar argument as the one in [26, Lemma 6.6], and is presented here for completion. Denote the set of the degrees of freedom functionals in $V^{\text{nc}}(\mathcal{T})$ as $\mathcal{N} = \{\gamma_i(\cdot)\}_{i \in \Lambda}$, and the nodal bases set is $\{\phi_i(\mathbf{x})\}$ corresponding to \mathcal{N} . Let $\omega_i = \text{supp } \phi_i$, where ω_i is either a single element τ or $\omega(e)$, then we consider the following projection $\Pi_i v_{\mathcal{T}}$ obtained through

$$(v_{\mathcal{T}} - \Pi_i v_{\mathcal{T}}, w)_{\omega_i} = 0, \quad \forall w \in V^{\text{nc}}(\omega_i) := \Pi_{\tau \in \omega_i} P_{\ell}(\tau) \cap H^{1,\text{nc}}(\mathcal{T}), \quad (3.8)$$

and then the interpolation is defined as:

$$I_{\mathcal{T}}v_{\mathcal{T}}(\mathbf{x}) := \sum_{i \in \Lambda} \gamma_i(\Pi_i v_{\mathcal{T}}) \phi_i(\mathbf{x}). \tag{3.9}$$

If $v_{\mathcal{T}} \in V^{nc}(\omega_i)$, we have locally $\Pi_i v_{\mathcal{T}} = v_{\mathcal{T}}$ on ω_i , thus $\gamma_i(\Pi_i v_{\mathcal{T}}) = \gamma_i(v_{\mathcal{T}})$, and $I_{\mathcal{T}}$ is a projection.

Moreover, $v_{\mathcal{T}} \in V^{nc}(\omega_i)$ implies that $\sum_{e \in \mathcal{E}^{int}(\omega_i)} h_{\tau}^{-1} \|Q_b \llbracket v_{\mathcal{T}} \rrbracket\|_e^2 = 0$, as well as $v_{\mathcal{T}} - \Pi_i v_{\mathcal{T}} = 0$ by (3.8), hence by a scaling argument and the equivalence of norms on a finite-dimensional space, we have

$$\sum_{\tau \in \omega_i} \|D^a(v_{\mathcal{T}} - \Pi_i v_{\mathcal{T}})\|_{\tau}^2 \lesssim \sum_{e \in \mathcal{E}^{int}(\omega_i)} h_{\tau}^{1-2|a|} \|Q_b \llbracket v_{\mathcal{T}} \rrbracket\|_e^2, \tag{3.10}$$

where we note that if a nodal basis ϕ_i has support $\omega_i = \tau$, both sides of the inequality above will be 0, as the projection (3.8) modulo out the element bubbles.

Next, consider on a $\tau \in \mathcal{T}$, and all local degrees of freedom $\{\gamma_i(\cdot)\}_{i=1}^{N_{\tau}}$ of which the nodal basis having support ω_i overlaps with $\tau \cup \partial\tau$, we have on τ

$$v_{\mathcal{T}} - I_{\mathcal{T}} = (v_{\mathcal{T}} - \Pi_1 v_{\mathcal{T}}) - \sum_{i=1}^{N_{\tau}} \gamma_i(\Pi_1 v_{\mathcal{T}} - \Pi_i v_{\mathcal{T}}) \phi_i. \tag{3.11}$$

On $\tau \cup \partial\tau$, $\gamma_i(v_{\mathcal{T}})$ is either defined on $e \subset \partial\tau$ (see [35, Section 3.2])

$$\gamma_i(v_{\mathcal{T}}) = \frac{1}{|e|} \int_e v_{\mathcal{T}} p \, ds, \forall p \in P_{\ell-1}(e),$$

or defined on τ ($\ell \geq 2$)

$$\gamma_i(v_{\mathcal{T}}) = \frac{1}{|\tau|} \int_{\tau} v_{\mathcal{T}} p \, ds, \forall p \in P_{\ell-2}(\tau),$$

and in both cases, we have by a simple scaling argument and by (3.10)

$$\left\| \gamma_i(\Pi_1 v_{\mathcal{T}} - \Pi_i v_{\mathcal{T}}) \right\|_{\tau} \lesssim \left\| \Pi_1 v_{\mathcal{T}} - \Pi_i v_{\mathcal{T}} \right\|_{\tau} \lesssim \sum_{e \in \mathcal{E}^{int}(\omega_i)} h_{\tau} \|Q_b \llbracket v_0 \rrbracket\|_e^2.$$

Lastly, combining the estimate above with (3.11), using the fact that $\|\phi_i\|_{\infty, \tau} \lesssim 1$, and estimate (3.10) yields the desired estimate. ■

3.2 | Reliability

In this section, the global reliability of the estimator in (3.2) is to be shown. The difference between the weak gradient and classical gradient can be controlled by the jump term which becomes a handy tool in our analysis.

Lemma 3.3. *For $v_{\mathcal{T}} = \{v_0, v_b\} \in V(\mathcal{T})$, it holds for any $\mathcal{M} \subseteq \mathcal{T}$*

$$\left\| A_{\tau}^{1/2} (\nabla_w v_{\mathcal{T}} - \nabla_h v_0) \right\|_{\mathcal{M}}^2 \lesssim \sum_{e \in \mathcal{E}(\mathcal{M})} A_e^{\max} h_{\tau}^{-1} \|Q_b \llbracket v_0 \rrbracket\|_e^2.$$

Proof. The proof follows from [23, Lemma 2.1] with coefficient added and a more localized version is presented. On $\tau \in \mathcal{T}$, let $\mathbf{q} = \nabla_w v_{\mathcal{T}} - \nabla_h v_0 \in P_{l-1}(\tau)$, applying integration by parts on $\nabla_h v_0$ and the weak gradient definition (2.9) on $\nabla_w v_{\mathcal{T}}$, we have

$$\int_{\tau} A_{\tau} (\nabla_w v_{\mathcal{T}} - \nabla_h v_0) \cdot \mathbf{q} \, dx = \int_{\partial\tau} (Q_b \{v_0\} - v_0) A_{\tau} \mathbf{q} \cdot \mathbf{n} \, ds$$

$$\begin{aligned}
&= \int_{\partial\tau} Q_b(\{v_0\} - v_0) A_\tau \mathbf{q} \cdot \mathbf{n} \, ds \\
&= \sum_{e \subset \partial\tau} \pm \frac{1}{2} \int_e Q_b \llbracket v_0 \rrbracket A_\tau \mathbf{q} \cdot \mathbf{n} \, ds,
\end{aligned}$$

where the plus or minus sign depends on whether the outward normal for $e \subset \partial\tau$ coincides with the globally defined normal for this edge/face. Now by a standard trace inequality and an inverse inequality, we have

$$\begin{aligned}
&\left\| A_\tau^{1/2} (\nabla_w v_{\mathcal{T}} - \nabla_h v_0) \right\|_\tau^2 \\
&\lesssim \sum_{e \subset \partial\tau} A_\tau^{1/2} h_\tau^{-1/2} \|Q_b \llbracket v_0 \rrbracket\|_e A_\tau^{-1/2} h_\tau^{1/2} \|A_\tau \mathbf{q} \cdot \mathbf{n}\|_e \\
&\lesssim \left(\sum_{e \subset \partial\tau} A_e^{\max} h_\tau^{-1} \|Q_b \llbracket v_0 \rrbracket\|_e^2 \right)^{1/2} \left\| A_\tau^{1/2} (\nabla_w v_{\mathcal{T}} - \nabla_h v_0) \right\|_\tau,
\end{aligned}$$

then the desired result follows by canceling a $\left\| A_\tau^{1/2} (\nabla_w v_{\mathcal{T}} - \nabla_h v_0) \right\|_\tau$ and summing up the element-wise estimate for every $\tau \in \mathcal{M}$. \blacksquare

Next we bound the stabilization term by the element-wise residual and the normal jump of the weak flux. We note that in [22], though focusing on a different model problem, the h^{-1} -weighted solution jump can be used as the sole error estimator to guarantee reliability and efficiency up to data oscillation. Nevertheless, the motivation here is to change the dependence of the error indicators on the local mesh size h from h^{-1} to h^s , $s > 0$, so that a contraction property of the estimator can be proved in Section 4.2 without any saturation assumptions, which is one of the keys to show the convergence of an adaptive algorithm.

Lemma 3.4. *Let u be the weak solution of (1.1) and $u_{\mathcal{T}} = \{u_0, u_b\} \in V_0(\mathcal{T})$ be the solution to (2.11), we have*

$$\begin{aligned}
\sum_{e \in \mathcal{E}} A_e^{\max} h_\tau^{-1} \|Q_b \llbracket u_0 \rrbracket\|_e^2 &\lesssim \left(\sum_{\tau \in \mathcal{T}} h_\tau^2 A_\tau^{-1} \|R(\nabla_w u_{\mathcal{T}})\|_{0,\tau}^2 \right. \\
&\quad \left. + \sum_{e \in \mathcal{E}} h_\tau (A_e^{\max})^{-1} \|J_{n,e}(A \nabla_w u_{\mathcal{T}})\|_e^2 \right), \tag{3.12}
\end{aligned}$$

where the constant depends on the shape regularity of \mathcal{T} .

Proof. For simplicity, we denote $A_e := A_e^{\max}$ in the proof. Let that $I_{\mathcal{T}} u_{\mathcal{T}} \in V^{\text{nc}}(\mathcal{T})$ be the nonconforming interpolation defined in Lemma 3.2, using the definition of the modified weak derivative (2.9) element-wisely, we have

$$\begin{aligned}
\sum_{e \in \mathcal{E}} A_e h_\tau^{-1} \|Q_b \llbracket u_0 \rrbracket\|_e^2 &= \sum_{e \in \mathcal{E}} A_e h_\tau^{-1} \langle Q_b \llbracket u_0 \rrbracket, Q_b \llbracket u_{\mathcal{T}} - I_{\mathcal{T}} u_{\mathcal{T}} \rrbracket \rangle_e \\
&= (f, u_{\mathcal{T}} - I_{\mathcal{T}} u_{\mathcal{T}})_{\mathcal{T}} - (A \nabla_w u_{\mathcal{T}}, \nabla_w (u_{\mathcal{T}} - I_{\mathcal{T}} u_{\mathcal{T}}))_{\mathcal{T}} \\
&= (f, u_{\mathcal{T}} - I_{\mathcal{T}} u_{\mathcal{T}})_{\mathcal{T}} + (\nabla_h \cdot (A \nabla_w u_{\mathcal{T}}), u_{\mathcal{T}} - I_{\mathcal{T}} u_{\mathcal{T}})_{\mathcal{T}} \\
&\quad - \sum_{\tau \in \mathcal{T}_h} \langle (A \nabla_w u_{\mathcal{T}}) \cdot \mathbf{n}, \{u_{\mathcal{T}} - I_{\mathcal{T}} u_{\mathcal{T}}\} \rangle_{\partial\tau}
\end{aligned}$$

$$\begin{aligned}
 &= (f + \nabla_h \cdot (A \nabla_w u_{\mathcal{T}}), u_{\mathcal{T}} - I_{\mathcal{T}} u_{\mathcal{T}})_{\mathcal{T}} \\
 &\quad - \sum_{e \in \mathcal{E}} \langle \llbracket (A \nabla_w u_{\mathcal{T}}) \cdot \mathbf{n} \rrbracket, \{u_{\mathcal{T}} - I_{\mathcal{T}} u_{\mathcal{T}}\} \rangle_e.
 \end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
 &\sum_{e \in \mathcal{E}} A_e h_{\tau}^{-1} \|Q_b \llbracket u_0 \rrbracket\|_e^2 \\
 &\leq \left(\sum_{\tau \in \mathcal{T}} h_{\tau}^2 A_{\tau}^{-1} \|f + \nabla \cdot (A_{\tau} \nabla_w u_{\mathcal{T}})\|_{\tau}^2 \right)^{1/2} \left(\sum_{\tau \in \mathcal{T}} h_{\tau}^{-2} \|A_{\tau}^{1/2} (u_{\mathcal{T}} - I_{\mathcal{T}} u_{\mathcal{T}})\|_{\tau}^2 \right)^{1/2} \\
 &\quad + \left(\sum_{e \in \mathcal{E}} h_{\tau} A_e^{-1} \|\llbracket A \nabla_w u_{\mathcal{T}} \cdot \mathbf{n} \rrbracket\|_e^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} h_{\tau}^{-1} A_e \|u_{\mathcal{T}} - I_{\mathcal{T}} u_{\mathcal{T}}\|_e^2 \right)^{1/2} \\
 &\lesssim \eta(\nabla_w u_{\mathcal{T}}, \mathcal{T}) \cdot \left(\sum_{e \in \mathcal{E}} h_{\tau}^{-1} A_e \|Q_b \llbracket u_0 \rrbracket\|_e^2 \right)^{1/2},
 \end{aligned}$$

lastly applying Lemma 3.2 yields the result. ■

The proof of the upper bound mainly follows the paradigm in [18, 37, 38]. Without loss of generality, the presentation is for $d = 3$. We shall use the Helmholtz decomposition of $\nabla u - \nabla_w u_{\mathcal{T}}$.

Theorem 3.5 (Upper Bound). *Let u be the solution of (1.1) and $u_{\mathcal{T}} \in V_0(\mathcal{T})$ be the solution of (2.11), then*

$$\|A^{1/2}(\nabla u - \nabla_w u_{\mathcal{T}})\|_{\mathcal{T}}^2 \leq C_U \eta^2(\nabla_w u_{\mathcal{T}}, \mathcal{T}), \tag{3.13}$$

where the constant C_U depends on the shape regularity of \mathcal{T} and the ratio of the coefficient A across elements.

Proof. We first give an outline of our proof. The following Helmholtz decomposition ([39, 40], see also [18, Lemma 4.2], Chapter I Theorem 3.4 and Remark 3.10 in [41]) commonly used for nonconforming elements is applied to $\nabla u - \nabla_w u_{\mathcal{T}}$:

$$\nabla u - \nabla_w u_{\mathcal{T}} = \nabla \psi + A^{-1} \nabla \times \xi, \tag{3.14}$$

where $\psi \in H_0^1(\Omega)$, and $\xi \in \mathbf{H}(\mathbf{curl}; \Omega)$. The decomposition satisfies

$$\left\| A^{1/2}(\nabla u - \nabla_w u_{\mathcal{T}}) \right\|_{\mathcal{T}}^2 = \left\| A^{1/2} \nabla \psi \right\|_{\mathcal{T}}^2 + \left\| A^{-1/2} \nabla \times \xi \right\|_{\mathcal{T}}^2. \tag{3.15}$$

Moreover, ξ can be chosen to be divergence-free so that it is piecewise \mathbf{H}^1 -smooth (e.g., see [42, Appendix A]) such that

$$\|A^{-1/2} \nabla \xi\|_{\mathcal{T}} \lesssim \|A^{-1/2} \nabla \times \xi\|_{\mathcal{T}}. \tag{3.16}$$

- 1 For ψ in (3.14), we shall construct a quasi-interpolant to $V^c(\mathcal{T})$ and use the orthogonality (3.6). This part is similar to the *a posteriori* error estimator of the conforming finite element which can be controlled by the element-wise residual and normal jump of the numerical flux.
- 2 For ξ in (3.14), we shall construct a similar quasi-interpolant, but this time leads to the jump of tangential derivatives. Comparing to results for WG [18], the element-wise $\nabla_h \times \nabla_w u_{\mathcal{T}}$ will be bounded by the stabilization term.

A key result (Lemma 3.4) is applied here to bound the stabilization term by the element-wise residual and the normal jump of the flux, where the quasi-interpolant to $V^{\text{nc}}(\mathcal{T})$ is used. As a result, we turn our focus onto the error without the stabilization as follows

$$\left\| A^{1/2}(\nabla u - \nabla_w u_{\mathcal{T}}) \right\|_{\mathcal{T}}^2 = \underbrace{(A(\nabla u - \nabla_w u_{\mathcal{T}}), \nabla \psi)_{\mathcal{T}}}_{=: \mathfrak{F}_C} + \underbrace{(\nabla u - \nabla_w u_{\mathcal{T}}, \nabla \times \xi)_{\mathcal{T}}}_{=: \mathfrak{F}_N}. \quad (3.17)$$

Let $\psi_{\mathcal{T}} \in V^c(\mathcal{T})$ be the robust Clément-type quasi-interpolation (e.g., see [43]) for ψ such that the following estimates holds,

$$\begin{aligned} A_{\tau} \|\psi - \psi_{\mathcal{T}}\|_{\tau}^2 &\lesssim h_{\tau}^2 \left\| A^{1/2} \nabla \psi \right\|_{\omega(\tau)}^2, \\ \text{and } A_e^{\max} \|\psi - \psi_{\mathcal{T}}\|_e^2 &\lesssim h_{\tau} \left\| A^{1/2} \nabla \psi \right\|_{\omega(e)}^2. \end{aligned} \quad (3.18)$$

Using the partial orthogonality (3.6), integrating by parts for \mathfrak{F}_C , Cauchy–Schwarz inequality, and estimates in (3.18), we have

$$\begin{aligned} \mathfrak{F}_C &= (A(\nabla u - \nabla_w u_{\mathcal{T}}), \nabla(\psi - \psi_{\mathcal{T}}))_{\mathcal{T}} \\ &= -(\nabla_h \cdot A(\nabla u - \nabla_w u_{\mathcal{T}}), \psi - \psi_{\mathcal{T}})_{\mathcal{T}} + \sum_{\tau \in \mathcal{T}} \langle (A(\nabla u - \nabla_w u_{\mathcal{T}}) \cdot \mathbf{n}, \psi - \psi_{\mathcal{T}})_{\partial\tau} \rangle \\ &= (f + \nabla_h \cdot (A \nabla_w u_{\mathcal{T}}), \psi - \psi_{\mathcal{T}})_{\mathcal{T}} - \sum_{e \in \mathcal{E}} \langle \llbracket (A \nabla_w u_{\mathcal{T}}) \cdot \mathbf{n} \rrbracket, \psi - \psi_{\mathcal{T}} \rangle_e \\ &\leq \left(\sum_{\tau \in \mathcal{T}} h_{\tau}^2 A_{\tau}^{-1} \|f + \nabla \cdot (A \nabla_w u_{\mathcal{T}})\|_{\tau}^2 \right)^{1/2} \left(\sum_{\tau \in \mathcal{T}} h_{\tau}^{-2} A_{\tau} \|\psi - \psi_{\mathcal{T}}\|_{\tau}^2 \right)^{1/2} \\ &\quad + \left(\sum_{e \in \mathcal{E}} A_e^{-1} h_{\tau} \llbracket A \nabla_w u_{\mathcal{T}} \cdot \mathbf{n} \rrbracket_e^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} A_e h_{\tau}^{-1} \|\psi - \psi_{\mathcal{T}}\|_e^2 \right)^{1/2} \\ &\lesssim \eta(\nabla_w u_{\mathcal{T}}, \mathcal{T}) \|A^{1/2} \nabla \psi\|_{\mathcal{T}} \leq \eta(\nabla_w u_{\mathcal{T}}, \mathcal{T}) \left\| A^{1/2}(\nabla u - \nabla_w u_{\mathcal{T}}) \right\|_{\mathcal{T}}, \end{aligned}$$

where the constant depends on the shape-regularity of \mathcal{T} .

For \mathfrak{F}_N in (3.17), we need a robust Clément-type interpolation $\xi_{\mathcal{T}} \in \mathbf{H}(\mathbf{curl}; \Omega)$ (see [42, Theorem 4.6]) satisfying:

$$\begin{aligned} A_{\tau}^{-1} \|\xi - \xi_{\mathcal{T}}\|_{\tau}^2 &\lesssim \sum_{\tau \subset \omega(\tau)} h_{\tau}^2 \left\| A^{-1/2} \nabla \xi \right\|_{\tau}^2, \\ \text{and } (A_e^{\min})^{-1} \|\xi - \xi_{\mathcal{T}}\|_e^2 &\lesssim \sum_{\tau \subset \omega(e)} h_{\tau}^2 \left\| A^{1/2} \nabla \xi \right\|_{\tau}^2. \end{aligned} \quad (3.19)$$

It is straightforward to verify that, by the definition of the modified weak gradient (2.9) and the fact that $\nabla \times \xi_{\mathcal{T}} \in \mathbf{H}(\text{div}; \Omega)$, we have

$$(\nabla_w u_{\mathcal{T}}, \nabla \times \xi_{\mathcal{T}})_{\mathcal{T}} = -(u_0, \nabla_h \cdot (\nabla \times \xi_{\mathcal{T}}))_{\mathcal{T}} + \sum_{\tau \in \mathcal{T}} \langle Q_b \{u_0\}, \nabla \times \xi_{\mathcal{T}} \cdot \mathbf{n} \rangle_{\partial\tau} = 0.$$

Consequently, as $\nabla u \in \mathbf{H}(\mathbf{curl}; \Omega)$ and $\xi_{\mathcal{T}}$ can be inserted to \mathfrak{F}_N obtain the following:

$$\mathfrak{F}_N = -(\nabla_w u_{\mathcal{T}}, \nabla \times \xi)_{\mathcal{T}} = -(\nabla_w u_{\mathcal{T}}, \nabla \times (\xi - \xi_{\mathcal{T}}))_{\mathcal{T}}. \quad (3.20)$$

Upon an integration by parts, $\nabla_h \times (\nabla_w u_{\mathcal{T}})$ will appear in each element, in general, this is not zero as $\nabla_w u_{\mathcal{T}}$ is computed in the sense of distribution. To handle this term, simply

notice that $\nabla_h \times (\nabla_h u_0) = 0$ element-wise, by Lemmas 3.3 and 3.4, and an inverse inequality

$$\begin{aligned} \sum_{\tau \in \mathcal{T}} h_\tau^2 \|A^{1/2} \nabla \times (\nabla_w u_{\mathcal{T}})\|_\tau^2 &= \sum_{\tau \in \mathcal{T}} h_\tau^2 \|A^{1/2} \nabla \times (\nabla_h u_0 - \nabla_w u_{\mathcal{T}})\|_\tau^2 \\ &\lesssim \|A^{1/2} (\nabla_h u_0 - \nabla_w u_{\mathcal{T}})\|_{\mathcal{T}} \\ &\lesssim \eta(\nabla_w u_{\mathcal{T}}, \mathcal{T}). \end{aligned} \tag{3.21}$$

Integrating by parts on (3.20), applying (3.21), we have

$$\begin{aligned} &(\nabla_w u_{\mathcal{T}}, \nabla \times (\xi - \xi_{\mathcal{T}}))_{\mathcal{T}} \\ &= (\nabla \times (\nabla_w u_{\mathcal{T}}), \xi - \xi_{\mathcal{T}})_{\mathcal{T}} - \sum_{e \in \mathcal{E}} \langle [\mathbf{n} \times \nabla_w u_{\mathcal{T}}], \xi - \xi_{\mathcal{T}} \rangle_e \\ &\leq \left(\sum_{\tau \in \mathcal{T}} h_\tau^2 A_\tau \|\nabla \times (\nabla_w u_{\mathcal{T}})\|_\tau^2 \right)^{1/2} \left(\sum_{\tau \in \mathcal{T}} h_\tau^{-2} A_\tau^{-1} \|\xi - \xi_{\mathcal{T}}\|_\tau^2 \right)^{1/2} \\ &\quad + \left(\sum_{e \in \mathcal{E}} h_e A_e^{\min} \|\llbracket \nabla_w u_{\mathcal{T}} \times \mathbf{n} \rrbracket\|_e^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}} h_e^{-1} (A_e^{\min})^{-1} \|\xi - \xi_{\mathcal{T}}\|_e^2 \right)^{1/2} \\ &\lesssim \eta(\nabla_w u_{\mathcal{T}}, \mathcal{T}) \|A^{-1/2} \nabla \xi\|_{\mathcal{T}}. \end{aligned}$$

Finally by (3.16), the reliability (3.13) follows. ■

By Lemma 3.4 and Theorem 3.5, the upper bound of the error (2.15) follows.

Corollary 3.6. *Let u be the solution of (1.1) and $u_{\mathcal{T}} \in V_0(\mathcal{T})$ be the solution of (2.11), then the error (2.15)*

$$\| \|u - u_{\mathcal{T}}\| \|_{\mathcal{T}}^2 \lesssim \eta^2(\nabla_w u_{\mathcal{T}}, \mathcal{T}), \tag{3.22}$$

where the constant depends on the shape regularity of \mathcal{T} and the ratio of the coefficient A across neighboring elements.

3.3 | Efficiency

The standard bubble function technique is opted (see [44]) to derive the efficiency bound, while the tangential jump part’s proof follows a standard argument of the *a posteriori* error estimation for standard WG discretization in [18]. As the proofs are standard, we only present the results here.

For $v_{\mathcal{T}} \in V(\mathcal{T})$ and $\tau \in \mathcal{T}$, the oscillation is defined to be

$$\text{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}, \tau) := h_\tau^2 \|(Id - Q_m)R(\nabla_w v_{\mathcal{T}})\|_\tau^2, \tag{3.23}$$

where Q_m denotes the L^2 projection onto the set of either the piecewise $P_m(\tau)$ on $\tau \in \mathcal{T}$, where $m = \ell - 2$ if $\ell \geq 2$ and $m = 0$ when $\ell = 1$.

For any subset $\mathcal{M} \subseteq \mathcal{T}$, we define

$$\text{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}, \mathcal{M}) := \sum_{\tau \in \mathcal{M}} \text{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}, \tau). \tag{3.24}$$

Note that, on $\tau \in \mathcal{T}$, using the properties of the L^2 -projection, the oscillation are dominated by the estimator; namely,

$$\text{osc}_{\mathcal{T}}^2(v_{\mathcal{T}}, \tau) \leq \eta_{\mathcal{T}}^2(v_{\mathcal{T}}, \tau), \text{ for } v_{\mathcal{T}} \in V(\mathcal{T}). \tag{3.25}$$

Theorem 3.7 (Local lower bound). *For all $\tau \in \mathcal{T}$, there holds*

$$C_L \eta(\nabla_w u_{\mathcal{T}}, \tau) \leq \|A^{1/2}(\nabla u - \nabla_w u_{\mathcal{T}})\|_{\mathcal{T}} + \text{osc}_{\mathcal{T}}^2(u_{\mathcal{T}}, \mathcal{T}), \quad (3.26)$$

where the constant C_L only depends on the shape regular of \mathcal{T} .

4 | AN ADAPTIVE MODIFIED WEAK GALERKIN METHOD

In this section, first we introduce an adaptive modified weak Galerkin method (AmWG). Next, a quasi-orthogonality is proved and is further exploited to derive the convergence of AmWG. At last, we shall present the discrete reliability and propose the quasi-optimality of the AmWG.

Henceforth, the polynomial degree is chosen to be $\ell = 1$, the reason for this is that we shall borrow some classical results for Crouzeix–Raviart element to establish a penalty parameter-free convergence. Notice that the penalty parameters in schemes based on discontinuous approximation spaces are indispensable not only for the coercivity (Remark 2.4), but also for the convergence of adaptive algorithms due to the lack of a direct orthogonality result (see e.g., [26]). For a similar nonconforming method [45], one still needs to choose a sufficiently large penalty parameter to prove the convergence. By bridging the connections between the lowest order WG method and Crouzeix–Raviart element, we are able to show the convergence without the presence of a sufficiently large penalty.

4.1 | Algorithm

In the **SOLVE** step, given a function $f \in L^2(\Omega)$ and a triangulation \mathcal{T} , the exact discrete solution is sought $u_{\mathcal{T}} = \text{SOLVE}(\mathcal{T}, f)$. In this step, we assume that the discrete linear system associated with problem (2.11) can be solved exactly.

In the **ESTIMATE** step, local error indicators $\{\eta(\nabla_w u_{\mathcal{T}}, \tau)\}_{\tau \in \mathcal{T}}$ and the global estimator $\eta(\nabla_w u_{\mathcal{T}}, \mathcal{T})$ are calculated.

Algorithm 1. An adaptive modified weak Galerkin finite element (AmWG) cycle $[u_J, \mathcal{T}_J] = \text{AmWG}(\mathcal{T}_0, f, \text{tol}, \theta)$

Input: $\mathcal{T}_0, f, \text{tol}, \theta \in (0, 1)$.

Output: \mathcal{T}_J, u_J .

- 1: $\eta = 1, k = 0$.
- 2: **while** $\eta \geq \text{tol}$ **do**
- 3: **SOLVE:** Solve (2.11) on \mathcal{T}_k to get the solution u_k ;
- 4: **ESTIMATE:** Compute $\eta = \eta(\nabla_w u_k, \mathcal{T}_k)$;
- 5: **MARK:** Seek a minimum cardinality $\mathcal{M}_k \subseteq \mathcal{T}_k$ such that

$$\eta^2(\nabla_w u_k, \mathcal{M}_k) \geq \theta \eta^2(\nabla_w u_k, \mathcal{T}_k); \quad (4.1)$$

- 6: **REFINE:** Bisect/quadsect elements in \mathcal{M}_k and the neighboring elements to form a conforming \mathcal{T}_{k+1} ;
 - 7: $k \leftarrow k + 1$
 - 8: **end while**
 - 9: $u_J = u_k; \mathcal{T}_J = \mathcal{T}_k$.
-

In the **MARK** step, a set of marked elements is obtained by the Dörfler marking strategy [46] applied on the error indicators $\{\eta(\nabla_w u_{\mathcal{T}}, \tau)\}_{\tau \in \mathcal{T}}$ on \mathcal{T} obtained in the **ESTIMATE** step.

In the **REFINE** step, different from traditional DG approaches which allow hanging nodes (e.g., [26]), the marked elements, as well as their neighbors, are refined using bisection ($d = 2, 3$) or red-green refinement ($d = 2$) while preserving the *conformity* of the triangulation.

In the paragraphs hereafter, the notation $\mathcal{T}_1 \leq \mathcal{T}_2$ stands for that \mathcal{T}_2 is a refinement of \mathcal{T}_1 following the marking strategy above, where $\mathcal{T}_1, \mathcal{T}_2 \in \mathcal{C}(\mathcal{T}_0)$, and here $\mathcal{C}(\mathcal{T}_0)$ denotes the set of triangulations which are conforming (no hanging nodes), shape regular and refined from an initial triangulation \mathcal{T}_0 .

While showing the lemmas related to the convergence of the AmWG, for $\mathcal{T}, \mathcal{T}_* \in \mathcal{C}(\mathcal{T}_0)$ and $\mathcal{T} \leq \mathcal{T}_*$, the set of refined elements in \mathcal{T} , which become new elements in \mathcal{T}_* , is denoted as

$$\mathcal{R}_{\mathcal{T} \rightarrow \mathcal{T}_*} := \{\tau \in \mathcal{T} : \tau \notin \mathcal{T}_*\} \subset \mathcal{T}.$$

Whenever the dependence of the weak gradient on two different meshes becomes relevant, the weak gradient's notation is changed accordingly to emphasize the mesh of the function defined on, for example, the piecewisely defined weak gradient is $\nabla_{w,*} v$ for any $v \in \mathcal{T}_*$. When its restriction to one element $\tau \in \mathcal{T}$ is of interest, $\nabla_{w,\tau} v_{\mathcal{T}}$ is used.

Remark 4.1. The reason why we opt for a notation $\eta(\nabla_w v_{\mathcal{T}}, \mathcal{M})$, not $\eta(v_{\mathcal{T}}, \mathcal{M})$ is as follows. As in the context of the convergence analysis, this chosen notation has a more consistent meaning when considering two meshes: one is refined from the other. Note that for two nested triangulation $\mathcal{T} \leq \mathcal{T}_*$, the weak gradient of a coarse function $v_{\mathcal{T}} \in V(\mathcal{T})$ on the fine mesh \mathcal{T}_* is different with the weak gradient on the coarse grid \mathcal{T} . To be specific, on $\tau_* \subset \tau \in \mathcal{T}$ with $\tau_* \notin \mathcal{T}$, $\nabla_{w,\tau_*} v_{\mathcal{T}}$ is different from $(\nabla_{w,\tau} v_{\mathcal{T}})|_{\tau_*}$. We note that this is different from that of the piecewise gradient ∇_h .

4.2 | Reduction of error estimator

By $\ell = 1$, the error estimator defined in (3.1) can be split as

$$\eta^2(\nabla_w v_{\mathcal{T}}, \tau) := F(f, \tau) + \tilde{\eta}^2(\nabla_w v_{\mathcal{T}}, \tau), \tag{4.2}$$

where

$$F(f, \tau) := h_{\tau}^2 A_{\tau}^{-1} \|f\|_{0,\tau}^2;$$

and

$$\tilde{\eta}^2(\nabla_w v_{\mathcal{T}}, \tau) := \sum_{e \subset \partial \tau} h_{\tau} \int_e ((A_e^{\max})^{-1} J_{n,e}^2(A \nabla_w v_{\mathcal{T}}) + A_e^{\min} J_{t,e}^2(\nabla_w v_{\mathcal{T}})) \, ds.$$

For any subset $\mathcal{M} \subset \mathcal{T}$, define

$$F(f, \mathcal{M}) = \sum_{\tau \in \mathcal{M}} F(f, \tau); \quad \tilde{\eta}^2(v_{\mathcal{T}}, \mathcal{M}) = \sum_{\tau \in \mathcal{M}} \tilde{\eta}^2(v_{\mathcal{T}}, \tau).$$

The next lemma shows the reduction of the error estimator after the mesh is refined. On a refined mesh, the effect of changing the finite element function for $\tilde{\eta}^2(v_{\mathcal{T}}, \mathcal{T})$ is as follows.

Lemma 4.2. *For $\mathcal{T}, \mathcal{T}_* \in \mathcal{C}(\mathcal{T}_0)$ with $\mathcal{T} \leq \mathcal{T}_*$, for any $\zeta \in (0, 1)$, $v \in V(\mathcal{T})$, and $v_* \in V(\mathcal{T}_*)$, there exists a constant C_E depending on the shape regularity \mathcal{T}_* such that*

$$\begin{aligned} & \tilde{\eta}^2(\nabla_{w,*} v_*, \mathcal{T}_*) \\ & \leq (1 + \zeta) \tilde{\eta}^2(\nabla_{w,\tau} v_{\mathcal{T}}, \mathcal{T}_*) + C_E (1 + \zeta^{-1}) \|A^{1/2}(\nabla_{w,*} v_* - \nabla_{w,\tau} v_{\mathcal{T}})\|_{\mathcal{T}_*}^2. \end{aligned} \tag{4.3}$$

We can also get the contraction of $\tilde{\eta}^2(\nabla_{w,\tau}u_{\mathcal{T}}, \mathcal{T})$ if the weak flux of the solution remains invariant and is interpolated into a finer mesh refined using the Dörfler marking strategy.

Lemma 4.3. For $\mathcal{T}, \mathcal{T}_* \in C(\mathcal{T}_0)$ with $\mathcal{T} \leq \mathcal{T}_*$. Let $u_{\mathcal{T}} \in V_0(\mathcal{T})$ be the solution to (2.11). For $\lambda \in (0, 1)$ defined in Lemma 4.5, we have

$$\tilde{\eta}^2(\nabla_{w,\tau}u_{\mathcal{T}}, \mathcal{T}_*) \leq \tilde{\eta}^2(\nabla_{w,\tau}u_{\mathcal{T}}, \mathcal{T}) - \lambda \tilde{\eta}^2(\nabla_{w,\tau}u_{\mathcal{T}}, \mathcal{R}_{\mathcal{T} \rightarrow \mathcal{T}_*}). \quad (4.4)$$

Here we skip the proof of Lemmas 4.2 and 4.3, since the corresponding techniques are quite standard and can be found, for example, in [47].

Hereafter the following short notations are adopted: on \mathcal{T}_k , $\nabla_{w,k}$ denotes the weak gradient, and ∇_k denotes the piecewise gradient ∇_h , for quantities involving two levels of meshes, the subscript follows that of the coarse one.

$$\begin{aligned} \varepsilon_k &= \|A^{1/2}(\nabla u - \nabla_{w,k}u_{\mathcal{T}_k})\|^2; \\ E_k &= \|A^{1/2}(\nabla_{w,k}u_{\mathcal{T}_k} - \nabla_{w,k+1}u_{\mathcal{T}_{k+1}})\|^2; \\ R_k &= \mathcal{R}_{\mathcal{T}_k \rightarrow \mathcal{T}_{k+1}}; \\ \eta_k &= \eta(\nabla_{w,k}u_{\mathcal{T}_k}, \mathcal{T}_k); \quad \eta_{R_k} = \eta(\nabla_{w,k}u_{\mathcal{T}_k}, \mathcal{R}_{\mathcal{T}_k \rightarrow \mathcal{T}_{k+1}}); \\ \tilde{\eta}_k &= \tilde{\eta}(\nabla_{w,k}u_{\mathcal{T}_k}, \mathcal{T}_k); \quad \tilde{\eta}_{R_k} = \tilde{\eta}(\nabla_{w,k}u_{\mathcal{T}_k}, \mathcal{R}_{\mathcal{T}_k \rightarrow \mathcal{T}_{k+1}}); \\ F_k &= F(f, \mathcal{T}_k); \quad F_{R_k} = F(f, \mathcal{R}_{\mathcal{T}_k \rightarrow \mathcal{T}_{k+1}}). \end{aligned}$$

The following lemma summarizes the contraction of $\tilde{\eta}^2(\cdot, \cdot)$ by using Lemmas 4.2 and 4.3.

Lemma 4.4. For the two consecutive triangulation $\mathcal{T}_k \leq \mathcal{T}_{k+1}$ in the AmWG cycle (1), and any $\zeta \in (0, 1)$, there exists a constant $\rho > 0$ depending on the shape regularity \mathcal{T}_{k+1} such that

$$\tilde{\eta}_{k+1}^2 \leq (1 + \zeta)(\tilde{\eta}_k^2 - \lambda \tilde{\eta}_{R_k}^2) + E_k/\rho. \quad (4.5)$$

Proof. Let $\mathcal{T} = \mathcal{T}_k$ and $\mathcal{T}_* = \mathcal{T}_{k+1}$ in Lemmas 4.2 and 4.3, respectively. Then the desired result (4.5) follows from letting $\rho^{-1} = C_E(1 + \zeta^{-1})$. ■

For the contraction of $F(\cdot, \cdot)$, there is no extra ζ factor as an artifact of Young's inequality, and this will play a key role in proving the convergence without any penalty parameter on the stabilization.

Lemma 4.5. Let \mathcal{T}_{k+1} be the refinement of \mathcal{T}_k produced in Algorithm 1. There exists a constant $\lambda \in (0, 1)$ satisfying

$$F_{k+1} \leq F_k - \lambda F_{R_k}. \quad (4.6)$$

Proof. For any $\tau \in \mathcal{T} \setminus \mathcal{T}_*$, we only need to consider the case where τ is subdivided into $\tau_*^1, \tau_*^2 \in \mathcal{T}_*$ with $h_{\tau_*^d}^d = h_{\tau_*^d}^d = \frac{1}{2}h_{\tau_*^d}^d$ ($d = 2, 3$), we have

$$\begin{aligned} F_{k+1} &= \sum_{\tau \in \mathcal{T}_{k+1}} h_{k+1}^2 A_{k+1}^{-1} \|f\|_{0,\tau}^2 \\ &= \sum_{\tau \in \mathcal{T}_{k+1} \cap \mathcal{T}_k} h_{k+1}^2 A_{k+1}^{-1} \|f\|_{0,\tau}^2 + \sum_{\tau \in \mathcal{T}_{k+1} \setminus \mathcal{T}_k} h_{k+1}^2 A_{k+1}^{-1} \|f\|_{0,\tau}^2 \\ &\leq \sum_{\tau \in \mathcal{T}_k} h_k^2 A_k^{-1} \|f\|_{0,\tau}^2 - \sum_{\tau \in \mathcal{T}_k \setminus \mathcal{T}_{k+1}} h_k^2 A_k^{-1} \|f\|_{0,\tau}^2 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{\tau \in \mathcal{T}_k \setminus \mathcal{T}_{k+1}} 2^{-1/d} h_k^2 A_k^{-1} \|f\|_{0,\tau}^2 \\
 &= F_k - \lambda F_{R_k},
 \end{aligned}$$

where $\lambda := 1 - 2^{-1/d} \in (0, 1)$. ■

The following lemma summarizes the contraction of error estimator $\eta^2(\cdot, \cdot)$ by using Lemmas 4.4 and 4.5.

Lemma 4.6. *There exist constants $\zeta \in (0, 1)$ and $\rho > 0$ such that*

$$\eta_{k+1}^2 \leq (1 + \zeta)(1 - \theta\lambda)\eta_k^2 - \zeta F_k + \zeta \lambda F_{R_k} + E_k/\rho, \tag{4.7}$$

where the parameters θ and λ are given in the marking strategy (4.1), Lemmas 4.5 and 4.4, respectively.

Proof. Making use of the simplified notation of $\eta(\cdot, \cdot)$ in (4.2), Lemmas 4.5 and 4.4, we have

$$\begin{aligned}
 \eta_{k+1}^2 &= \tilde{\eta}_{k+1}^2 + F_{k+1} \\
 &\leq (1 + \zeta) (\tilde{\eta}_k^2 - \lambda \tilde{\eta}_{R_k}^2) + F_k - F_{R_k} + E_k/\rho \\
 &\leq (1 + \zeta) (\eta_k^2 - \lambda \eta_{R_k}^2) - \zeta (F_k - \lambda F_{R_k}) + E_k/\rho.
 \end{aligned} \tag{4.8}$$

By (4.1) and \mathcal{T}_{k+1} being refined at least once from \mathcal{T}_k , we have $\eta_{R_k}^2 \geq \theta \eta_k^2$. In conjunction with (4.8), we obtain

$$\eta_{k+1}^2 \leq (1 + \zeta)(1 - \theta\lambda)\eta_k^2 - \zeta F_k + \zeta \lambda F_{R_k} + E_k/\rho,$$

which completes the proof. ■

4.3 | Quasi-orthogonality

In this section, we will show the contraction property of the energy error by using similar arguments in [28].

First, a canonical interpolation operator $I_{\mathcal{T}}^{\text{CR}}$ is defined any $v \in H_0^1(\Omega)$: $I_{\mathcal{T}}^{\text{CR}} v \in V^{\text{nc}}(\mathcal{T})$ satisfies

$$\int_e I_{\mathcal{T}}^{\text{CR}} v = \int_e v, \quad \forall e \in \mathcal{E}, \tag{4.9}$$

and the interpolation admits the following estimate:

$$\left\| v - I_{\mathcal{T}}^{\text{CR}} v \right\|_{\tau} \lesssim h_{\tau} \|\nabla v\|_{\tau}, \quad \forall \tau \in \mathcal{T}, \tag{4.10}$$

where the constant depends only on the shape regularity of τ .

Lemma 4.7. *Assume that $\sigma_{\mathcal{T}}$ is a constant vector on each element $\tau \in \mathcal{T}$, we have*

$$(\nabla_h v, \sigma_{\mathcal{T}})_{\mathcal{T}} = (\nabla_h I_{\mathcal{T}}^{\text{CR}} v, \sigma_{\mathcal{T}})_{\mathcal{T}}. \tag{4.11}$$

Proof. Note that $\sigma_{\mathcal{T}}$ is a constant vector on each $\tau \in \mathcal{T}$, $\nabla \cdot (\sigma_{\mathcal{T}}|_{\tau}) = 0$. By applying integration by parts, we will get the desired result. ■

For $v_0 \in V^{\text{DG}}(\mathcal{T})$, it is embedded into $V(\mathcal{T})$ by $(v_0, Q_b\{v_0\})$. Denote the interpolant $I_{\mathcal{T}}^{\text{CR}}$ for $V^{\text{nc}}(\mathcal{T})$ satisfying

$$\int_e I_{\mathcal{T}}^{\text{CR}} v_0 = \int_e Q_b\{v_0\}, \quad \forall e \in \mathcal{E}. \quad (4.12)$$

Lemma 4.8. For any $v_0 \in V^{\text{DG}}(\mathcal{T})$ and $v_{\mathcal{T}} = \{v_0, Q_b\{v_0\}\} \in V(\mathcal{T})$, the interpolation defined in (4.12) satisfies

$$\nabla_{w,\tau} v_{\mathcal{T}} = \nabla_h I_{\mathcal{T}}^{\text{CR}} v_0. \quad (4.13)$$

Proof. For any constant vector $\sigma_{\mathcal{T}}$, using the definition (2.3) leads to

$$\begin{aligned} (\nabla_{w,\tau} v_{\mathcal{T}}, \sigma_{\mathcal{T}})_{\tau} &= (v_0, \nabla \cdot \sigma_{\mathcal{T}}) + \langle Q_b\{v_0\}, \sigma_{\mathcal{T}} \cdot \mathbf{n} \rangle_{\partial\tau} \\ &= \langle I_{\mathcal{T}}^{\text{CR}} v_{\mathcal{T}}, \sigma_{\mathcal{T}} \cdot \mathbf{n} \rangle_{\partial\tau} \\ &= (\nabla_h I_{\mathcal{T}}^{\text{CR}} v_0, \sigma_{\mathcal{T}})_{\tau}. \end{aligned}$$

As $\nabla_{w,\tau} v_{\mathcal{T}}, \nabla_h I_{\mathcal{T}}^{\text{CR}} v_0$ are constant on each element $\tau \in \mathcal{T}$, we get (4.13). \blacksquare

Lemma 4.9 (Quasi-orthogonality). Let $u \in H_0^1(\Omega)$ is the weak solution of (1.1). Let \mathcal{T}_{k+1} be the refinement of \mathcal{T}_k produced in Algorithm 1, $u_k = \{u_0^k, u_b^k\} \in V_0(\mathcal{T}_k)$ and $u_{k+1} = \{u_0^{k+1}, u_b^{k+1}\} \in V_0(\mathcal{T}_{k+1})$ be the solutions to mWG discretization (2.11), respectively. For any positive constant $\delta \in (0, 1)$, we have

$$(1 - \delta)\varepsilon_{k+1} \leq \varepsilon_k - E_k + (C_1 F_{R_k})/\delta, \quad (4.14)$$

with constant C_1 depending on the shape regularity of \mathcal{T}_{k+1} .

Proof. Let $v = u - u_0^k$. Notice that $\nabla_{w,k+1} u_{k+1} - \nabla_{w,k} u_k$ is a piecewise constant on $\tau \in \mathcal{T}_{k+1}$, we have

$$\begin{aligned} (A(\nabla u - \nabla_{w,k+1} u_{k+1}), \nabla_{w,k+1} u_{k+1} - \nabla_{w,k} u_k)_{\mathcal{T}_{k+1}} \\ = (A\nabla_{k+1} I_{k+1}^{\text{CR}} v, \nabla_{w,k+1} u_{k+1} - \nabla_{w,k} u_k)_{\mathcal{T}_{k+1}}. \end{aligned}$$

As $Q_b \llbracket I_{k+1}^{\text{CR}} v \rrbracket_{e_{k+1}} = 0, \forall e_{k+1} \in \mathcal{E}_{k+1}$, we get

$$(A\nabla_{w,k+1} u_{k+1}, \nabla_h I_{k+1}^{\text{CR}} v)_{\mathcal{T}_{k+1}} = (f, \nabla_h I_{k+1}^{\text{CR}} v),$$

and

$$\begin{aligned} (A\nabla_{w,k} u_k, \nabla_h I_{k+1}^{\text{CR}} v)_{\mathcal{T}_{k+1}} &= (A\nabla_{w,k} u_k, \nabla_k I_k^{\text{CR}} (I_{k+1}^{\text{CR}} v))_{\mathcal{T}_{k+1}} \\ &= (f, \nabla_k I_k^{\text{CR}} (I_{k+1}^{\text{CR}} v))_{\mathcal{T}_{k+1}}. \end{aligned}$$

For any $\tau \in \mathcal{T}_k \cap \mathcal{T}_{k+1}$, we have $I_k^{\text{CR}} v|_{\tau} = I_{k+1}^{\text{CR}} v|_{\tau}$, then

$$(A\nabla_{k+1} I_{k+1}^{\text{CR}} v, \nabla_{w,k+1} u_{k+1} - \nabla_{w,k} u_k)_{\mathcal{T}_{k+1}} = 0.$$

Applying Lemmas 4.7,4.8, the triangle inequality, Lemma 3.3, and Lemma 3.4 leads to

$$\begin{aligned} (A\nabla_{k+1} I_{k+1}^{\text{CR}} v, \nabla_{w,k+1} u_{k+1} - \nabla_{w,k} u_k)_{\mathcal{T}_{k+1}} \\ = (f, (I - I_k^{\text{CR}}) I_{k+1}^{\text{CR}} v)_{\mathcal{T}_{k+1}} \\ \leq \sum_{\tau \in \mathcal{T}_{k+1} \setminus \mathcal{T}_k} h_{k+1} \|f\|_{\tau_{k+1}} \cdot h_{k+1}^{-1} \|(I - I_k^{\text{CR}}) I_{k+1}^{\text{CR}} v\|_{\tau_{k+1}} \end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{\tau \in \mathcal{T}_k \setminus \mathcal{T}_{k+1}} h_k \|f\|_{\tau_k} \cdot \|\nabla_{k+1} J_{k+1}^{\text{CR}} v\|_{\mathcal{T}_{k+1}} \\ &\leq \sqrt{C_1} F_{R_k}^{1/2} \cdot \varepsilon_{k+1}. \end{aligned} \tag{4.15}$$

It follows the Young's inequality and (4.15) that

$$\begin{aligned} \varepsilon_{k+1} &= \varepsilon_k - E_k - 2(A(\nabla u - \nabla_{w,k+1} u_{k+1}), \nabla_{w,k+1} u_{k+1} - \nabla_{w,k} u_k)_{\mathcal{T}_{k+1}} \\ &\leq \varepsilon_k - E_k + \delta \varepsilon_{k+1} + \delta \eta_{k+1} + (C_1 F_{R_k})/\delta. \end{aligned} \quad \blacksquare$$

4.4 | Convergence of AmWG

In the following theorem, the convergence of Algorithm 1 is proved. The main idea is to use the negative term on the right to cancel the positive terms, and to use the reduction factor $\lambda \in (0, 1)$ in (4.6) and (4.7).

Theorem 4.10. *Given a marking parameter $\theta \in (0, 1)$ and an initial mesh \mathcal{T}_0 . Let u be the solution of (1.1), $\{\mathcal{T}_k, u_k, \eta(\nabla_w u_k, \mathcal{T}_k)\}_{k \geq 0}$ be a sequence of meshes, finite element approximations and error estimators produced by Algorithm 1 with $\ell = 1$, then there exist constants $\sigma \in (0, 1)$, $\rho > 0$, $C_2 > 0$ depending only on the shape regularity of \mathcal{T}_0 , the marking parameter θ , and δ , such that if*

$$0 < \delta < \min\left(\frac{\rho(1 - (1 + \zeta)(1 - \theta\lambda))}{C_U}, 1\right),$$

then

$$(1 - \delta)\varepsilon_{k+1} + \rho \eta_{k+1}^2 + C_2 F_{k+1} \leq \sigma \left((1 - \delta)\varepsilon_k + \rho \eta_k^2 + C_2 F_k \right),$$

where the constant C_U is given by Theorem 3.5.

Proof. By adding $\rho \eta_{k+1}^2$ to both sides of (4.14), then applying Lemma 4.6, we have

$$\begin{aligned} (1 - \delta)\varepsilon_{k+1} + \rho \eta_{k+1}^2 \\ \leq \varepsilon_k + \rho(1 + \zeta)(1 - \theta\lambda)\eta_k^2 + (\zeta \lambda \rho + C_1/\delta) F_{R_k} - \rho \zeta F_k, \end{aligned} \tag{4.16}$$

for any constant $\delta \in (0, 1)$. Let $C_2 > 0$ be a to-be-determined constant, by adding $C_2 F_{k+1}$ in the both sides of (4.16) and applying Lemma 4.5, we obtain

$$\begin{aligned} (1 - \delta)\varepsilon_{k+1} + \rho \eta_{k+1}^2 + C_2 F_{k+1} \\ \leq \varepsilon_k + \rho(1 + \zeta)(1 - \theta\lambda)\eta_k^2 + (\zeta \lambda \rho + C_1/\delta - \lambda C_2) F_{R_k} + (C_2 - \rho \zeta) F_k. \end{aligned} \tag{4.17}$$

The inequality above (4.17) along with a sufficiently large C_2 satisfying

$$\zeta \lambda \rho + C_1/\delta - \lambda C_2 \leq 0, \tag{4.18}$$

combining Theorem 3.5 with (4.17), yields

$$\begin{aligned} (1 - \delta)\varepsilon_{k+1} + \rho \eta_{k+1}^2 + C_2 F_{k+1} \\ \leq \varepsilon_k + \rho(1 + \zeta)(1 - \theta\lambda)\eta_k^2 + (C_2 - \rho \zeta) F_k \\ \leq \sigma_1(1 - \delta)\varepsilon_k + (C_U - C_U \sigma_1(1 - \delta) + \rho(1 + \zeta)(1 - \theta\lambda))\eta_k^2 + (C_2 - \rho \zeta) F_k, \end{aligned}$$

according to

$$\rho \sigma_1 = C_U - C_U \sigma_1(1 - \delta) + \rho(1 + \zeta)(1 - \theta\lambda),$$

choose

$$\sigma_1 = \frac{C_U + \rho(1 + \zeta)(1 - \theta\lambda)}{C_U + \rho - C_U\delta}.$$

Choosing ζ satisfies $(1 + \zeta)(1 - \theta\lambda)(1 + \delta/\rho) \in (0, 1)$ and the requirement $0 < \delta < \min\left(\frac{\rho(1-(1+\zeta)(1-\theta\lambda))}{C_U}, 1\right)$ lead to $\sigma_1 \in (0, 1)$. By (4.18), we obtain $C_2 - \rho\zeta > 0$. Now letting $\sigma_2 = (C_2 - \rho\zeta)/C_2$ results $\sigma_2 \in (0, 1)$, and

$$(1 - \delta)\varepsilon_{k+1} + \rho\eta_{k+1}^2 + C_2F_{k+1} \leq \sigma_1(1 - \delta)\varepsilon_k + \rho\sigma_1\eta_k^2 + C_2\sigma_2F_k.$$

We complete the proof by setting $\sigma = \max\{\sigma_1, \sigma_2\} \in (0, 1)$. ■

By recursion, the decay of the error plus the estimator is as follows.

Corollary 4.11. *Under the hypotheses of Theorem 4.10, then we have*

$$(1 - \delta)\varepsilon_{k+1} + \rho\eta_{k+1}^2 + C_2F_{k+1} \leq C_0\sigma^k,$$

where the constant ρ, δ are given in Theorem 4.10, and $C_0 = (1 - \delta)e_0 + \rho\eta_0^2 + C_2F_0$. As a result, The AmWG in Algorithm 1 will terminate in finite steps.

4.5 | Discrete reliability

In this section, we prove the discrete reliability. Let \mathcal{T}_{k+1} be a refinement from \mathcal{T}_k , we recall the projection operator $J_{k+1} : V^{\text{nc}}(\mathcal{T}_k) \rightarrow V^{\text{nc}}(\mathcal{T}_{k+1})$ (see [28, Section 5]).

Lemma 4.12 ([28, Lemma 5.1]). *For any $v_k \in V^{\text{nc}}(\mathcal{T}_k)$, it holds that*

$$\|\nabla_{k+1}(J_{k+1}v_k - v_k)\|_{\mathcal{T}} \lesssim \left(\sum_{\tau \in R_k} \sum_{e \subset \partial\tau} h_\tau \|\mathcal{J}_{t,e}(\nabla_k v_k)\|_e^2 \right)^{1/2}. \quad (4.19)$$

Remark 4.13. Lemma 4.12 was presented in [28] for Stokes problem with a vector function in the Crouzeix–Raviart space, with a bound using $\|[(\nabla_k v_k)\mathbf{t}_e]\|_e$ on an edge (2D). However, since the proof only relies on the scaling of the nodal basis function, and a partition of unity property of the basis on an edge, both of which holds in 3D tetrahedral nodal basis associated with faces, the result holds for scalar functions by choosing only 1 non-trivial component in the vectorial result, and acknowledging the fact that $\|[(\nabla_k v_k) \times \mathbf{n}_e]\|_e = \|[\text{Proj}_e(\nabla_k v_k)]\|_e$ if e is a face on $\partial\tau$.

Lemma 4.14. *The following discrete reliability holds with constant C_{dr} depending on the shape regularity of the mesh*

$$E_k \leq C_{dr}\eta_{R_k}^2. \quad (4.20)$$

Proof. By Lemmas 4.7 and 4.8, we obtain

$$\begin{aligned} E_k &= \|A^{1/2}(\nabla_{w,k+1}u_{k+1} - \nabla_{w,k}u_k)\|_{\mathcal{T}_{k+1}}^2 \\ &= (A(\nabla_{w,k+1}u_{k+1} - \nabla_{w,k}u_k), \nabla_{k+1}I_{k+1}^{\text{CR}}u_0^{k+1} - \nabla_k I_k^{\text{CR}}u_0^k)_{\mathcal{T}_{k+1}} \\ &= (A(\nabla_{w,k+1}u_{k+1} - \nabla_{w,k}u_k), \nabla_{k+1}J_{k+1}I_{k+1}^{\text{CR}}u_0^{k+1} - \nabla_{k+1}J_{k+1}I_k^{\text{CR}}u_0^k)_{\mathcal{T}_{k+1}} \\ &\quad + (A(\nabla_{w,k+1}u_{k+1} - \nabla_{w,k}u_k), \nabla_{k+1}J_{k+1}I_k^{\text{CR}}u_0^k - \nabla_k I_k^{\text{CR}}u_0^k)_{\mathcal{T}_{k+1}}. \end{aligned} \quad (4.21)$$

For the first term on the right-hand side of the equation above, denote $v_{k+1}^{\text{CR}} = I_{k+1}^{\text{CR}} u_0^{k+1} - J_{k+1} I_k^{\text{CR}} u_0^k$, it follows from $Q_b \llbracket v_{k+1}^{\text{CR}} \rrbracket_{e_{k+1}} = 0, \forall e_{k+1} \in \mathcal{E}_{k+1}$ that

$$(A \nabla_{w,k+1} u_{k+1}, \nabla_{k+1} v_{k+1}^{\text{CR}})_{\mathcal{T}_{k+1}} = (f, v_{k+1}^{\text{CR}}),$$

and $(\nabla_{k+1} v_{k+1}^{\text{CR}}, \boldsymbol{\sigma})_{\tau} = (\nabla_k I_k^{\text{CR}} v_{k+1}^{\text{CR}}, \boldsymbol{\sigma})_{\tau}$ for a constant vector $\boldsymbol{\tau}$ on $\forall \tau \in \mathcal{T}_k \setminus \mathcal{T}_{k+1}$

$$(A \nabla_{w,k} u_k, \nabla_{k+1} v_{k+1}^{\text{CR}})_{\mathcal{T}_{k+1}} = (A \nabla_{w,k} u_k, \nabla_k I_k^{\text{CR}} v_{k+1}^{\text{CR}})_{\mathcal{T}_k} = (f, I_k^{\text{CR}} v_{k+1}^{\text{CR}}).$$

Combining both further implies

$$\begin{aligned} & (A(\nabla_{w,k+1} u_{k+1} - \nabla_{w,k} u_k), \nabla_k v_k^{\text{CR}})_{\mathcal{T}_{k+1}} \\ &= (f, (I - I_k^{\text{CR}}) v_{k+1}^{\text{CR}}) \\ &\lesssim F_{R_k}^{1/2} \cdot \|A^{1/2} \nabla_{k+1} (I_{k+1}^{\text{CR}} u_0^{k+1} - J_{k+1} I_k^{\text{CR}} u_0^k)\|_{\mathcal{T}_{k+1}} \\ &\leq F_{R_k}^{1/2} \|A^{1/2} (\nabla_{w,k+1} u_{k+1} - \nabla_{w,k} u_k)\|_{\mathcal{T}_{k+1}} \\ &\quad + \|A^{1/2} (\nabla_{k+1} J_{k+1} I_k^{\text{CR}} u_0^k - \nabla_k I_k^{\text{CR}} u_0^k)\|_{\mathcal{T}_{k+1}}. \end{aligned} \tag{4.22}$$

For the second term in (4.21), together with Lemma 4.8, applying the Cauchy–Schwarz inequality implies

$$\begin{aligned} & (A(\nabla_{w,k+1} u_{k+1} - \nabla_{w,k} u_k), \nabla_{k+1} J_{k+1} I_k^{\text{CR}} u_0^k - \nabla_k I_k^{\text{CR}} u_0^k)_{\mathcal{T}_{k+1}} \\ &\leq \|A^{1/2} (\nabla_{w,k+1} u_{k+1} - \nabla_{w,k} u_k)\|_{\mathcal{T}_{k+1}} \\ &\quad \cdot \|A^{1/2} (\nabla_{k+1} J_{k+1} I_k^{\text{CR}} u_0^k - \nabla_k I_k^{\text{CR}} u_0^k)\|_{\mathcal{T}_{k+1}}. \end{aligned} \tag{4.23}$$

After inserting (4.22) and (4.23) into (4.21), using the Young’s inequality implies

$$E_k \lesssim F_{R_k} + \|A^{1/2} \nabla_{k+1} (J_{k+1} I_k^{\text{CR}} u_0^k - I_k^{\text{CR}} u_0^k)\|_{\mathcal{T}_{k+1}}^2. \tag{4.24}$$

At last, using Lemma 4.12 leads to the desired result. ■

4.6 | The optimality of the AmWG

In this section, the optimality of the AmWG Algorithm 1 will be shown. First a C ea-type lemma can be obtained as follows.

Lemma 4.15. *There exists a constant C_3 depending only on the shape regularity of \mathcal{T} such that*

$$\begin{aligned} & \|A^{1/2} (\nabla u - \nabla_w u_{\mathcal{T}})\|_{\mathcal{T}}^2 + F(f, \mathcal{T}) \\ &\leq C_4 \inf_{v_{\mathcal{T}} \in V(\mathcal{T})} (\|A^{1/2} (\nabla u - \nabla_w v_{\mathcal{T}})\|_{\mathcal{T}}^2 + F(f, \mathcal{T})). \end{aligned} \tag{4.25}$$

Proof. The application of Strang’s lemma [48] yields

$$\begin{aligned} & \|A^{1/2} (\nabla u - \nabla_w u_{\mathcal{T}})\|_{\mathcal{T}}^2 \\ &= \|A^{1/2} (\nabla u - \nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} u_0)\|_{\mathcal{T}}^2 \\ &\lesssim \|A^{1/2} (\nabla u - \nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0)\|_{\mathcal{T}}^2 \\ &\quad + \sup_{v_{\mathcal{T}} \in V(\mathcal{T})} \frac{(A \nabla u, \nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0) - (f, I_{\mathcal{T}}^{\text{CR}} v_0)}{\|\nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0\|}. \end{aligned} \tag{4.26}$$

We need to define the following higher order conforming finite element space

$$V^c(\mathcal{T}) := \{v \in H_0^1(\Omega), v|_\tau \in (P_3(K))^d, d = 2, 3, \forall \tau \in \mathcal{T}\}, \quad (4.27)$$

there exists an interpolation $Y_{\mathcal{T}} : V^{\text{nc}}(\mathcal{T}) \rightarrow V^c(\mathcal{T})$ with following properties (see [28, Section 6])

$$\begin{aligned} \int_e (w_{\mathcal{T}} - Y_{\mathcal{T}} w_{\mathcal{T}}) \cdot c_e ds &= 0, \quad \forall c_e \in P_1(e), \\ \int_{\tau} (w_{\mathcal{T}} - Y_{\mathcal{T}} w_{\mathcal{T}}) dx &= 0, \end{aligned} \quad (4.28)$$

for $w_{\mathcal{T}} \in V^{\text{nc}}(\mathcal{T})$, edge/face e and $\tau \in \mathcal{T}$. We also have

$$\|w_{\mathcal{T}} - Y_{\mathcal{T}} w_{\mathcal{T}}\|_{\mathcal{T}} + h_{\tau} \|\nabla Y_{\mathcal{T}} w_{\mathcal{T}}\|_{\mathcal{T}} \lesssim h_{\tau} \|\nabla_h w_{\mathcal{T}}\|_{0, \omega(\tau)}. \quad (4.29)$$

For any $v_{\mathcal{T}} = \{v_0, v_b\} \in V(\mathcal{T})$, the following decomposition holds:

$$\begin{aligned} (A \nabla u, \nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0) - (f, I_{\mathcal{T}}^{\text{CR}} v_0) \\ = (A \nabla u - A \nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} u_0, \nabla_{\mathcal{T}} (I_{\mathcal{T}}^{\text{CR}} v_0 - Y_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0)) \\ - (f, I_{\mathcal{T}}^{\text{CR}} v_0 - Y_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0) + (A \nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} u_0, \nabla_{\mathcal{T}} (I_{\mathcal{T}}^{\text{CR}} v_0 - Y_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0)), \end{aligned}$$

By the properties (4.28) and (4.29), we have

$$\begin{aligned} (A \nabla u, \nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0) - (f, I_{\mathcal{T}}^{\text{CR}} v_0) \\ \lesssim \|\nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0\| \cdot \|A^{1/2} (\nabla u - \nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0)\|_{\mathcal{T}} + F^{1/2}(f, \mathcal{T}) \cdot \|\nabla_{\mathcal{T}} I_{\mathcal{T}}^{\text{CR}} v_0\|. \end{aligned} \quad (4.30)$$

After inserting (4.30) into (4.26), we use Young's inequality to have the desired result (4.25). \blacksquare

Let \mathbb{T}_N be the set of all partitions \mathcal{T} which is refined from \mathcal{T}_0 and $\#\mathcal{T} \leq N$. For a given partition \mathcal{T} , we introduce the following semi-norm:

$$|u, f|_s^2 = \sup_{N > s} N^s \inf_{\mathcal{T} \in \mathbb{T}_N} \left(\inf_{v_{\mathcal{T}} \in V(\mathcal{T})} \|A^{1/2} (\nabla u - \nabla_w v_{\mathcal{T}})\|_{\mathcal{T}}^2 + F(f, \mathcal{T}) \right), \quad (4.31)$$

and the approximation class is then defined as follows, for $s > 0$:

$$\mathbb{A}_s := \{(u, f) : |u, f|_s < +\infty\}. \quad (4.32)$$

In this case, we recall all ingredients needed for the optimality of the adaptive procedure:

- (1) Quasi-orthogonality in Lemma 4.9:

$$(A(\nabla u - \nabla_{w, k+1} u_{k+1}), \nabla_{w, k+1} u_{k+1} - \nabla_{w, k} u_k)_{\mathcal{T}_{k+1}} \leq \sqrt{C_1} F_{R_k}^{1/2} \cdot \varepsilon_{k+1};$$

- (2) Discrete Reliability in Lemma 4.14:

$$E_k \leq C_{dr} \eta_{R_k}^2;$$

- (3) The lower bound when the polynomial degree $\ell = 1$:

$$C_L \eta (\nabla_w u_{\mathcal{T}}, \mathcal{T})^2 \leq \|A^{1/2} (\nabla u - \nabla_w u_{\mathcal{T}})\|_{\mathcal{T}}^2 + F(f, \mathcal{T}).$$

Thanks to these preparations, following [29], the optimality result is as follows:

Theorem 4.16. Let u be the solution of (1.1), $\{\mathcal{T}_N, u_N, \eta(\nabla_w u_k, \mathcal{T}_N)\}_{N \geq 0}$ be a sequence of meshes, finite element approximations and error estimators produced by Algorithm 1. For $(u, f) \in \mathbb{A}_s$ with

$$\theta \in \left(0, \frac{\min(1, C_L)}{\min(1, C_L) + C_1 + 1}\right), \tag{4.33}$$

then it holds that

$$\|A^{1/2}(\nabla u - \nabla_w u_{\mathcal{T}_N})\|_{\mathcal{T}_N}^2 + F(f, \mathcal{T}_N) \lesssim |u, f|_s^2 (\#\mathcal{T}_N - \#\mathcal{T}_0)^{-2s}. \tag{4.34}$$

5 | NUMERICAL EXPERIMENTS

In this section, with the aid of the MATLAB software package iFEM [49], we implement the following numerical experiments to verify the convergence and quasi-optimality of the Algorithm 4.1.

Example 5.1. In this example, we choose a square domain $\Omega = (-1, 1)^2$ and coefficient $A = \mathbf{I}$, the exact solution of (1.1) is $u(x, y) = \frac{y(x^2-1)(y^2-1)}{x^2+y^2+0.01}$.

On the left of Figure 1 shows the initial mesh \mathcal{T}_0 for Example 5.1; on the right of Figure 1 shows the refined mesh after $k = 18$ iterations for the Example 5.1 with $\theta = 0.5$.

Figure 2 shows the rate of $\ln \#\mathcal{T}_k$ versus $\ln \|A^{1/2}(\nabla u - \nabla_w u_{\mathcal{T}_k})\|_{\mathcal{T}_k}$ with different marking parameters $\theta = 0.3, 0.5$, and 0.7 , where $\#\mathcal{T}_k$ and u_k represent the number of elements and the corresponding solution, respectively, gotten from the Algorithm 4.1.

Example 5.2. In this example, we choose the L-shape domain $\Omega = (-1, 1)^2 / ([0, 1) \times (-1, 0])$ and coefficient $A = \mathbf{I}$, the exact solution of (1.1) is $u(x, y) = r^{2/3} \sin(\frac{2\theta}{3})$.

On the left of Figure 3 shows the initial mesh \mathcal{T}_0 for Example 5.2; on the right of Figure 3 shows the refined mesh after $k = 20$ iterations for the Example 5.2 with $\theta = 0.7$.

Figure 4 shows the rate of $\ln \#\mathcal{T}_k$ versus $\ln \|A^{1/2}(\nabla u - \nabla_w u_{\mathcal{T}_k})\|_{\mathcal{T}_k}$ with different marking parameters $\theta = 0.3, 0.5$, and 0.7 , where $\#\mathcal{T}_k$ and u_k represent the number of elements and the corresponding solution, respectively, gotten from the Algorithm 4.1.

The right one of Figures 1 and 3 show that the mesh is locally refined. And the curves in Figures 2 and 4 indicate that the convergence and the quasi-optimality of the Algorithm 4.1, namely

$$\|A^{1/2}(\nabla u - \nabla_w u_{\mathcal{T}_k})\|_{\mathcal{T}_k} \leq C(\#\mathcal{T}_k)^{-1/2}.$$

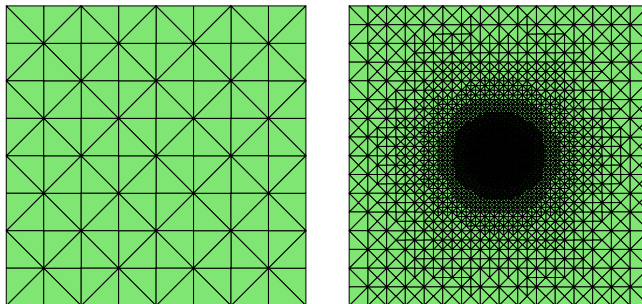


FIGURE 1 The initial mesh (left); adaptively refined mesh after $k = 18$ iterations (right) for Example 5.1.

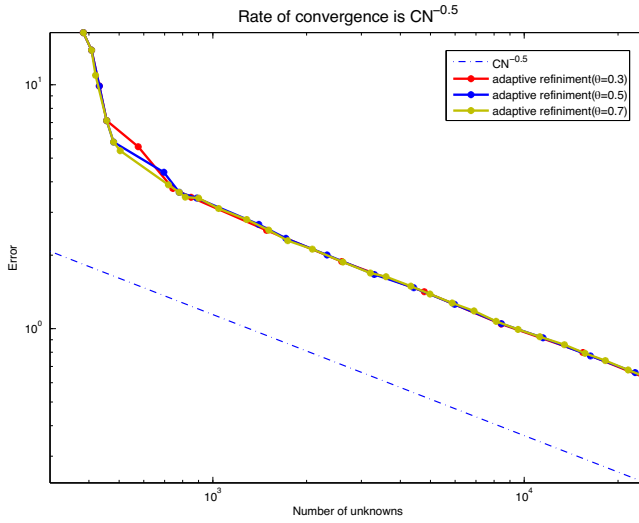


FIGURE 2 Quasi-optimality of the adaptive mesh refinements with marking parameters $\theta = 0.3, 0.5, 0.7$.

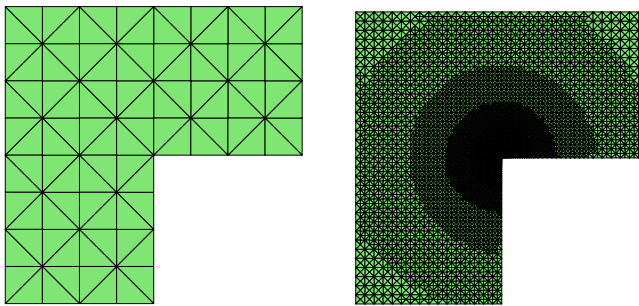


FIGURE 3 The initial mesh (left); adaptively refined mesh after $k = 20$ iterations (right) for Example 5.2.

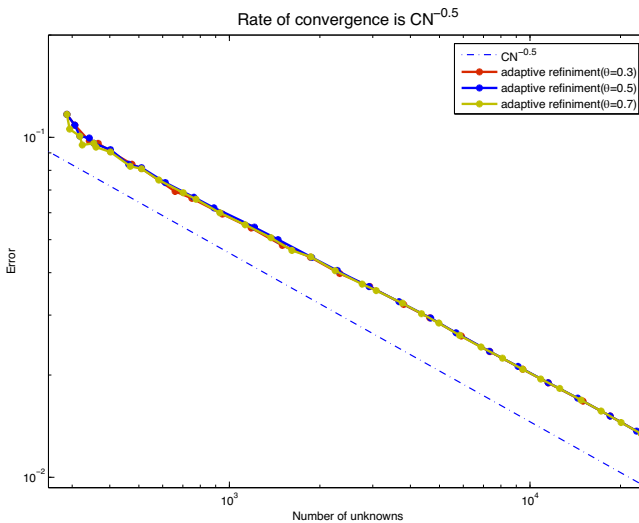


FIGURE 4 Quasi-optimality of the adaptive mesh refinements with marking parameters $\theta = 0.3, 0.5, 0.7$.

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DATA AVAILABILITY STATEMENT

Data will be available at <https://github.com/lyc102/ifem/tree/master/research> after publication.

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